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Math 8385: Calculus of Variations

Take-home Final

(1) I would like to make a little change of notations in the problem so that they are more convenient for me to write. Specifically, I would like to write "u" instead of "x" in the arguments of the functional L, and reserve the name $x = (x_1, \dots, x_d)$ for Cartesian coordinates on $\mathbb{R}^d$. Denote

$$u = (u_1, u_2, \dots, u_d) \in \mathbb{R}^d,$$
$$p = (p_1, p_2, \dots, p_d) \in \mathbb{R}^d,$$
$$r = (r_1, r_2, \dots, r_d) \in \mathbb{R}^d.$$

Let $L = L(t, u, p, r) \in C^3([a,b] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$. For $u \in C^2([a,b], \mathbb{R}^d)$, consider

$$I(u) = \int_a^b L(t, u, \dot{u}, \ddot{u})\, dt,$$

with boundary conditions (BC): $u(a) = u_0$, $\dot{u}(a) = p_0$, $u(b) = u_1$, $\dot{u}(b) = p_1$.

We'll derive the Euler-Lagrange equations for $I(u)$.

We see that if u satisfies the boundary condition (BC) then so does $u + s\varphi$ where $s \in \mathbb{R}$ and $\varphi \in C_0^2([a,b], \mathbb{R}^d)$. Therefore, if u is a critical point of I then $\frac{d}{ds}\Big|_{s=0} I(u+s\varphi) = 0$. By the chain rule of differentiation, we have

$$0 = \frac{d}{ds}\Big|_{s=0} I(u+s\varphi) = \int_a^b \frac{d}{ds}\Big|_{s=0} L(t, u+s\varphi, \dot{u}+s\dot{\varphi}, \ddot{u}+s\ddot{\varphi})\, dt$$

$$= \int_a^b \Big[L_{u_i}(t, u+s\varphi, \dot{u}+s\dot{\varphi}, \ddot{u}+s\ddot{\varphi})\Big|_{s=0} \varphi_i + L_{p_j}(t, u+s\varphi, \dot{u}+s\dot{\varphi}, \ddot{u}+s\ddot{\varphi})\Big|_{s=0} \dot{\varphi}_j$$
$$+ L_{r_k}(t, u+s\varphi, \dot{u}+s\dot{\varphi}, \ddot{u}+s\ddot{\varphi})\Big|_{s=0} \ddot{\varphi}_k \Big]\, dt$$

$$= \int_a^b [L_{u_i}(t,u,\dot u,\ddot u)\varphi_i + L_{p_j}(t,u,\dot u,\ddot u)\dot\varphi_j + L_{r_k}(t,u,\dot u,\ddot u)\ddot\varphi_k]\,dt$$

Thus, $$\underbrace{\int_a^b L_{u_i}(t,u,\dot u,\ddot u)\varphi_i\,dt}_{\{1\}} + \underbrace{\int_a^b L_{p_j}(t,u,\dot u,\ddot u)\dot\varphi_j\,dt}_{\{2\}} + \underbrace{\int_a^b L_{r_k}(t,u,\dot u,\ddot u)\ddot\varphi_k\,dt}_{\{3\}} = 0 \quad (1)$$

In the following, we'll use the integration-by-part formula. Thus, we can assume that $u \in C^4([a,b],\mathbb{R}^d)$ so that all differentiations in t are valid.

We have $$\{2\} = \underbrace{L_{p_j}(t,u,\dot u,\ddot u)\varphi_j(t)\Big|_{t=a}^{t=b}}_{=0 \text{ because } \varphi_j(a)=\varphi_j(b)=0} - \int_a^b \frac{d}{dt}[L_{p_j}(t,u,\dot u,\ddot u)]\dot\varphi_j\,dt$$

$$= -\int_a^b \frac{d}{dt}[L_{p_j}(t,u,\dot u,\ddot u)]\varphi_j\,dt \qquad (2)$$

