Connectivity of the set of invertible elements of a complex Banach algebra

Tuan Pham

04/17/2015

1 Remark 1

Let B be a finite dimensional Banach algebra over $\mathbb{C}$. Then $B^\times$, the set of all invertible elements of B, is connected.

Proof. Denote by 1 the unit element of B. Let $a \in B^\times$. We show that 1 and a can be connected by a path in $B^\times$. Let A be the subalgebra of B generated by 1 and a. That is

$$A = \{P(a) : \ P(t) \in \mathbb{C}[t]\}.$$

Because $A^\times \subset B^\times$, it suffices to find a path in $A^\times$ that connects 1 with a.

First, we determine all multiplicative functionals of A. Since A is a linear subspace of B, it is finite dimensional. Then there exists $n \in \mathbb{N}$ such that $1, a, a^2, ..., a^n$ are linearly dependent. Assume n is the smallest number with this property. Then there exists a monic polynomial $q(t) \in \mathbb{C}[t]$ with degree n such that $q(a) = 0$. Let $\lambda_1, \lambda_2, ..., \lambda_m \in \mathbb{C}$ be the distinct roots of $q(t)$. A multiplicative functional ϕ of A is determined by $\phi(a)$ because

$$\phi(P(a) = P(\phi(a)) \quad \forall P(t) \in \mathbb{C}[t].$$

We have $q(\phi(a)) = \phi(q(a)) = \phi(0) = 0$. Thus, $\phi(a) \in \{\lambda_1, \lambda_2, ..., \lambda_m\}$.

For each $1 \le j \le m$, we define a map $\phi_j : A \to \mathbb{C}$, $\phi_j(P(a)) = P(\lambda_j)$ for all polynomials $P(t) \in \mathbb{C}[t]$. We show that ϕ_j is well-defined. Suppose $P(a) = Q(a)$ for some $P(t), Q(t) \in \mathbb{C}[t]$. Then $(P - Q)(a) = 0$. Because $q(t)$ is the minimal polynomial of a, it divides $P(t) - Q(t)$. Then λ_j is a root of $P(t) - Q(t)$, which implies $P(\lambda_j) = Q(\lambda_j)$. Thus, ϕ_j is well-defined. It is clear from the definition of ϕ_j that it is a multiplicative functional of A with $\phi_j(a) = \lambda_j$. Therefore, $\phi_1, \phi_2, ..., \phi_m$ are all multiplicative functionals of A.

Next, we show that $a \in A^\times$. Because A is a commutative algebra, an element of A is invertible if and only if it does not vanish any of $\phi_1, \phi_2, ..., \phi_m$. Now suppose by contradiction that $\lambda_j = 0$ for some $1 \le j \le m$. Then $q(t)$ is of the form

$$q(t) = t^n + \alpha_{n-1}t^{n-1} + ... + \alpha_1 t.$$

Then $0 = q(a) = a(t^{n-1} + \alpha_{n-1}t^{n-2} + ... + \alpha_1)$. Because $a \in B^\times$, $a^{n-1} + \alpha_{n-1}a^{n-2} + ... + \alpha_1 = 0$. Then $1, a, a^2, ..., a^{n-1}$ are linearly dependent. This contradicts the minimality of n. Therefore, $\phi_j(a) = \lambda_j \neq 0$ for all $1 \leq j \leq m$. That is $a \in A^\times$.

A path in $A^\times$ that connects 1 with a can be constructed as follows. Let $\theta_1, \theta_2, ..., \theta_m \in [0, 2\pi)$ be the arguments of $\lambda_1, \lambda_2, ..., \lambda_m$ respectively. Take $\theta \in (0, \pi)\backslash\{\pi - \theta_1, ..., \pi - \theta_m, 2\pi - \theta_1, ..., 2\pi - \theta_m\}$. Define a map $h : [0, 1] \to A$,

$$h(t) = \begin{cases} 1 + 3te^{i\theta}a, & \text{if } 0 \leq t \leq \frac{1}{3} \\ 2 - 3t + e^{i\theta}a, & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3} \\ e^{i\theta(3-3t)}a, & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases}$$

Since $\theta + \theta_j \in (0, 3\pi)\backslash\{\pi, 2\pi\}$, $e^{i\theta}\lambda_j$ is not a real number for every $j = 1, 2, ..., m$. Then

$$\begin{aligned} \phi_j(1 + 3te^{i\theta}a) &= \phi_j(1) + 3te^{i\theta}\phi_j(a) = 1 + 3te^{i\theta}\lambda_j \neq 0, \\ \phi_j(2 - 3t + e^{i\theta}a) &= \phi_j(2 - 3t) + e^{i\theta}\phi_j(a) = 2 - 3t + e^{i\theta}\lambda_j \neq 0, \\ \phi_j(e^{i\theta(3-3t)}a) &= e^{i\theta(3-3t)}\phi_j(a) = e^{i\theta(3-3t)}\lambda_j \neq 0. \end{aligned}$$

Then $h(t) \in A^\times$ for all $t \in [0, 1]$. It is clear that h is continuous and $h(0) = 1$, $h(1) = a$. Therefore, h is a path in $A^\times$ that connects 1 with a. □

