Name: Tuan Pham
ID: 4652218

Math 8386: Calculus of Variations
Take-home final

① Problem 3.4, Jost–Li Jost, page 181.

Let $A \subset \mathbb{R}^d$ be a measurable set and $1 \leq p \leq q \leq r < \infty$, $0 \leq \alpha \leq 1$ be given such that $\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{r}$. Consider $f \in L^p(A) \cap L^r(A)$. We show that $f \in L^q(A)$ and $\|f\|_{L^q(A)} \leq \|f\|_{L^p(A)}^{\alpha} \|f\|_{L^r(A)}^{1-\alpha}$.

Put $\beta = \frac{p}{\alpha q}$ and $\gamma = \frac{r}{(1-\alpha)q} > 0$. Then $\beta, \gamma \in (0, \infty]$ and $\frac{1}{\beta} + \frac{1}{\gamma} = 1$.

Moreover, $q = \alpha q + (1-\alpha) q = \frac{p}{\beta} + \frac{r}{\gamma}$. Hence,

$$\int_A |f|^q dx = \int_A |f|^{\frac{p}{\beta}} |f|^{\frac{r}{\gamma}} dx \overset{\text{Hölder}}{\leq} \left(\int_A \left(|f|^{\frac{p}{\beta}}\right)^{\beta} dx \right)^{\frac{1}{\beta}} \left(\int_A \left(|f|^{\frac{r}{\gamma}}\right)^{\gamma} dx \right)^{\frac{1}{\gamma}}$$

$$= \left(\int_A |f|^p dx \right)^{\frac{\alpha q}{p}} \left(\int_A |f|^r dx \right)^{\frac{(1-\alpha)q}{r}}.$$

Thus, $\left(\int_A |f|^q dx \right)^{\frac{1}{q}} \leq \left(\int_A |f|^p dx \right)^{\frac{\alpha}{p}} \left(\int_A |f|^r dx \right)^{\frac{1-\alpha}{r}}$.

Therefore, $\|f\|_{L^q(A)} \leq \|f\|_{L^p(A)}^{\alpha} \|f\|_{L^r(A)}^{1-\alpha} < \infty$.

② Problem 3.2, Jost–Li Jost, page 182.

Let $A \subset \mathbb{R}^d$ be a measurable set and (f_n) be a bounded sequence in $L^p(A)$. Suppose that (f_n) converges almost everywhere to a function $f : A \to \mathbb{R}$.

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First, we show that $f \in L^p(A)$. Because f is the pointwise limit of a sequence of measurable functions, f is also measurable. We have

$$|f(x)|^p = \lim_{n\to\infty} |f_n(x)|^p = \liminf_{n\to\infty} |f_n(x)|^p \quad \text{a.e. } x \in A.$$

By Fatou's lemma,

$$\int_A |f(x)|^p dx = \int_A \liminf_{n\to\infty} |f_n(x)|^p dx \le \liminf_{n\to\infty} \int_A |f_n(x)|^p dx = \liminf_{n\to\infty} \|f_n\|^p_{L^p(A)}. \quad (*)$$

Because (f_n) is a bounded sequence in $L^p(A)$, there exists a number $C > 0$ such that $\|f_n\|_{L^p(A)} \le C$ for all $n \in \mathbb{N}$. Then $(*)$ implies

$$\int_A |f(x)|^p dx \le C^p < \infty.$$

Therefore, $f \in L^p(A)$.

Secondly, we show that (f_n) does not necessarily converge to f in $L^p(A)$. Let us construct a counterexample as follows. Consider $A = B_1$, which is the open unit ball in $\mathbb{R}^d$, $p \in [1, \infty)$ and $f_n(x) = \begin{cases} n^{d/p} & \text{if } |x| < \frac{1}{n}, \\ 0 & \text{if } |x| > \frac{1}{n}. \end{cases}$

Then (f_n) converges pointwise to the function $f \equiv 0$ in $\mathbb{R}^d \setminus \{0\}$. Moreover,

$$\int_A |f_n|^p dx = \int_{B_{1/n}} n^d dx = n^d |B_{1/n}| = |B_1| \quad \forall n \in \mathbb{N}.$$

Thus, (f_n) is a bounded sequence in $\mathbb{R}^d$. However,

$$\|f_n - 0\|_{L^p(A)} = \|f_n\|_{L^p(A)} = |B_1|^{\frac{1}{p}} \not\to 0 \quad \text{as } n \to \infty.$$

③ Problem 5.1, Jost–Li Jost, page 224.

Consider the variational problem $F(u) = \int_{-1}^{1} f(u, u(x), u'(x))\,dx \to \min$

among $u \in W^{1,4}((-1,1))$ in three following cases.