We have $$\{3\} = \underbrace{L_{r_k}(t,u,\dot u,\ddot u)\dot\varphi_k(t)\Big|_{t=a}^{t=b}}_{=0 \text{ because } \dot\varphi_k(a)=\dot\varphi_k(b)=0} - \int_a^b \frac{d}{dt}[L_{r_k}(t,u,\dot u,\ddot u)]\dot\varphi_k\,dt$$

$$= \underbrace{-\frac{d}{dt}[L_{r_k}(t,u,\dot u,\ddot u)]\varphi_k(t)\Big|_{t=a}^{t=b}}_{=0 \text{ because } \varphi_k(a)=\varphi_k(b)=0} + \int_a^b \frac{d^2}{dt^2}[L_{r_k}(t,u,\dot u,\ddot u)]\varphi_k\,dt$$

$$= \int_a^b \frac{d^2}{dt^2}[L_{r_k}(t,u,\dot u,\ddot u)]\varphi_k\,dt \qquad (3)$$

Now substituting (2) and (3) into (1), we get

$$\int_a^b \left\{ L_{u_i}(t,u,\dot u,\ddot u) - \frac{d}{dt}[L_{p_i}(t,u,\dot u,\ddot u)] + \frac{d^2}{dt^2}[L_{r_i}(t,u,\dot u,\ddot u)] \right\}\varphi_i(t)\,dt = 0.$$

Because this identity is true for all $\varphi = (\varphi_1,\dots,\varphi_d) \in C_0^2([a,b],\mathbb{R}^d)$, by the Fundamental lemma of Calculus of Variations, we have

$$L_{u_i}(t,u,\dot u,\ddot u) - \frac{d}{dt}\left[L_{p_i}(t,u,\dot u,\ddot u)\right] + \frac{d^2}{dt^2}\left[L_{r_i}(t,u,\dot u,\ddot u)\right] = 0 \quad \forall t\in(a,b), \ \forall\, 1\le i\le d.$$

These are the Euler-Lagrange equations for the functional $I(u)$.

Every critical point u which is in $C^4([a,b],\mathbb{R}^d)$ must satisfy these equations.

(2) We restate the Noether's theorem with time-variance (a corrected version of Theorem 1.5.2, page 29, Jost - Li Jost) as follows.

Consider the variational integral $I(x) = \int_a^b F(t, x(t), \dot x(t))\,dt$, with $F\in C^2([a,b]\times\mathbb{R}^d\times\mathbb{R}^d)$. Suppose there is a smooth one-parameter family of differentiable maps $\bar h_s = (h_s^0, h_s) : [a,b]\times\mathbb{R}^d \to \mathbb{R}\times\mathbb{R}^d$, for $s\in(-\varepsilon,\varepsilon)$, with

$$\bar h_0(t,z) = (t,z) \quad \forall (t,z)\in[a,b]\times\mathbb{R}^d$$

and satisfying

$$\int_{h_s^0(t_0)}^{h_s^0(t_1)} F\left(t_s, x_s(t_s), \frac{dx_s}{dt_s}\right)dt_s = \int_{t_0}^{t_1} F(t, x(t), \dot x(t))\,dt$$

$\forall [t_0,t_1]\subset[a,b]$, $\forall x\in C^2([a,b],\mathbb{R}^d)$, where

$t_s = h_s^0(t)$, $x_s(t_s) = h_s(x(t))$.

Then for any solution x to the Euler-Lagrange equations of I,

$$F_p(t,x,\dot x)\left.\frac{d}{ds}\right|_{s=0} h_s(x(t)) + \left[F(t,x,\dot x) - \dot x F_p(t,x,\dot x)\right]\left.\frac{d}{ds}\right|_{s=0} h_s^0(t) \qquad (1)$$

is constant with respect to $t\in[a,b]$.

Let $\Phi\in C^2(\mathbb{R}\times\mathbb{R}^3)$ and a family of maps $\bar h_s : \mathbb{R}\times\mathbb{R}^3\to\mathbb{R}$,

$$\bar h_s(t,x_1,x_2,x_3) = (t+s,\ x_1\cos s + x_2\sin s,\ -x_1\sin s + x_2\cos s,\ x_3+s)$$

such that $\Phi \circ \bar{h}_s(t,x) \equiv \Phi(t,x)$ in $\mathbb{R}\times\mathbb{R}^3$, for all $s\in(-\varepsilon,\varepsilon)$. Let $f:[0,\infty)\to[0,\infty)$ be a C^2 function. Consider the variational integral, which is an energy functional of an expanding screw motion,

