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Math 8701

Complex Analysis, Problem Set 1

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① Find the fifth root of $3+3i$.

Proof

We express $3+3i$ in polar coordinate:

$$3+3i = 3\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 3\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

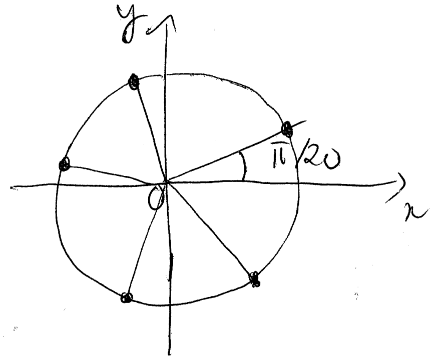
Thus,
$$3+3i = 3\sqrt{2}\left[\cos\left(\frac{\pi}{4} + k2\pi\right) + i\sin\left(\frac{\pi}{4} + k2\pi\right)\right]$$

for any $k \in \mathbb{Z}$.

Thus the fifth roots of $3+3i$ are

$$\sqrt[5]{3+3i} = \sqrt[5]{3\sqrt{2}}\left[\cos\left(\frac{\pi}{20} + k\frac{2\pi}{5}\right) + i\sin\left(\frac{\pi}{20} + k\frac{2\pi}{5}\right)\right]$$

for any $k \in \mathbb{Z}$.



Because cosine and sine are periodic with period 2π, we actually have only 5 fifth roots corresponding to $k = 0, 1, 2, 3, 4$.

(2) Problem 3, Ahlfors p. 9: prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| = 1 \qquad (*)$$

if either $|a|=1$ or $|b|=1$. What exception must be made if $|a|=|b|=1$?

Proof. If $|a|=1$ and $|b|\neq 1$ then $|\bar{a}b| = |b| \neq 1$. Thus $1-\bar{a}b\neq 0$.

Similarly, if $|b|=1$ and $|a|\neq 1$ then $1-\bar{a}b\neq 0$. Thus in both case, the fraction $\dfrac{a-b}{1-\bar{a}b}$ is well-defined.

We have $|a-b|^2 = (a-b)\overline{(a-b)}$ (definition of modulus)

$$= (a-b)(\bar{a}-\bar{b})$$

$$= a\bar{a} - b\bar{a} - \bar{a}b + b\bar{b}$$

$$= |a|^2 + |b|^2 - 2\operatorname{Re}(\bar{a}b) \quad (\text{since } b\bar{a} = \overline{\bar{a}b})$$

Now if $|a| = 1$ then

$$|a-b|^2 = 1 + |b|^2 - 2\operatorname{Re}(\bar{a}b) = 1 + |\bar{a}b|^2 - 2\operatorname{Re}(\bar{a}b)$$

Put $z = \bar{a}b$. We get $1+|z|^2 - 2\operatorname{Re} z = 1 + z\bar{z} - z - \bar{z} = (1-z)(1-\bar{z})$

$$= (1-z)\overline{(1-z)}$$

$$= |1-z|^2 = |1-\bar{a}b|^2$$

Thus $|a-b|^2 = |1-\bar{a}b|^2$, or $|a-b| = |1-\bar{a}b|$.

Similarly, if $|b|=1$ we ~~get~~ also get $|a-b|=|1-\bar{a}b|$. Thus we have the conclusions "$|a|=1 \Rightarrow |a-b|=|1-\bar{a}b|$" and

"$|b|=1 \Rightarrow |a-b|=|1-\bar{a}b|$".

If one of a and b has modulus different from 1, then $1-\bar{a}b$

as already shown; Thus $\dfrac{|a-b|}{|1-\bar{a}b|}=1$, or equivalently $\left|\dfrac{a-b}{1-\bar{a}b}\right|=1$.

In case $|a|=|b|=1$, we cannot take the quotient if and only if $|a-b|=|1-\bar{a}b|=0$, which is equivalent to $a=b$. Thus the statement

"$|a|=|b|=1 \Rightarrow \left|\dfrac{a-b}{1-\bar{a}b}\right|=1$" is only false if $a=b$.