(i) $f(x,u,v) = (1-v)^2 u^2$,
 $u(-1) = 0$, $u(1) = 1$.

(ii) $f(x,u,v) = (2x-v)^2 u^2$,
 $u(-1) = 0$, $u(1) = 1$.

(iii) $f(x,u,v) = (u^2-\alpha)^2 + (v^2-1)^2$ where $\alpha \in \mathbb{R}$.

Note that we have modified (iii) to make the notation meaningful: the textbook gave $f(x,u,v) = (u^2-\alpha) + (v^2-1)$. Also, the textbook perhaps misses the boundary conditions for Case (iii). Without the Dirichlet boundary conditions, we cannot represent $sc^- F$ via $q^- f$, the quasi-convexification of f.

In each case, we first determine $sc^- F$, the relaxed function of F with respect to the weak topology on $W^{1,4}((-1,1))$, and then state the relaxed variational problem. By definition,

$$(sc^- F)(u) = \sup \left\{ \Phi(u) : \begin{array}{l} \Phi \text{ is weakly lower semicontinuous on } W^{1,4}((-1,1)) \\ \text{and } \Phi(u) \le F(u) \ \forall u \in W^{1,4}((-1,1)) \end{array} \right\}.$$

Because F depends on (x, u, u') instead of u' alone, Theorem 5.2.1 in Jost–Li Jost, page 213, which says $(sc^- F)(u) = \int_{-1}^{1} (cv^- f)(u'(x))\,dx$, is inapplicable.

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However, we have a very similar result thanks to Theorem 9.8, Dacorogna "Direct methods in the Calculus of Variations", page 432 and Theorem 8.11, page 382. It is explained as follows.

In Cases (i), (ii), (iii), we can find numbers $c_0, c_1, c_2 \in \mathbb{R}$ such that

$$c_0 |v|^4 + c_1 \leq |f(x,u,v)| \leq c_2(1+u^4+v^4) \quad \forall x \in (-1,1)\ \forall u,v \in \mathbb{R}.$$

In fact, c_0 and c_1 can be chosen to be zero. The number c_2 can be chosen by simple estimation:

Case (i): $|f(x,u,v)| = (1-v)^2 u^2 \leq (2+2v^2)u^2 = 2u^2+2u^2v^2$

$$\leq (u^4+1)+(u^4+v^4)$$

$$\leq 2(1+u^4+v^4).$$

We can choose $c_2 = 2$.

Case (ii): $|f(x,u,v)| = (2x-v)^2u^2 \leq (8x^2+2v^2)u^2 \leq 8u^2+2u^2v^2$

$$\leq 4(1+u^4)+(u^4+v^4)$$

$$\leq 5(1+u^4+v^4).$$

We can choose $c_2 = 5$.

Case (iii): $|f(x,u,v)| = (u^2-\alpha)^2+(v^2-1)^2$

$$\leq 2(u^4+\alpha^2)+2(v^4+1)$$

$$\leq (2+2\alpha^2)(1+u^4+v^4).$$

We can choose $c_2 = 2+2\alpha^2$.

Denote by qf the quasiconvex envelope of f with respect to variable v. It is also the convex envelope of f with respect to v (Lemma 5.2.3, Jost-Li Jost).

Case (i): $q^{-}f(x,u,v) = f(x,u,v) = (1-v)^2u^2$

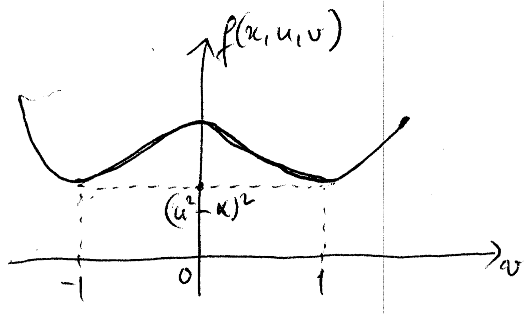
because $\dfrac{\partial^2 f}{\partial v^2} = 2u^2 \geq 0$.

Case (ii): $q^{-}f(x,u,v) = f(x,u,v) = (2x-v)^2u^2$

because $\dfrac{\partial^2 f}{\partial v^2} = 2u^2 \geq 0$.

Case (iii):

$$q^{-}f(x,u,v) = \begin{cases} (u^2-\alpha)^2 & \text{if } -1 \leq v \leq 1, \\ (u^2-\alpha)^2 + (v^2-1)^2 & \text{otherwise.} \end{cases}$$



Put $G(u) = \int_{-1}^{1} q^{-}f(x, u(x), u'(x))\,dx$. Then $G(u) \leq F(u)$ for all $u \in W^{1,4}((-1,1))$.