$$I(x) = \int_{t_1}^{t_2} \left[\frac{1}{2} f(r)|\dot{x}|^2 - \Phi(t,x(t))\right] dt,$$

where t_1, t_2 are constants and $r = r(x(t)) = \sqrt{x_1(t)^2 + x_2(t)^2}$. We want to use Noether's theorem to find a first integral of motion. Put

$$h_s(x_1,x_2,x_3) = (x_1\cos s + x_2 \sin s,\ -x_1 \sin s + x_2 \cos s,\ x_3 + s),$$
$$h_s^0(t) = t+s.$$

Then $\bar{h}_s = (h_s^0, h_s)$ and $\bar{h}_0(t,x_1,x_2,x_3) = (t,x_1,x_2,x_3)$. We want to apply Noether's theorem for $a=t_1$, $b=t_2$,

$$F(t,x,p) = \frac{1}{2} f(r)|p|^2 - \Phi(t,x),$$

where $r = r(x) = \sqrt{x_1^2+x_2^2}$.

Since $f\in C^2([0,\infty))$, $F\in C^2([a,b]\times\mathbb{R}^3\times\mathbb{R}^3)$. In order to apply Noether's theorem, we need to show that for any function $x=(x_1,x_2,x_3)\in C^2([t_1,t_2],\mathbb{R}^3)$ and any subinterval $[\alpha,\beta]\subset[t_1,t_2]$,

$$\int_{h_s^0(\alpha)}^{h_s^0(\beta)} F\left(t_s, x_s(t_s), \frac{dx_s}{dt_s}\right) dt_s = \int_\alpha^\beta F(t,x(t),\dot{x}(t))\, dt, \qquad (2)$$

where $t_s = h_s^0(t)$ and $x_s(t_s) = h_s(x(t))$.

We have $x_s(t_s) = h_s\circ x(t) = \big(x_1(t)\cos s + x_2(t)\sin s,\ -x_1(t)\sin s + x_2(t)\cos s,\ x_3(t)+s\big)$ (3)

$$\frac{dx_s}{dt_s}(t_s) = \frac{d}{dt}(h_s \circ x(t)) \underbrace{\frac{dt}{dt_s}}_{=1} = \left(\frac{dx_1}{dt}\cos s + \frac{dx_2}{dt}\sin s, -\frac{dx_1}{dt}\sin s + \frac{dx_2}{dt}\cos s, \frac{dx_3}{dt}\right).$$

Thus,
$$\left|\frac{dx_s}{dt_s}(t_s)\right|^2 = \left(\frac{dx_1}{dt}\cos s + \frac{dx_2}{dt}\sin s\right)^2 + \left(-\frac{dx_1}{dt}\sin s + \frac{dx_2}{dt}\cos s\right)^2 + \left(\frac{dx_3}{dt}\right)^2$$
$$= \left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2$$
$$= \left|\frac{dx}{dt}\right|^2 = |\dot{x}|^2 \qquad (4)$$

Also,
$$r(x_s(t_s)) \overset{(3)}{=\!=} \sqrt{(x_1(t)\cos s + x_2(t)\sin s)^2 + (-x_1(t)\sin s + x_2(t)\cos s)^2}$$
$$= \sqrt{x_1(t)^2 + x_2(t)^2}$$
$$= r(x(t)) \qquad (5)$$

We have
$$LHS(2) = \int_{\alpha+s}^{\beta+s} \left[\frac{1}{2} f(r(x_s(t_s))) \left|\frac{dx_s}{dt_s}\right|^2 - \Phi(t_s, x_s(t_s))\right] dt_s$$
$$\overset{(4),(5)}{=\!=} \int_{\alpha}^{\beta} \left[\frac{1}{2} f(r(t)) |\dot{x}|^2 - \Phi(h_s^0(t), h_s(x(t)))\right] dt$$
$$= \int_{\alpha}^{\beta} \frac{1}{2} f(r(t)) |\dot{x}|^2 dt - \int_{\alpha}^{\beta} \underbrace{\Phi(\bar{h}_s(t,x))}_{=\Phi(t,x)} dt$$
$$= \int_{\alpha}^{\beta} \left[\frac{1}{2} f(r(t)) |\dot{x}|^2 - \Phi(t, x(t))\right] dt$$
$$= RHS(2).$$