(3) Problem 4, Ahlfors p. 9 : Find the conditions under which

4/5 the equation $az+b\bar{z}+c=0$ in one complex unknown has exactly one solution, and compute that solution.

Proof We write a, b, c, z and $\bar{z}$ in familiar form

$$z = x+iy$$
$$\bar{z} = x-iy$$

$\boxed{4}$

$$a = a_1 + i a_2$$

$$b = b_1 + i b_2$$

$$c = c_1 + i c_2$$

Thus

$$
\begin{aligned}
az &= (a_1 x - a_2 y) + i\,(a_2 x + a_1 y) \\
+ \quad b\bar{z} &= (b_1 x + b_2 y) + i\,(b_2 x - b_1 y) \\
c &= \qquad c_1 \qquad + i \quad c_2
\end{aligned}
$$

$$az + b\bar{z} + c = \left[(a_1 + b_1)x - (a_2 - b_2)y + c_1\right] + i\left[(a_2 + b_2)x + (a_1 - b_1)y + c_2\right]$$

Thus $az + b\bar{z} + c = 0$ if and only if

$$
\begin{cases}
(a_1 + b_1)x - (a_2 - b_2)y + c_1 = 0 \\
(a_2 + b_2)x + (a_1 - b_1)y + c_2 = 0
\end{cases}
$$

By Cramer's rule, the above system of linear equations has unique solution (x, y) if and only if

$$D = \begin{vmatrix} a_1 + b_1 & -(a_2 - b_2) \\ a_2 + b_2 & a_1 - b_1 \end{vmatrix} \neq 0$$

or equivalently $a_1^2 - b_1^2 + a_2^2 - b_2^2 \neq 0$, or equivalently $|a| \neq |b|$.

Whenever $|a| \neq |b|$, the ~~equation~~ system of equations has unique solution which is given by

$$x = \frac{\begin{vmatrix} -(a_2-b_2) & c_1 \\ a_1-b_1 & c_2 \end{vmatrix}}{D} = \frac{-c_2(a_2-b_2) - c_1(a_1-b_1)}{a_1^2+a_2^2-b_1^2-b_2^2}$$

$$y = \frac{\begin{vmatrix} a_1+b_1 & c_1 \\ a_2+b_2 & c_2 \end{vmatrix}}{D} = \frac{c_2(a_1+b_1) - c_1(a_2+b_2)}{a_1^2+a_2^2-b_1^2-b_2^2}$$

you can rewrite all of this to get $z = \frac{b\bar{c} - c\bar{a}}{|a|^2-|b|^2}$

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4 Problem 1, Ahlfors p. 11: If $|a|<1$ and $|b|<1$, prove that $\left|\frac{a-b}{1-\bar{a}b}\right| < 1$.

Proof If $|a|<1$ and $|b|<1$, we know that $|\bar{a}b| = |\bar{a}||b| = |a||b| < 1$. Thus $1-\bar{a}b \neq 0$. Then we have the equivalence

$$\begin{aligned}
\left|\frac{a-b}{1-\bar{a}b}\right| < 1 &\iff |a-b| < |1-\bar{a}b| \\
&\iff |a-b|^2 < |1-\bar{a}b|^2 \\
&\iff |a|^2 - 2\operatorname{Re}(\bar{a}b) + |b|^2 < 1 - 2\operatorname{Re}(\bar{a}b) + |\bar{a}b|^2 \\
&\iff |a|^2 + |b|^2 < 1 + |a|^2|b|^2 \\
&\iff 1 + |a|^2|b|^2 - |a|^2 - |b|^2 > 0 \\
&\iff (1-|a|^2)(1-|b|^2) > 0, \text{ which is true.}
\end{aligned}$$

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(5) Problem 4, Ahlfors p. 11: show that there are complex numbers z satisfying $|z-a|+|z+a|=2|c|$ if and only if $|a|\leq|c|$. If this condition is fullfilled, what are the smallest and largest values of $|z|$?