2 Remark 2

In an infinite dimensional Banach algebra over $\mathbb{C}$, the set of invertible elements may not be connected. For example, let

$$B = \{f : \Omega \to \mathbb{C} \text{ holomorphic, can be continuously extended to } \bar{\Omega}\},$$

where Ω is the annulus $\{z \in \mathbb{C} : \frac{1}{2} < |z| < 1\}$. Then B is a Banach algebra with multiplication being the pointwise multiplication of functions, the unit element being the constant function 1, and the norm being

$$\|f\| = \sup_{z\in\Omega} |f(z)|.$$

The functions $..., z^{-2}, z^{-1}, 1, z, z^2, ...$ are invertible in B. But no two of them belong to the same connected component of $B^\times$. Indeed, suppose by contradiction that z^m and z^n belong to the same connected component of $B^\times$ for some $m, n \in \mathbb{Z}, m \neq n$.

Since $B^\times$ is open in B, it is locally path-connected. Then z^m and z^n can be connected by a piecewise linear path in $B^\times$. That is a continuous map $h : [0, 1] \to B^\times$, $h(0)(z) = z^m, h(1)(z) = z^n$, which consists of finitely many line segments in $B^\times$. Write $h(t)(z)$ as $h(t, z)$. Because $h(t, .) \in B^\times$, $h(t, .)$ does not have a zero in Ω. Let γ be the circle centered at 0 with radius $\frac{3}{4}$ in the complex plane. For every $f \in B^\times$, $\frac{f'(z)}{f(z)}$ is a smooth function on γ.

$$\int_\gamma \frac{f'(z)}{f(z)}dz = \int_0^1 \frac{f'(z(t))z'(t)}{f(z(t))}dt \overset{w(t)=f(z(t))}{=\!=\!=\!=} \int_0^1 \frac{w'(t)}{w(t)}dt = \int_\Gamma \frac{1}{w}dw,$$

where Γ is the image of γ under f. It is a closed rectifiable curve. We know that $\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{1}{w}dw$ is the winding number of Γ with respect to 0. Thus,

$$\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f'(z)}{f(z)}dz \in \mathbb{Z} \quad \forall f \in B^{\times}.$$

Applying this result for $f(z) = h(t,z)$, we get

$$\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\frac{\partial h}{\partial z}(t,z)}{h(t,z)}dz \in \mathbb{Z} \quad \forall t \in [0,1].$$

Because h is a piecewise linear path in $B^{\times}$, $\frac{\partial h}{\partial z}$ is continuous on $[0,1]\times\gamma$. Then the map $\phi : [0,1] \to \mathbb{Z}$,

$$\phi(t) = \frac{1}{2\pi i}\int\limits_{\gamma}\frac{\frac{\partial h}{\partial z}(t,z)}{h(t,z)}dz$$

is continuous. Then it must be a constant function. On the other hand,

$$\begin{aligned}\phi(0) &= \frac{1}{2\pi i}\int\limits_{\gamma}\frac{\frac{\partial h}{\partial z}(0,z)}{h(0,z)}dz = \frac{1}{2\pi i}\int\limits_{\gamma}\frac{mz^{m-1}}{z^m}dz = m,\\ \phi(1) &= \frac{1}{2\pi i}\int\limits_{\gamma}\frac{\frac{\partial h}{\partial z}(1,z)}{h(1,z)}dz = \frac{1}{2\pi i}\int\limits_{\gamma}\frac{nz^{n-1}}{z^n}dz = n.\end{aligned}$$

This implies $\phi(0) \neq \phi(1)$, which is a contradiction.

The above example is an illustration for Lorch's therem [Ric60, Theorem 1.4.14], which states:

> Let B be a commutative Banach algebra over $\mathbb{C}$. Then $B^{\times}$ either is connected or has infinitely many connected components.

The connected component of $B^{\times}$ containing 1 is called the *principal component* of $B^{\times}$. Denote it by G. We see that G is a normal subgroup of $B^{\times}$. Further description of G can be found in [Pal94, Theorem 2.1.12]. For example, G is the subgroup of $B^{\times}$ generated by $\{e^x : x \in B\}$. Here the exponential map $\exp : B \to B^{\times}$ is defined as

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots = \sum_{k=0}^{\infty}\frac{1}{k!}x^k.$$

If B is commutative, we see that $G = \{e^x : x \in B\} = \exp(B)$. Another form of Lorch's theorem is the following [Pal94, Theorem 2.1.12]:

> Let B be a commutative Banach algebra over $\mathbb{C}$. Then the quotient group $B^{\times}/G$ is torsion-free.

References

[Ric60] C. Rickart: *General Theory of Banach Algebras*, University Series in Higher Mathematics. Toronto-London-New York: D. van Nostrand Co., Inc., Princeton, N.J., 1960.

[Pal94] T. Palmer, *Banach algebras and the general theory of *-algebras*, Vol. 1, Encyclopedia of Mathematics and its Applications 49. Cambridge: Cambridge University Press, 1994.