Therefore, (2) is proved. Then by Noether's theorem, we have equation (1):

$$F_p(t,x,\dot{x}) \frac{d}{ds}\Big|_{s=0} h_s(x(t)) + \left[F(t,x,\dot{x}) - \dot{x} F_p(t,x,\dot{x})\right] \frac{d}{ds}\Big|_{s=0} h_s^0(t) \equiv \text{const} \qquad (6)$$

(with respect to $t \in [t_1, t_2]$)

Because $F(t,x,p) = \frac{1}{2} f(r)|p|^2 - \Phi(t,x)$, $F_p(t,x,p) = f(r)p$. Thus,

$$F_p(t,x,\dot{x}) = f(r)\dot{x} \qquad (7)$$

Moreover, $\left.\frac{d}{ds}\right|_{s=0} \xi_s^0(t) = \left.\frac{d}{ds}\right|_{s=0} (t+s) = 1$, $\qquad (8)$

$$\left.\frac{d}{ds}\right|_{s=0} h_s(x(t)) = \left.\frac{d}{ds}\right|_{s=0} (x_1(t)\cos s + x_2(t)\sin s,\ -x_1(t)\sin s + x_2(t)\cos s,\ x_3(t)+s)$$

$$= (-x_1(t)\sin s + x_2(t)\cos s,\ -x_1(t)\cos s - x_2(t)\sin s,\ 1)\Big|_{s=0}$$

$$= (x_2(t), -x_1(t), 1) \qquad (9)$$

Substituting (7), (8), (9) into (6), we get

$$LHS(6) = f(r)\dot{x}\cdot(x_2(t), -x_1(t), 1) + \left\{\left[\frac{1}{2} f(r)|\dot{x}|^2 - \Phi(t,x)\right] - \dot{x} f(r)\cdot\dot{x}\right\}$$

$$= f(r)(\dot{x}_1(t)x_2(t) - \dot{x}_2(t)x_1(t) + \dot{x}_3(t)) - \left(\frac{1}{2} f(r)|\dot{x}|^2 + \Phi(t,x)\right).$$

Therefore, the first integral of the Euler-Lagrange equations of $I(x)$ obtained from Noether's theorem is

$$f(r)(\dot{x}_1 x_2 - \dot{x}_2 x_1 + \dot{x}_3) - \frac{1}{2} f(r)|\dot{x}|^2 - \Phi(t,x) \equiv \text{Const} \quad \text{in } t\in[t_1,t_2].$$

(3) For any two functions $F, G \in C^1(\mathbb{R}^{2n})$, we define the Poisson bracket

$$\{F,G\} := \frac{\partial G}{\partial x_j}\frac{\partial F}{\partial p_j} - \frac{\partial G}{\partial p_j}\frac{\partial F}{\partial x_j},$$

where $F = F(x,p)$, $G = G(x,p)$, $x = (x_1,\dots,x_n)$, $p = (p_1,\dots,p_n)$.

Consider a local diffeomorphism $\Psi: U \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $\Psi(x,p) = (\xi,\pi)$, where $\xi = (\xi_1,\dots,\xi_n)$ and $\pi = (\pi_1,\dots,\pi_n)$. We'll show that Ψ is a canonical transformation if and only if $\{F,G\}\circ\Psi = \{F\circ\Psi, G\circ\Psi\}$ for all $F, G \in C^1(\mathbb{R}^{2n})$. Recall the definition: Ψ is called a canonical transformation if the following identities hold.

$$\frac{\partial p_i}{\partial \pi_j} = \frac{\partial \xi_j}{\partial x_i}, \qquad \frac{\partial x_i}{\partial \pi_j} = -\frac{\partial \xi_j}{\partial p_i},$$

$$\frac{\partial p_i}{\partial \xi_j} = -\frac{\partial \pi_j}{\partial x_i}, \qquad \frac{\partial x_i}{\partial \xi_j} = \frac{\partial \pi_j}{\partial p_i}.$$