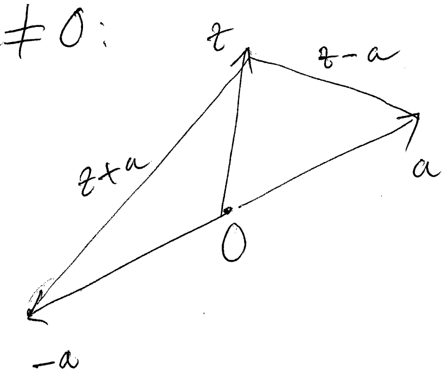
<u>Proof</u> Suppose that z satisfies $2|c|=|z-a|+|z+a|$. Then

$$2|c| \geq |(z-a)-(z+a)| = 2|a| \quad \text{(Triangle inequality)}$$

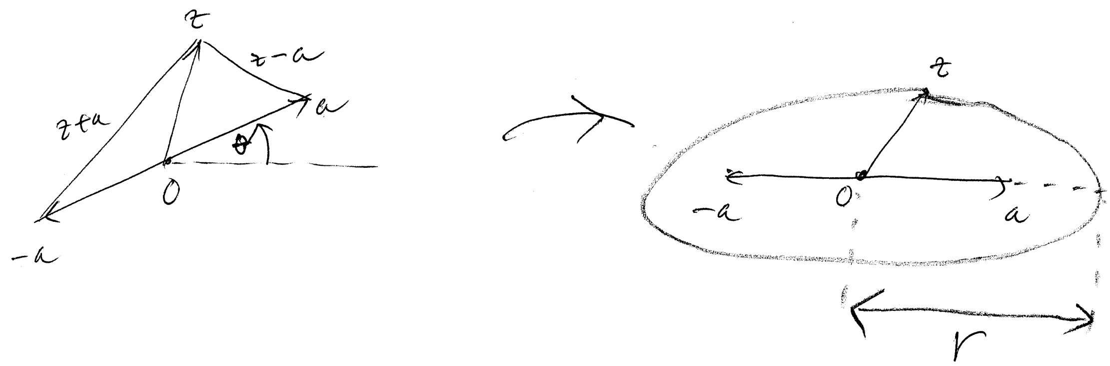
Thus $|c|\geq|a|$ is a necessary ~~for~~ condition for the existence of solution z.

- If $a=0$ then the equation becomes $2|z|=2|c|$. It has solutions, which are all points on the circle centered at origin with radius $|c|$. Moreover $\max|z| = \min|z| = |c|$.

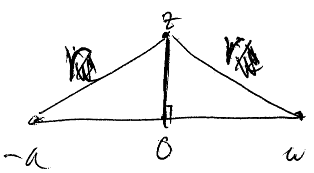
- If $a\neq 0$:



Then the set of all z satisfying the equation is the set of all point z in the plane such that the ~~sum~~ sum of the distance from z to two fixed points $-a$ and a is $2|c|$. Thus the set of all such z's is an ellipse whose foci are $-a$ and a, and major axis $|c|$. We are asked to find the smallest and largest values of $|z|$, i.e. the smallest and largest distance from the origin to a point on that ellipse. Thus what we are concerning is $|a|$ and $|c|$ rather than a and c. By rotating an angle $-\theta$ where $\theta = \arg(a)$, we can assume $a \in \mathbb{R}$, $a > 0$.



Put $r = |c|$. Then z belongs to an ellipse whose semi-major axis is r with foci $-a$ and a. Thus the maximum of $|z|$ is the ~~max $|z| = r$~~ larger between the semi-minor axis and semi-major axis, and the minimum of $|z|$ is the smaller one. The two axes are r and $\sqrt{r^2 - a^2}$.