By Theorem 9.8, Dacorogna, for each $u \in W^{1,4}((-1,1))$, there exists a sequence (v_n) in $u + W_0^{1,4}((-1,1))$ such that $u_n \to u$ in $W^{1,4}((-1,1))$ and $\lim_{n\to\infty} F(u_n) = G(u)$.

By Theorem 8.11, Dacorogna, G is weakly lower semicontinuous in $W^{1,4}((-1,1))$.

Therefore, by Lemma 5.12, Jost–Li-Jost (the necessary and sufficient conditions for $sc^{-}F$), $G = sc^{-}F$. We then get the following results.

Case (i): $(sc^{-}F)(u) = F(u) = \int_{-1}^{1} (1-u'(x))^2 u(x)^2\,dx$.

Case (ii): $(sc^{-}F)(u) = F(u) = \int_{-1}^{1} (2x-u'(x))^2 u(x)^2\,dx$.

Case (iii):

$$(sc^- F)(u) = \int_{-1}^{1} [(u(x)^2 - 1)^2 + Q(u'(x))]\, dx,$$

wher $Q(v) = \begin{cases} (v^2 - 1)^2 & \text{if } |v| \geq 1, \\ 0 & \text{if } |v| \leq 1. \end{cases}$

Now we state the relaxed variational problems. In Cases (i), (ii), (iii), $sc^- F \geq 0$. Thus,

$$0 \leq \inf\{(sc^- F)(u) : u \in W^{1,4}((-1,1)) \text{ satisfying certain Dirichlet boundary conditions}\}$$

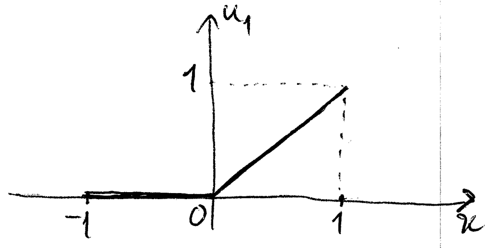
$$\leq \inf\{F(u) : u \in W^{1,4}((-1,1)) \text{ satisfying certain the same boundary conditions}\}.$$

Put $\alpha = \inf(sc^- F)(u) \geq 0$. Suppose by contradiction that $\alpha < \inf F(u)$. Then the constant functional $H(u) \equiv \frac{\alpha + \inf F(u)}{2}$ is weakly lower semicontinuous and satisfies

$$\begin{cases} H(u) < F(u) & \forall u \in W^{1,4}((-1,1)), \\ H(u_0) > (sc^- F)(u_0) & \text{for some } u_0. \end{cases}$$

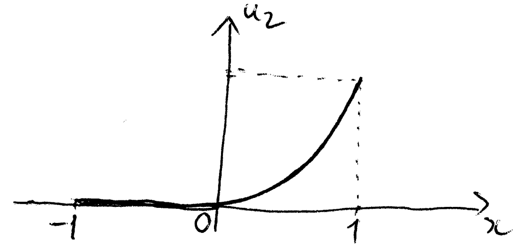
This contradicts the definition of $sc^- F$. Therefore, $\inf (sc^- F)(u) = \inf F(u)$. The variational problem $F(u) \to \min$ under certain Dirichlet boundary conditions is equivalent to the relaxed problem $(sc^- F)(u) \to \min$ under the same boundary conditions.

We now make a few aside remarks. For Cases (i) and (ii), $\min F(u) = 0$. Because $sc^- F = F$, there is no difference between the original problem and the relaxed problem.



$$u_1(x) = \begin{cases} 0 & \text{if } -1<x<0, \\ x & \text{if } 0<x<1, \end{cases}$$

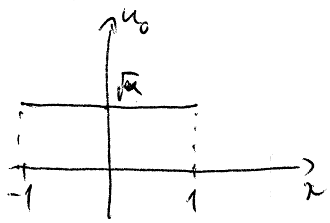
is a minimizer in Case (i).



$$u_2(x) = \begin{cases} 0 & \text{if } -1<x<0, \\ x^2 & \text{if } 0<x<1, \end{cases}$$

is a minimizer in Case (ii).

In Case (iii), we may impose different boundary conditions. There is one choice in which the problem $(sc^- F)(u) \to \min$ has a solution while the problem $F(u) \to \min$ does not. Consider $\alpha \geq 0$ and $u(-1) = u(1) = \sqrt{\alpha}$. A minimizer of $sc^- F$ is $u_0(x) = \sqrt{\alpha}$ for all $x \in (-1,1)$. Indeed,



$$\begin{aligned}(sc^- F)(u_0) &= (u_0(x)^2 - \alpha)^2 + Q(u_0'(x)) \\ &= (\sqrt{\alpha}^2 - \alpha)^2 + Q(0) \\ &= 0.\end{aligned}$$

Thus, $\inf F(u) = 0$. However, there is no $u \in W^{1,4}((-1,1))$ with $u(-1) = u(1) = \sqrt{\alpha}$ such that

$$F(u) = \int_{-1}^{1} \left[(u(x)^2 - \alpha)^2 + (u'(x)^2 - 1)^2 \right] dx = 0.$$

④ Problem 6.2, Jost – Li Jost, page 240.