Equivalently,
$$\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial p} \\ \frac{\partial \pi}{\partial x} & \frac{\partial \pi}{\partial p} \end{pmatrix}^{-1} = \begin{pmatrix} \left(\frac{\partial \pi}{\partial p}\right)^T & -\left(\frac{\partial \xi}{\partial p}\right)^T \\ -\left(\frac{\partial \pi}{\partial x}\right)^T & \left(\frac{\partial \xi}{\partial x}\right)^T \end{pmatrix}.$$

Multiplying both sides by $\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial p} \\ \frac{\partial \pi}{\partial x} & \frac{\partial \pi}{\partial p} \end{pmatrix} = D\Psi(x,p)$, we get

$$\begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial p} \\ \frac{\partial \pi}{\partial x} & \frac{\partial \pi}{\partial p} \end{pmatrix} \begin{pmatrix} \left(\frac{\partial \pi}{\partial p}\right)^T & -\left(\frac{\partial \xi}{\partial p}\right)^T \\ -\left(\frac{\partial \pi}{\partial x}\right)^T & \left(\frac{\partial \xi}{\partial x}\right)^T \end{pmatrix}$$

Thus, Ψ is a canonical transformation if and only if we have 4 identities:

$$\frac{\partial \xi}{\partial x}\left(\frac{\partial \pi}{\partial p}\right)^T - \frac{\partial \xi}{\partial p}\left(\frac{\partial \pi}{\partial x}\right)^T = I_n,$$

$$-\frac{\partial \xi}{\partial x}\left(\frac{\partial \xi}{\partial p}\right)^T + \frac{\partial \xi}{\partial p}\left(\frac{\partial \xi}{\partial x}\right)^T = 0,$$

$$\frac{\partial \pi}{\partial x}\left(\frac{\partial \pi}{\partial p}\right)^T - \frac{\partial \pi}{\partial p}\left(\frac{\partial \pi}{\partial x}\right)^T = 0,$$

$$-\frac{\partial \pi}{\partial x}\left(\frac{\partial \xi}{\partial p}\right)^T + \frac{\partial \pi}{\partial p}\left(\frac{\partial \xi}{\partial x}\right)^T = I_n.$$

In terms of indices, those equations are

$$\frac{\partial \xi_k}{\partial x_j}\frac{\partial \pi_l}{\partial p_j} - \frac{\partial \xi_k}{\partial p_j}\frac{\partial \pi_l}{\partial x_j} = \delta_{kl}, \qquad (1)$$

$$-\frac{\partial \xi_k}{\partial x_j}\frac{\partial \xi_l}{\partial p_j} + \frac{\partial \xi_l}{\partial x_j}\frac{\partial \xi_k}{\partial p_j} = 0, \qquad (2)$$

$$\frac{\partial \pi_k}{\partial x_j}\frac{\partial \pi_l}{\partial p_j} - \frac{\partial \pi_l}{\partial x_j}\frac{\partial \pi_k}{\partial p_j} = 0, \qquad (3)$$

$$-\frac{\partial \pi_k}{\partial x_j}\frac{\partial \xi_l}{\partial x_j} + \frac{\partial \pi_l}{\partial x_j}\frac{\partial \xi_k}{\partial p_j} = \delta_{kl}, \qquad (4)$$

for all $1 \leq k, l \leq n$.

For <u>any</u> local diffeomorphism $\psi : U \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and $F, G \in C^1(\mathbb{R}^{2n})$, we have

$$\{F, G\} \circ \psi(x,p) = \frac{\partial G}{\partial x_j}(\psi(x,p))\,\frac{\partial F}{\partial p_j}(\psi(x,p)) - \frac{\partial G}{\partial p_j}(\psi(x,p))\,\frac{\partial F}{\partial x_j}(\psi(x,p)).$$

In other words,

$$\{F, G\} \circ \psi(x,p) = \frac{\partial G}{\partial x_j}(\xi,\pi)\,\frac{\partial F}{\partial p_j}(\xi,\pi) - \frac{\partial G}{\partial p_j}(\xi,\pi)\,\frac{\partial F}{\partial x_j}(\xi,\pi) \qquad (5)$$