Thus
$$\max |z| = \cancel{\max\{|c|,}\ r = |c|$$
$$\min |z| = \sqrt{r^2 - a^2} = \sqrt{|c|^2 - |a|^2}$$

(6) Problem 1, Ahlfors p. 17: when does $az+b\bar{z}+c=0$ represent a line?

Proof We write $z=x+iy$, $a=a_1+ia_2$, $b=b_1+ib_2$, $c=c_1+ic_2$.
As in (3), the equation $az+b\bar{z}+c=0$ is equivalent to the folling system of equations

$$\begin{cases} A_1x+B_1y+C_1=0 \\ A_2x+B_2y+C_2=0 \end{cases} \qquad (I)$$

where $A_i = a_i+b_i$, $B_1=-a_2+b_2$, $B_2=a_1-b_1$, $C_1=c_1$, $C_2=c_2$.

If (I) represents a line, it must have infinitely many solutions. A necessary condition for that is

$$D=\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = A_1B_2-A_2B_1=0 \qquad (1)$$

$$D_x=\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} = B_1C_2-B_2C_1=0 \qquad (2)$$

$$D_y=\begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} = A_1C_2-A_2C_1=0 \qquad (3)$$

The conditions (1), (2), (3) guarantee that (I) has either no solution or infinitely many solutions.

Given (1), (2) and (3), the system (I) has no solution if and only if either $A_1x+B_1y+C_1=0$ or $A_2x+B_2y+C_2=0$ has no solution. Indeed, suppose that both of them have solutions; then each one represents a line or the whole plane; if one of them represent the whole plane then the intersection of two solution sets will be at least a line (contradiction!); if both of them represent lines then (1) implies that these lines ~~are~~ either coincide or parallel; if they are coincide then the intersection between them is the line itself (contradiction!); if they are parallel lines, that ~~contradd~~ contradicts $D_x=D_y=0$. Therefore, (I) has no solution (given (1), (2), (3)) if $A_1x+B_1y\cancel{\neq}+C_1=0$ or $A_2x+B_2y+C_2=0$ has no solution, which is quivalent to "$(A_1=B_1=0,\ C_1\neq 0)$ or $(A_2=B_2=0,\ C_2\neq 0)$".

Now, given (1), (2) and (3), if system (I) have solution set of the entire plane ~~if~~ then $C_1=C_2=0$ (by substituting $x=y=0$); if ~~either~~ one of the coefficients A_1, A_2, B_1, B_2 is nonzero (say A_1 for example) then by ~~substituting $x = A_1^{-1}, y = 0$ we see that (~~ $(x, y) = (A_1^{-1}, 0)$ is not a solution of (I). Therefore (I) has the solution set of the entire plane if and only if $A_1 = A_2 = B_1 = B_2 = C_1 = C_2 = 0$, which is equivalent to

$$\begin{cases} a_1 + b_1 = 0 \\ a_2 + b_2 = 0 \\ -a_2 + b_2 = 0 \\ a_1 - b_1 = 0 \\ c_1 = 0 \\ c_2 = 0 \end{cases}$$

$\Rightarrow a_2 = b_2 = 0$

$\Rightarrow a_1 = b_1 = 0$

$\Leftrightarrow a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 0$

$\Leftrightarrow a = b = c = 0$

Therefore, (I) has solution set of a line if and only if

$\begin{cases} (1) \\ (2) \\ (3) \end{cases}$ and not ("$A_1 = B_1 = 0, C_1 \neq 0$" or "$A_2 = B_2 = 0, C_2 \neq 0$")

and not ("$a = b = c = 0$")