Let X be a first countable topological space, and $F_n, G_n : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ be maps for $n \in \mathbb{N}$. Suppose that the sums $F_n + G_n$, $F + G$ are always well-defined and that we have

$$F = \Gamma\text{-}\lim_{n\to\infty} F_n, \qquad G = \Gamma\text{-}\lim_{n\to\infty} G_n, \qquad (1)$$

$$H = \Gamma\text{-}\lim_{n\to\infty} (F_n + G_n). \qquad (2)$$

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First, we show $F(x)+G(x) \leq H(x)$ for every $x \in X$. Suppose by contradiction that there exists $x_0 \in X$ such that $F(x_0)+G(x_0) > H(x_0)$. Because of (2), there exists a sequence (x_n) converging to x_0 such that $H(x_0) = \lim_{n\to\infty} (F_n(x_n) + G_n(x_n))$.

Because of (1),

$$\liminf_{n\to\infty} F_n(x_n) \geq F(x_0); \quad \liminf_{n\to\infty} G_n(x_n) \geq G(x_0).$$

Therefore,
$$\begin{aligned} H(x_0) = \liminf_{n\to\infty}(F_n(x_n)+G_n(x_n)) &\geq \liminf_{n\to\infty} F_n(x_n) + \liminf_{n\to\infty} G_n(x_n) \\ &\geq F(x_0) + G(x_0). \end{aligned}$$

This is a contradiction.

Next, we give an example for the case in which $F(x)+G(x) < H(x)$ for all $x \in X$. Consider $X = \mathbb{R}$ with the usual topology, and $F_n(x) = -G_n(x) = \sin(nx)$ for $n \in \mathbb{N}$, $x \in \mathbb{R}$. Then

$$H = \Gamma\text{-}\lim_{n\to\infty} (F_n + G_n) = \Gamma\text{-}\lim_{n\to\infty}(0) \equiv 0.$$

We want to show that $\Gamma\text{-}\lim_{n\to\infty} F_n = \Gamma\text{-}\lim_{n\to\infty} G_n = -1$. If this can be done then

$$F(x) + G(x) = (-1) + (-1) < 0 = H(x) \quad \forall x \in \mathbb{R}.$$

For every sequence (x_n) in $\mathbb{R}$, we have

$$\liminf_{n\to\infty} F_n(x_n) = \liminf_{n\to\infty} \sin(nx_n) \geq -1,$$
$$\liminf_{n\to\infty} G_n(x_n) = \liminf_{n\to\infty} (-\sin(nx_n)) \geq -1.$$

Fix $x_0 \in \mathbb{R}$. We want to find two sequences (a_n) and (b_n) converging to x_0 such that

$$\lim_{n\to\infty} \sin(na_n) = -1, \qquad (3)$$

$$\lim_{n\to\infty} (-\sin(nb_n)) = -1. \qquad (4)$$

If for any given x_0 we can find (a_n) converging to x_0 and satisfying (3), then we can also find (b_n) converging to x_0 and satisfying (4). Indeed, let (c_n) be a sequence converging to $-x_0$ such that $\lim_{n\to\infty} \sin(nc_n) = -1$; then we choose $b_n = -c_n$. Therefore, it suffices to consider (3) only. We see that $2\pi \approx 6.28\ldots < 7$. Thus, for each $n \in \mathbb{N}$, $n \geq 49$, there exists $k_n \in \mathbb{Z}$ such that

$$nx_0 + \frac{\pi}{2} \leq 2\pi k_n \leq nx_0 + \sqrt{n} + \frac{\pi}{2}. \qquad (5)$$

Choose $a_n = \frac{1}{n}\left(-\frac{\pi}{2} + 2\pi k_n\right)$. Then $na_n = -\frac{\pi}{2} + 2\pi k_n$ and thus

$$\sin(na_n) = \sin\left(-\frac{\pi}{2} + 2\pi k_n\right) = -1 \quad \forall n \in \mathbb{N}.$$

Moreover, (5) implies $nx_0 \leq -\frac{\pi}{2} + 2\pi k_n \leq nx_0 + \sqrt{n}$. Dividing all by n, we get

$$x_0 \leq a_n \leq x_0 + \frac{1}{\sqrt{n}}.$$

Thus, $a_n \to x_0$ as $n \to \infty$.