We have

$$\{F\circ\psi, G\circ\psi\}(x,p) = \frac{\partial (G\circ\psi)}{\partial x_j}(x,p)\,\frac{\partial (F\circ\psi)}{\partial p_j}(x,p) - \frac{\partial (G\circ\psi)}{\partial p_j}(x,p)\,\frac{\partial (F\circ\psi)}{\partial x_j}(x,p) \qquad (6)$$

By the chain rule,

$$\frac{\partial (G\circ\psi)}{\partial x_j}(x,p) = \frac{\partial G}{\partial x_k}(\psi(x,p))\,\frac{\partial \xi_k}{\partial x_j} + \frac{\partial G}{\partial p_k}(\psi(x,p))\,\frac{\partial \pi_k}{\partial x_j}$$

$$= \frac{\partial G}{\partial x_k}(\xi,\pi)\,\frac{\partial \xi_k}{\partial x_j} + \frac{\partial G}{\partial p_k}(\xi,\pi)\,\frac{\partial \pi_k}{\partial x_j},$$

$$\frac{\partial (F\circ\psi)}{\partial p_j}(x,p) = \frac{\partial F}{\partial x_\ell}(\psi(x,p))\,\frac{\partial \xi_\ell}{\partial p_j} + \frac{\partial F}{\partial p_\ell}(\psi(x,p))\,\frac{\partial \pi_\ell}{\partial p_j}$$

$$= \frac{\partial F}{\partial x_\ell}(\xi,\pi)\,\frac{\partial \xi_\ell}{\partial p_j} + \frac{\partial F}{\partial p_\ell}(\xi,\pi)\,\frac{\partial \pi_\ell}{\partial p_j}.$$

Thus,

$$\frac{\partial (G\circ\psi)}{\partial x_j}(x,p)\,\frac{\partial (F\circ\psi)}{\partial p_j}(x,p) = \underbrace{\frac{\partial G}{\partial x_k}(\xi,\pi)\,\frac{\partial F}{\partial x_\ell}(\xi,\pi)\,\frac{\partial \xi_k}{\partial x_j}\,\frac{\partial \xi_\ell}{\partial p_j}}_{\{1\}} +$$

$$+ \underbrace{\frac{\partial G}{\partial p_k}(\xi,\pi)\,\frac{\partial F}{\partial p_\ell}(\xi,\pi)\,\frac{\partial \pi_k}{\partial x_j}\,\frac{\partial \pi_\ell}{\partial p_j}}_{\{2\}} +$$

$$+ \underbrace{\frac{\partial G}{\partial x_k}(\xi,\pi)\,\frac{\partial F}{\partial p_\ell}(\xi,\pi)\,\frac{\partial \xi_k}{\partial x_j}\,\frac{\partial \pi_\ell}{\partial p_j}}_{\{3\}} + \underbrace{\frac{\partial G}{\partial p_k}(\xi,\pi)\,\frac{\partial F}{\partial x_\ell}(\xi,\pi)\,\frac{\partial \pi_k}{\partial x_j}\,\frac{\partial \xi_\ell}{\partial x_j}}_{\{4\}} \qquad (7)$$

Similarly,

$$\frac{\partial (G\circ\psi)}{\partial p_j}(x,p) = \frac{\partial F}{\partial x_k}(\xi,\pi)\,\frac{\partial \xi_k}{\partial p_j} + \frac{\partial G}{\partial p_k}(\xi,\pi)\,\frac{\partial \pi_k}{\partial p_j},$$

$$\frac{\partial (F\circ\psi)}{\partial x_j}(x,p) = \frac{\partial F}{\partial x_\ell}(\xi,\pi)\,\frac{\partial \xi_\ell}{\partial x_j} + \frac{\partial F}{\partial p_\ell}(\xi,\pi)\,\frac{\partial \pi_\ell}{\partial x_j}.$$