, which is equivalent to

$$\begin{cases} (1) \\ (2) \\ (3) \end{cases} \text{ and } |a|^2+|b|^2+|c|^2 \neq 0 \text{ and not}(\text{"}A_1=B_1=0, C_1\neq 0\text{"})$$
$$\text{and not}(\text{"}A_2=B_2=0, C_2\neq 0\text{"}),$$

which is equivalent to

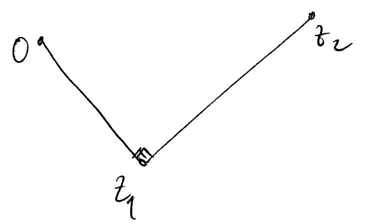
$$\begin{cases} (1) \\ (2) \\ (3) \end{cases} \text{ and } |a|^2+|b|^2+|c|^2 \neq 0 \text{ and } (A_1^2+B_1^2\neq 0 \text{ or } C_1=0)$$
$$\text{and } (A_2^2+B_2^2\neq 0 \text{ or } C_2=0)$$

which is equivalent to

$$\begin{cases} a_1^2+a_2^2-b_1^2-b_2^2=0 \\ -c_2(a_2-b_2)-c_1(a_1-b_1)=0 \\ c_2(a_1+b_1)-c_1(a_2+b_2)=0 \\ a_1^2+b_1^2+a_2^2+b_2^2+c_1^2+c_2^2\neq 0 \\ (a_1+b_1)^2+(a_2-b_2)^2\neq 0 \text{ or } c_1=0 \\ (a_2+b_2)^2+(a_1-b_1)^2\neq 0 \text{ or } c_2=0 \end{cases}$$

(7) Problem 5, Ahlfors p. 17: Show that all circles that pass through a and $1/\bar{a}$ intersect the circle $|z|=1$ at right angle.

Proof We will use the following property:

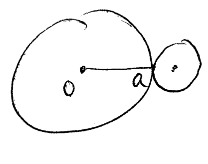


« the line passing through 0 and z_0, and the line passing through z_1 and z_2, are perpendicular if and only if $Re\left(\frac{z_2 - z_1}{z_1}\right) = 0$. (provided that $z_1 \neq 0$)

or equivalently $Re\left(\frac{z_2}{z_1}\right) = 1$. »

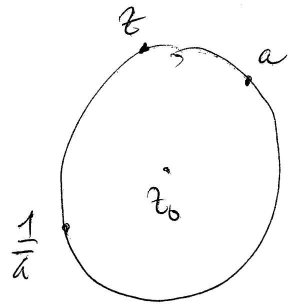
This is a corollary from a more general statement in Ahlfors, p. 17.

We see that $\left|a\left(\frac{1}{\bar{a}}\right)\right| = 1$ thus ~~the modulus of~~ one of a and $\frac{1}{\bar{a}}$ has modulus ≤ 1 and ~~one~~ the other has modulus ≥ 1. That means one ~~is~~ in the unit circle and one is out of the unit circle. Thus any circle passing through them must have a common point with the unit circle. Here we eliminate the case $|a| = 1$ because these two circle may not intersect as shown in the figure; and in this case



two circles do not meet at right angle.

~~Thus~~ Let z_0 be the center of the circle passing through a and $\frac{1}{\bar{a}}$.



~~z is out~~ Let z be ~~an intersection point of this circle and the un~~ some point on this circle. Then $|z - z_0| = |a - z_0|$. Thus

$$|z - z_0|^2 = |a - z_0|^2$$

or
$$(z - z_0)(\bar{z} - \bar{z}_0) = (a - z_0)(\bar{a} - \bar{z}_0)$$

or
$$|z|^2 - 2\,\mathrm{Re}(\bar{z}_0 z) + |z_0|^2 = |a|^2 - 2\,\mathrm{Re}(\bar{z}_0 a) + |z_0|^2$$

or
$$2\,\mathrm{Re}(\bar{z}_0 z) = -|a|^2 + |z|^2 + 2\,\mathrm{Re}(\bar{z}_0 a) \qquad (*)$$

