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Math 8701: Complex Analysis

Problem set 10

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(1) Problem 1, Ahlfors, p. 197.

Let (a_n) and (A_n) be two sequences of complex numbers such that $a_n \to \infty$. We'll show that there exists an entire function $f(z)$ which satisfies $f(a_n) = A_n$.

First, we have the following observations:

1) For the existence of such a function $f(z)$, the sequences (a_n) and (A_n) must satisfy some consistency, namely if $a_i = a_j$ then $A_i = A_j$. Thus, at the first step, we should remove all duplicated entries in the sequence (a_n), and at the same time, remove the corresponding entries in (A_n). The remaining of (a_n) form a subsequence which has to ~~converge~~ go to ∞. Therefore, what we are facing now is a sequence (a_n) whose entries are pairwise distinct and approaches ~~zero~~ infinity. In other words, we can add one more hypothesis, namely all a_n's are pairwise distinct.

2) If one of the a_n's is zero, say a_i, we can interchange a_i with a_1 to make $a_1 = 0$. Assume that we had solved the problem in case that all

entries of (a_n) are nonzero. We put $\widetilde{A}_n = \dfrac{A_n - A_1}{a_n}$ for all $n \geq 2$.

Let $\tilde{f}(z)$ be an entire function such that $\tilde{f}(a_n) = \widetilde{A}_n$ for all $n \geq 2$. We put $f(z) = z\tilde{f}(z) + A_1$. Then $f(z)$ is also an entire function and

$$f(a_1) = f(0) = A_1,$$

$$f(a_n) = a_n \tilde{f}(a_n) + A_1 = a_n \widetilde{A}_n + A_1 = A_n, \quad \forall n \geq 2.$$

Therefore, it suffices to solve only the case $a_n \neq 0$ for all $n \geq 1$.

3) The idea is that for each $n \in \mathbb{N}$, we find an entire function $h_n(z)$ such that $h_n(a_i) = \begin{cases} 0 & \text{if } i \neq n \\ 1 & \text{if } i = n \end{cases}$

Then the function $f(z)$ that we need is $\quad f(z) = \sum_{n=1}^{\infty} A_n h_n(z). \qquad (*)$

How to find $h_n(z)$? Let $g(z)$ be an entire function that has a_n's for zero.

$$g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{z}{a_n}\right)^n}$$

Then all a_n's are zeros of order one of $g(z)$. Thus $g'(a_n) \neq 0$ for all $n \geq 1$.

We can choose $\quad h_n(z) = \dfrac{1}{g'(a_n)} \dfrac{g(z)}{z - a_n} e^{\gamma_n (z - a_n)}$

where γ_n be a complex number which will be used to produce convergence of the series $(*)$. Since $g(a_n) = 0$, $\lim_{z \to a_n} h_n(z) = 1$. ✓ Thus a_n is a removable singularity of $h_n(z)$. Thus $h_n(z)$ is an entire function. Moreover, $h_n(a_n) = 1$ and

$h_n(a_i) = 0$ for all $i \neq n$. Thus, the problem reduces to finding complex numbers γ_n such that the series $\sum_{n=1}^{\infty} A_n h_n(z)$ converges to an entire function. By Weierstrass's theorem, what we actually want is the uniform convergence on compact sets of this series.

Fix $R > 0$. We consider the compact set $\{z : |z| \leq R\}$. On this set, $g(z)$ is bounded, say $|g(z)| \leq M_R$. Also, since $a_n \to \infty$, there exists $n_R \in \mathbb{N}$ such that $|a_n| > 2R+1$ for all $n \geq n_R$. Consequently,

$$|z - a_n| \geq |a_n| - |z| > (2R+1) - R = R+1 > 1$$

We put $$f_n(z) = A_n h_n(z) = \frac{A_n}{g'(a_n)} \frac{g(z)}{z-a_n} e^{\gamma_n (z-a_n)}$$

Then $$|f_n(z)| \leq \underbrace{\left|\frac{A_n}{g'(a_n)}\right|}_{c_n \in \mathbb{R},\ c_n \geq 0} \frac{|g(z)|}{|z-a_n|} \left|e^{\gamma_n(z-a_n)}\right|$$

$$\leq c_n M_R \left|e^{\gamma_n(z-a_n)}\right| \quad \text{for all } |z| \leq R.$$