Thus,

$$\frac{\partial (G\circ\psi)}{\partial p_j}(x,p)\,\frac{\partial (F\circ\psi)}{\partial x_j}(x,p) = \underbrace{\frac{\partial G}{\partial x_k}(\xi,\pi)\,\frac{\partial F}{\partial x_\ell}(\xi,\pi)\,\frac{\partial \xi_k}{\partial p_j}\,\frac{\partial \xi_\ell}{\partial x_j}}_{\{5\}} +$$

$$+ \underbrace{\frac{\partial G}{\partial p_k}(\xi,\pi)\,\frac{\partial F}{\partial p_\ell}(\xi,\pi)\,\frac{\partial \pi_k}{\partial p_j}\,\frac{\partial \pi_\ell}{\partial x_j}}_{\{6\}} +$$

$$+ \underbrace{\frac{\partial G}{\partial x_k}(\xi,\pi)\,\frac{\partial F}{\partial p_\ell}(\xi,\pi)\,\frac{\partial \xi_k}{\partial p_j}\,\frac{\partial \pi_\ell}{\partial x_j}}_{\{7\}} + \underbrace{\frac{\partial G}{\partial p_k}(\xi,\pi)\,\frac{\partial F}{\partial x_\ell}(\xi,\pi)\,\frac{\partial \pi_k}{\partial p_j}\,\frac{\partial \xi_\ell}{\partial x_j}}_{\{8\}} \qquad (8)$$

Substituting (7) and (8) into (6), we get

$$\{F\circ\Psi, G\circ\Psi\}(x,p) = \{1\}+\{2\}+\{3\}+\{4\} - (\{5\}+\{6\}+\{7\}+\{8\})$$

$$= (\{1\}-\{5\}) + (\{2\}-\{6\}) + (\{3\}-\{7\}) + (\{4\}-\{8\})$$

$$= \frac{\partial G}{\partial x_k}(\xi,\pi)\frac{\partial F}{\partial x_\ell}(\xi,\pi)\underbrace{\left(\frac{\partial \xi_k}{\partial x_j}\frac{\partial \xi_\ell}{\partial p_j} - \frac{\partial \xi_\ell}{\partial x_j}\frac{\partial \xi_k}{\partial p_j}\right)}_{\{9\}} +$$

$$+ \frac{\partial G}{\partial p_k}(\xi,\pi)\frac{\partial F}{\partial p_\ell}(\xi,\pi)\underbrace{\left(\frac{\partial \pi_k}{\partial x_j}\frac{\partial \pi_\ell}{\partial p_j} - \frac{\partial \pi_\ell}{\partial x_j}\frac{\partial \pi_k}{\partial p_j}\right)}_{\{10\}} +$$

$$+ \frac{\partial G}{\partial x_k}(\xi,\pi)\frac{\partial F}{\partial p_\ell}(\xi,\pi)\underbrace{\left(\frac{\partial \xi_k}{\partial x_j}\frac{\partial \pi_\ell}{\partial p_j} - \frac{\partial \xi_k}{\partial p_j}\frac{\partial \pi_\ell}{\partial x_j}\right)}_{\{11\}} +$$

$$+ \frac{\partial G}{\partial p_k}(\xi,\pi)\frac{\partial F}{\partial x_\ell}(\xi,\pi)\underbrace{\left(\frac{\partial \pi_k}{\partial x_j}\frac{\partial \xi_\ell}{\partial p_j} - \frac{\partial \pi_k}{\partial p_j}\frac{\partial \xi_\ell}{\partial x_j}\right)}_{\{12\}}. \qquad (9)$$

<u>Assume that Ψ is a canonical transformation</u> (i.e. Eqs. (1)–(4) hold).

By (2), $\{9\} = 0$.

By (3), $\{10\} = 0$.

By (1), $\{11\} = \delta_{k\ell}$,

By (4), $\{12\} = \delta_{k\ell}$.