We know that $z = \frac{1}{\bar{a}}$ is on this circle. Thus $(*)$ applies for $z = \frac{1}{\bar{a}}$:

$$2\,\mathrm{Re}\left(\frac{\bar{z}_0}{\bar{a}}\right) = -|a|^2 + \frac{1}{|a|^2} + 2\,\mathrm{Re}(\bar{z}_0 a)$$

or
$$2\,\mathrm{Re}\left(\frac{1}{|a|^2}\bar{z}_0 a\right) = \frac{1}{|a|^2} + 2\,\mathrm{Re}(\bar{z}_0 a) - |a|^2$$

or
$$\frac{1}{|a|^2}\left(2\,\mathrm{Re}(\bar{z}_0 a) - 1\right) = 2\,\mathrm{Re}(\bar{z}_0 a) - |a|^2$$

~~or~~ $\cancel{\left(\frac{1}{|a|^2} - 1\right)\left(2\,\mathrm{Re}(\bar{z}_0 a) + 1\right)}$ Put $\alpha = 2\,\mathrm{Re}(\bar{z}_0 a)$, $\beta = |a|^2$

we have $\frac{1}{\beta}(\alpha - 1) = \alpha - \beta$. Thus $\alpha = 1 + \beta$. Thus

$$\boxed{2\operatorname{Re}(\bar{z}_0 a) = |a|^2 + 1} \quad (**)$$

Now let z be an intersection point of the circle centered at z_0 and the unit circle. Then $|z| = 1$, $\bar{z} = \frac{1}{z}$ and $(*)$ gives

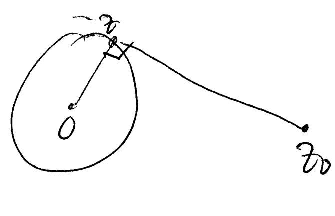
$$2\operatorname{Re}(\bar{z}_0 z) = -|a|^2 + 1 + 2\operatorname{Re}(\bar{z}_0 a) \overset{(**)}{=} 2$$

Thus
$$1 = \operatorname{Re}(\bar{z}_0 z) = \operatorname{Re}\left(\frac{\bar{z}_0}{\bar{z}}\right) = \operatorname{Re}\left(\frac{z_0}{z}\right)$$

Thus
$$\cancel{\operatorname{Re}\left(\frac{z_0}{z}\right)} \ \operatorname{Re}\left(\frac{z_0 - z}{z}\right) = \operatorname{Re}\left(\frac{z_0}{z}\right) - 1 = 0$$

thus $z_0 - z$ is perpendicular to z

Thus the two circles meet at right angle.



(8) Problem 2, Ahlfors p. 28

Verify Cauchy-Riemann's equations for $f(z) = z^2$ and $g(z) = z^3$.

<u>For $f(z) = z^2$</u>

Put $z = x + iy$. Then $f(z) = (x+iy)^2 = (x^2 - y^2) + 2xy\,i$
$$= u(x,y) + i v(x,y)$$

where $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$.

We see that $u(x,y)$ and $v(x,y)$ are continuously differentiable at any order and

$$\begin{cases} \dfrac{\partial u}{\partial x} = 2x = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -2y = -\dfrac{\partial v}{\partial x} \end{cases}$$

Thus the C-R equations are satisfied. Therefore f is differentiable on $\mathbb{C}$

<u>For $g(z) = z^3$</u>

$$\begin{aligned} g(z) = (x+iy)^3 &= x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3 \\ &= x^3 + 3ix^2y - 3xy^2 - iy^3 \\ &= (x^3 - 3xy^2) + i(3x^2y - y^3) \\ &= u(x,y) + i\,v(x,y) \end{aligned}$$

where $u(x,y) = x^3 - 3xy^2$ and $v(x,y) = 3x^2y - y^3$. Thus

$\cancel{\dfrac{\partial u}{\partial x} = 3x^2}$ $u(x,y)$ and $v(x,y)$ are continuously differentiable at any order and

$$\begin{cases} \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2 = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -6xy = -\dfrac{\partial v}{\partial x} \end{cases}$$

Thus the C-R equations are satisfied, and g is differentiable on $\mathbb{C}$.

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(9) Problem 4, Ahlfors p. 28

S|S Suppose that $f : \mathbb{C} \to \mathbb{C}$ is analytic and $|f(z)| \equiv C$ (const).