We put $\beta_n = \gamma_n a_n$, or equivalently $\gamma_n = \frac{\beta_n}{a_n}$. Then

$$\left|e^{\gamma_n(z-a_n)}\right| = \left|e^{\beta_n\left(\frac{z}{a_n}-1\right)}\right| = \exp\left(\operatorname{Re}\left(\beta_n\left(\frac{z}{a_n}-1\right)\right)\right)$$

$$= \exp\left(\beta_n\left(\operatorname{Re}\frac{z}{a_n}-1\right)\right) \quad (\text{assuming } \beta_n \in \mathbb{R})$$

$$\leq \exp\left(\beta_n\left(\frac{|z|}{|a_n|}-1\right)\right) \quad (\text{assuming } \beta_n \geq 0)$$

$$\leq \exp\left(\beta_n\left(\frac{R}{|a_n|}-1\right)\right) \leq \exp\left(\beta_n\left(-\frac{1}{2}\right)\right) = \exp\left(-\frac{\beta_n}{2}\right) \text{ for all } |z|\leq R.$$

Thus, $|f_n(z)| \leq M_R\, c_n \exp\left(-\frac{\beta_n}{2}\right)$ for all $|z|\leq R$ and $n \geq n_R$.

We want to choose for each $n \in \mathbb{N}$, $\beta_n \geq 0$ such that the series

$$\sum_{n=1}^{\infty} c_n \exp\left(-\frac{\beta_n}{2}\right) < \infty.$$

We choose $\beta_n = 2(c_n + n - 1)$. Indeed, $\exp\left(-\frac{\beta_n}{2}\right) = \exp\left(1-c_n-n\right) = e^{1-c_n}e^{-n}$

Since $c_n \geq 0$, $e^{c_n-1} \geq c_n$. Then $c_n \exp\left(-\frac{\beta_n}{2}\right) = c_n e^{1-c_n} e^{-n} \leq e^{-n}$. Thus,

$$\sum_{n=1}^{\infty} c_n \exp\left(-\frac{\beta_n}{2}\right) \leq \sum_{n=1}^{\infty} e^{-n} < \infty$$

With the chosen β_n, γ_n will be given by

$$\gamma_n = \frac{2}{a_n}\left(\left|\frac{A_n}{g'(a_n)}\right| + n - 1\right).$$

② Problem 3, Ahlfors, p. 198.

We'll find the genus of function $f(z) = \cos\sqrt{z}$.

Assume that the function $\sqrt{z}$ was chosen as the principal branch. Then it is analytic on $\Omega = \mathbb{C}\setminus\{z : \operatorname{Im} z = 0, \operatorname{Re} z \leq 0\}$. Thus $f(z)$ is analytic on Ω. We know that cosine has a power series representation

$$\cos w = 1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \frac{w^6}{6!} + \cdots = g(w^2),$$

where $g(z) := 1 - \frac{z}{2!} + \frac{z^2}{4!} - \frac{z^3}{6!} + \cdots$

The radius of convergence of $g(z)$ is the square root of the radius of convergence of $\cos z$, which was proven ∞. Thus $g(z)$ is an entire function. By the definition of $f(z)$ and $g(z)$, we have

$$f(z) = \cos\sqrt{z} = g(z) \quad \forall z \in \Omega.$$

Thus $g(z)$ is a (unique) analytic continuation of $f(z)$ on $\mathbb{C}$. In otherwords, all singularities of $f(z)$ are removable. Thus, we can identify

$$f(z) = g(z) = 1 - \frac{z}{2!} + \frac{z^2}{4!} - \frac{z^3}{6!} + \cdots \qquad \forall z \in \mathbb{C}.$$

Next, we'll find all zeros of $f(z)$. Suppose that z is a zero of $f(z)$ and w is a square root of z. Then

$$\begin{aligned} f(z) = 0 &\Leftrightarrow f(w^2) = 0 \\ &\Leftrightarrow \cos w = 0 \\ &\Leftrightarrow w = \frac{\pi}{2}(2k+1) \quad \text{for } k \in \mathbb{Z} \end{aligned}$$

Therefore, all zeros of $f(z)$ are $a_k = \frac{\pi^2}{4}(2k+1)^2$, for $k \in \mathbb{Z}$.

We see that $a_k > 0$ for all k and

$$\sum_{k=1}^{\infty} \frac{1}{a_k} = \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} < \infty,$$

$$\sum_{k=1}^{\infty} \frac{1}{a_k} = \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(-2k+1)^2} < \infty.$$

Thus the series $\sum_{k \in \mathbb{Z}} \frac{1}{a_k}$ Converges. That means $f(z)$ is an entire function of genus $h=0$. Thus, we can write

$$f(z) = e^{\gamma(z)} \prod_{k\in\mathbb{Z}} \left(1 - \frac{z}{a_k}\right),$$

for some ~~cont~~ entire function $\gamma(z)$.

