FINAL EXAM

Math 8385: Calculus of Variations

due 4:30 PM Weds. 11 December, 2013

(1) Note: this problem is different from the first-order integrand we considered this semester.

Let $L(t, x, p, r)$ be $C^3([a,b] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ and for $x : [a,b] \to \mathbb{R}^d$ consider

$$(*) \quad I(x) := \int_a^b L(t, x, \dot{x}, \ddot{x})\, dt$$

with boundary conditions

$$(BC) \quad x(a) = x_0, \ \dot{x}(a) = p_0, \ x(b) = x_1, \ \dot{x}(b) = p_1.$$

Derive the Euler-Lagrange equations for $(*)$, and show that they are satisfied by any C^4 minimizer $x : [a,b] \longrightarrow \mathbb{R}^d$.

(2) Let $\Phi : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ be the potential energy, $\Phi \in C^2(\mathbb{R} \times \mathbb{R}^3)$ and assume Φ is invariant under the expanding screw motions

$$\bar{h}_s(t, x_1, x_2, x_3) := (t+s,\ x_1 \cos s + x_2 \sin s,\ -x_1 \sin s + x_2 \cos s,\ x_3 + s)$$

that is, $\Phi \circ \bar{h}_s \equiv \Phi$ on $\mathbb{R} \times \mathbb{R}^3$ $\forall$ $-\varepsilon < s < \varepsilon$. Let $f : [0, \infty) \to (0, \infty)$ be a C^2 function of the radial variable r in cylindrical coordinates (r, θ, z): $x_1 = r\cos\theta$, $x_2 = r \sin\theta$, $x_3 = z$.

Consider the integral

$$I(x) := \int_{t_1}^{t_2} \left[\frac{1}{2} f(r) |\dot{x}|^2 - \Phi(t, x(t)) \right] dt.$$

Apply Noether's Theorem to find a first integral, that is, a nontrivial quantity which is constant along any solution of the Euler–Lagrange equations for I.

(3) Define the Poisson bracket $\{F, G\}$ of two functions $F, G \in C^1(\mathbb{R}^{2n})$:

$$\{G, F\} := \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial x_i},$$

where $(x_1, \dots, x_n, p_1, \dots, p_n)$ are coordinates for $\mathbb{R}^{2n}$. Let $\Psi : U \subset \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ be a local diffeomorphism.

Show that Ψ is a canonical transformation if and only if $\forall\, F, G \in C^1(\mathbb{R}^{2n})$ there holds

$$\{F, G\} \circ \Psi = \{F \circ \Psi, G \circ \Psi\}.$$

Hint: you might choose to use the notation

$z = (z_1, \dots, z_{2n}) = (x_1, \dots, x_n, p_1, \dots, p_n) = (x, p) = \Psi(\xi, \pi) =$
$= \Psi(\xi_1, \dots, \xi_n, \pi_1, \dots, \pi_n) = \Psi(\zeta).$