Therefore, (9) reduces to

$$\{F\circ\Psi, G\circ\Psi\}(x,p) = \frac{\partial G}{\partial x_k}(\xi,\pi)\frac{\partial F}{\partial p_\ell}(\xi,\pi)\,\delta_{k\ell} - \frac{\partial G}{\partial p_k}(\xi,\pi)\frac{\partial F}{\partial x_\ell}(\xi,\pi)\,\delta_{k\ell}$$

$$\overset{(5)}{=} \{F, G\}\circ\Psi(x,p).$$

- Assume that $\{F,G\}\circ\Psi = \{F\circ\Psi, G\circ\Psi\}$ $\forall F,G \in C^1(\mathbb{R}^{2n})$

By (5) and (9) we have

$$\frac{\partial G}{\partial x_j}(\xi,\pi)\frac{\partial F}{\partial p_j}(\xi,\pi) - \frac{\partial G}{\partial x_j}(\xi,\pi)\frac{\partial F}{\partial x_j}(\xi,\pi) =$$

$$= \frac{\partial G}{\partial x_k}(\xi,\pi)\frac{\partial F}{\partial x_l}(\xi,\pi)\left(\frac{\partial \xi_k}{\partial x_j}\frac{\partial \xi_l}{\partial p_j} - \frac{\partial \xi_l}{\partial x_j}\frac{\partial \xi_k}{\partial p_j}\right) + \frac{\partial G}{\partial p_k}(\xi,\pi)\frac{\partial F}{\partial p_l}(\xi,\pi)\left(\frac{\partial \pi_k}{\partial x_j}\frac{\partial \pi_l}{\partial p_j} - \frac{\partial \pi_l}{\partial x_j}\frac{\partial \pi_k}{\partial p_j}\right)$$

$$+ \frac{\partial G}{\partial x_k}(\xi,\pi)\frac{\partial F}{\partial p_l}(\xi,\pi)\left(\frac{\partial \xi_k}{\partial x_j}\frac{\partial \pi_l}{\partial p_j} - \frac{\partial \xi_k}{\partial p_j}\frac{\partial \pi_l}{\partial x_j}\right) + \frac{\partial G}{\partial p_k}(\xi,\pi)\frac{\partial F}{\partial x_l}(\xi,\pi)\left(\frac{\partial \pi_k}{\partial x_j}\frac{\partial \xi_l}{\partial p_j} - \frac{\partial \pi_k}{\partial p_j}\frac{\partial \xi_l}{\partial x_j}\right).$$

$$\forall F,G \in C^1(\mathbb{R}^{2n}) \qquad (10).$$

For any $1 \le k,l \le n$, we choose

$$G(x,p) = G^{(1)}(x,p) := x_k, \quad F(x,p) = F^{(1)}(x,p) := p_l.$$

Then (10) reduces to $\underbrace{\delta_{kj}\,\delta_{lj}}_{=\delta_{kl}} = \frac{\partial \xi_k}{\partial x_j}\frac{\partial \pi_l}{\partial p_j} - \frac{\partial \xi_k}{\partial p_j}\frac{\partial \pi_l}{\partial x_j}$. Thus, (1) is satisfied.

Next, for any $1 \le k,l \le n$, we choose

$$G(x,p) = G^{(2)}(x,p) := p_k, \quad F(x,p) = F^{(2)}(x,p) := x_l.$$

Then (10) reduces to $\underbrace{-\delta_{kj}\,\delta_{lj}}_{=-\delta_{kl}} = \frac{\partial \pi_k}{\partial x_j}\frac{\partial \xi_l}{\partial p_j} - \frac{\partial \pi_k}{\partial p_j}\frac{\partial \xi_l}{\partial x_j}$. Thus, (4) is satisfied.

Next, for any $1 \le k,l \le n$, we choose

$$G(x,p) = G^{(3)}(x,p) := x_k, \quad F(x,p) = F^{(3)}(x,p) := x_l.$$

Then (10) reduces to $\frac{\partial \xi_k}{\partial x_j}\frac{\partial \xi_l}{\partial p_j} - \frac{\partial \xi_l}{\partial x_j}\frac{\partial \xi_k}{\partial p_j} = 0$. Thus, (2) is satisfied.

Next, for any $1 \le k, l \le n$, we choose

$$G(x,p) = G^{(4)}(x,p) := p_k, \quad F(x,p) = F^{(4)}(x,p) := p_l.$$

Then (16) reduces to $\frac{\partial \pi_l}{\partial x_j}\frac{\partial \pi_l}{\partial p_j} - \frac{\partial \pi_l}{\partial x_j}\frac{\partial \pi_k}{\partial p_j} = 0$. Thus, (3) is satisfied.

Because Ψ satisfies (1), (2), (3), (4), it is a canonical transformation.