③ Problem 1, Ahlfors, p. 200.

We will prove the formula of Gauss:

$$(2\pi)^{\frac{n-1}{2}} \Gamma(z) = n^{z-\frac{1}{2}} \Gamma\left(\frac{z}{n}\right) \Gamma\left(\frac{z+1}{n}\right) \cdots \Gamma\left(\frac{z+n-1}{n}\right).$$

Put $w = z/n \in \mathbb{C}$, the above identity is equivalent to

$$(2\pi)^{\frac{n-1}{2}} \Gamma(nw) = n^{nw-\frac{1}{2}} \Gamma(w) \Gamma\left(w + \tfrac{1}{n}\right) \cdots \Gamma\left(w + \tfrac{n-1}{n}\right) \qquad (*)$$

This is a generalization of the identity $\sqrt{\pi}\,\Gamma(2w) = 2^{2z-1}\Gamma(w)\Gamma(w+\tfrac{1}{2})$, which Ahlfors proved. Thus, we should mimick his approach to prove $(*)$, namely, taking the logarithms of both sides of $(*)$, and then consider the second derivatives.

From the definition of $\Gamma(z)$, we have the following identity, which was proven in Ahlfors.

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{m=0}^{\infty} \frac{1}{(z+m)^2} \qquad (**)$$

Replacing z by nw, we get

$$\frac{1}{n}\frac{d}{dw}\left(\frac{\Gamma'(nw)}{\Gamma(nw)}\right) = \sum_{m=0}^{\infty} \frac{1}{(nw+m)^2} \qquad (***)$$

Replacing z in $(**)$ by $w+\frac{k}{n}$ for each $k=0,1,\dots,n-1$, we get

$$\frac{d}{dw}\left(\frac{\Gamma'\left(w+\frac{k}{n}\right)}{\Gamma\left(w+\frac{k}{n}\right)}\right)=\sum_{m=0}^{\infty}\frac{1}{\left(w+\frac{k}{n}+m\right)^2}=n^2\sum_{m=0}^{\infty}\frac{1}{(nw+mn+k)^2}$$

Taking the sum of both sides for $k=0,1,\dots,n-1$, we get

$$\frac{d}{dw}\left(\sum_{k=0}^{n-1}\frac{\Gamma'\left(w+\frac{k}{n}\right)}{\Gamma\left(w+\frac{k}{n}\right)}\right)=n^2\sum_{k=0}^{n-1}\sum_{m=0}^{\infty}\frac{1}{(nw+mn+k)^2}$$

$$=n^2\sum_{m=0}^{\infty}\sum_{k=0}^{n-1}\frac{1}{(nw+mn+k)^2}$$

$$=n^2\sum_{q=0}^{\infty}\frac{1}{(nw+q)^2}$$

Comparing this result with $(***)$, we get

$$\frac{d}{dw}\left(\sum_{k=0}^{n-1}\frac{\Gamma'\left(w+\frac{k}{n}\right)}{\Gamma\left(w+\frac{k}{n}\right)}\right)=n^2\,\frac{1}{n}\,\frac{d}{dw}\left(\frac{\Gamma'(nw)}{\Gamma(nw)}\right)=\frac{d}{dw}\left(\frac{1}{n}\,\frac{\Gamma'(nw)}{\Gamma(nw)}\right)$$

We realize that the terms to be differentiated are logarithmic derivatives, i.e.

$$\frac{d^2}{dw^2}\left(\sum_{k=0}^{n-1}\log\Gamma\left(w+\frac{k}{n}\right)\right)=\frac{d^2}{dw^2}\left(\log\Gamma(nw)\right)$$

Thus, $$\frac{d^2}{dw^2}\left(\log\Gamma(nw)-\log\left(\prod_{k=0}^{n-1}\Gamma\left(w+\frac{k}{n}\right)\right)\right)=0$$

Thus the difference between two logarithms is of the form $aw+b$ where a,b are independent of w. Thus,

$$\Gamma(nw) = e^{aw+b}\Gamma(w)\Gamma\left(w+\tfrac{1}{n}\right)\cdots\Gamma\left(w+\tfrac{n-1}{n}\right). \qquad (1)$$