We will show that f must be a constant function:

Put $f(z) = u(x,y) + iv(x,y)$ where u and v are real functions.

Then $C^2 = |f(z)|^2 = |u(x,y)|^2 + |v(x,y)|^2$. (*) ~~Suppose that f i~~ If

$C = 0$ then $|f(z)| = 0$ and $f(z) \equiv 0$, ✓ which is a constant function. Now

we consider the case $C \neq 0$. We have $u(x,y)$ and $v(x,y)$ are

continuously differentiable and they satisfies C-R's equations at all points

$(x,y) \in \mathbb{R}^2$: $\quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \quad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

Taking the derivative with respect to x at both side, we get

$$\frac{\partial u}{\partial x} u + \frac{\partial v}{\partial x} v = 0 \qquad (1)$$

Similarly, with y, we get

$$\frac{\partial u}{\partial y} u + \frac{\partial v}{\partial y} v = 0 \qquad (2)$$

Now applies the C-R relations, (1) and (2) gives

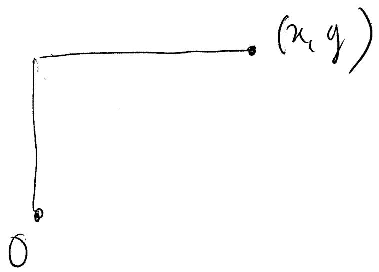
$$\begin{cases} u\dfrac{\partial u}{\partial x} - v\dfrac{\partial u}{\partial y} = 0 \\ v\dfrac{\partial u}{\partial x} + u\dfrac{\partial u}{\partial y} = 0 \end{cases}$$

We consider $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ above as two unknowns of a linear system of equations. The determinant of this system is

$$D = \det\begin{pmatrix} u & -v \\ v & u \end{pmatrix} = u^2+v^2 = C^2 > 0$$

Thus the system has unique solution $\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = (0,0)$.

Now we will show that $u = u(x,y)$ is a constant function. For each $(x,y) \in \mathbb{R}^2$, we'll show that $u(x,y) = u(0,0)$. By the Mean Value Theorem, we have



$$u(x,y) - u(0,y) = x\frac{\partial u}{\partial x}(\xi, y)$$

(where ξ is some point between 0 and x)

$$= 0$$

Thus $u(x,y) = u(0,y)$. Moreover,

$$u(0,y) - u(0,0) = y\frac{\partial u}{\partial y}(0, \xi) = 0$$

(where ξ is some point between 0 and y)

Thus $u(x,y) = u(0,0)$ $\forall (x,y) \in \mathbb{R}^2$.

Similarly, we have $\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) = 0$ by C-R's equations, and thus $v(x,y) = v(0,0)$ $\forall (x,y) \in \mathbb{R}^2$. Therefore $f(z) = u(x,y) + i v(x,y)$

$= u(0,0) + i v(0,0)$

$\forall z \in \mathbb{C}$. ✓

(10) Problem 7, Ahlfor p.28

Let $u: \mathbb{R}^2 \to \mathbb{R}$ be a harmonic function, which is defined to be a function in $C^2(\mathbb{R}^2)$ and $\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

We'll ~~show~~ verify that the formal differential equation $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$.

We have
$$\frac{\partial^2 u}{\partial z \partial \bar{z}} \overset{(\text{def})}{=} \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial \bar{z}}\right) \overset{\text{def}}{=} \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}\right)$$

$$\overset{\text{def}}{=} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}\right) + i \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial^2 u}{\partial x^2} - i \frac{\partial^2 u}{\partial x \partial y} + i \frac{\partial^2 u}{\partial y \partial x} - i^2 \frac{\partial^2 u}{\partial y^2}$$

$$= \underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}_{= \Delta u = 0} + i \underbrace{\left(\frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial x \partial y}\right)}_{= 0 \text{ by Clauraus's Theorem}}$$

$$= 0.$$

missing:
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