To find a, we notice that $\Gamma(k) = (k-1)!$ for all $k \in \mathbb{N}$. Replacing w in (1) by 1, we get

$$\Gamma(n) = (n-1)! = e^{a+b}\Gamma(1)\Gamma\left(1+\tfrac{1}{n}\right)\Gamma\left(1+\tfrac{2}{n}\right)\cdots\Gamma\left(1+\tfrac{n-1}{n}\right). \qquad (2)$$

Replacing w in (1) by $1+\frac{1}{n}$, we get

$$\Gamma(n+1) = n! = e^{a+b+a/n}\,\Gamma\left(1+\tfrac{1}{n}\right)\Gamma\left(1+\tfrac{2}{n}\right)\cdots\Gamma\left(1+\tfrac{n-1}{n}\right)\Gamma(2) \qquad (3)$$

Dividing (3) by (2), we get $n = e^{a/n}\dfrac{\Gamma(2)}{\Gamma(1)} = e^{a/n}$. Thus $e^a = n^n$. Then (1) becomes $\Gamma(nw) = e^b n^{nw}\Gamma(w)\Gamma\left(w+\frac{1}{n}\right)\cdots\Gamma\left(w+\frac{n-1}{n}\right)$. Thus,

$$e^b = \frac{\Gamma(nw)}{n^{nw}\Gamma(w)\Gamma\left(w+\frac{1}{n}\right)\cdots\Gamma\left(w+\frac{n-1}{n}\right)} \qquad (4)$$

By comparing (4) to $(*)$, what we need to show is that $e^b = n^{-\frac{1}{2}}(2\pi)^{-\frac{n-1}{2}}$, or equivalently

$$e^{2b} = \frac{1}{n(2\pi)^{n-1}}. \qquad (5)$$

From the definition of the Gamma function,

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{m=1}^{\infty}\left(1+\frac{z}{m}\right)^{-1}e^{z/m}$$

Thus

$$z\Gamma(z) = e^{-\gamma z}\prod_{m=1}^{\infty}\left(1+\frac{z}{m}\right)^{-1}e^{z/m}$$

Therefore, $\lim_{z\to 0} z\Gamma(z) = 1$. Then we have

$$\lim_{w\to 0}\frac{\Gamma(nw)}{\Gamma(w)} = \lim_{w\to 0}\frac{nw\,\Gamma(nw)}{nw\,\Gamma(w)} = \frac{1}{n}.$$

Taking the limits of both sides of (4) as $w \to 0$, we get

$$e^{b} = \lim_{w\to 0}\frac{\Gamma(nw)}{\Gamma(w)}\left(\lim_{w\to 0}\frac{1}{n^{nw}}\right)\lim_{w\to 0}\frac{1}{\cancel{\Gamma(w)}\,\Gamma(w+\frac{1}{n})\cdots\Gamma(w+\frac{n-1}{n})}$$

$$= \frac{1}{n}\,\frac{1}{\Gamma(\frac{1}{n})\Gamma(\frac{2}{n})\cdots\Gamma(\frac{n-1}{n})}$$

Taking the square, we get

$$e^{2b} = \frac{1}{n^2}\prod_{k=1}^{n-1}\frac{1}{\Gamma(\frac{k}{n})\Gamma(1-\frac{k}{n})}$$

By the identity $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin \pi z}$, we have $\Gamma(\frac{k}{n})\Gamma(1-\frac{k}{n}) = \frac{\pi}{\sin\frac{k\pi}{n}}$. Thus,

$$e^{2b} = \frac{1}{n^2}\prod_{k=1}^{n-1}\frac{\sin\frac{k\pi}{n}}{\pi} = \frac{1}{n^2\pi^{n-1}}\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}. \qquad (6)$$

By comparing (6) to (5), what we need to show is that

$$\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} = \frac{n}{2^{n-1}}$$

Since the left hand member is always positive, this identity is equivalent to

$$\prod_{k=1}^{n-1}\sin^2\frac{k\pi}{n} = \frac{n^2}{(2^{n-1})^2}$$

$$\iff \prod_{k=1}^{n-1}\frac{1-\cos\frac{2k\pi}{n}}{2} = \frac{n^2}{(2^{n-1})^2}$$

$$\iff \prod_{k=1}^{n-1}\left(1-\cos\frac{2k\pi}{n}\right) = \frac{n^2}{2^{n-1}} \qquad (7)$$

Therefore, we need to verify (7).

The Chebyshev polynomials $T_n(x)$, for $n \geq 0$, are very closely related to trigonometric terms like $\cos\frac{2k\pi}{n}$, and would be useful for the computation of LHS(7).

$$\begin{cases} T_0(x) = 1 \\ T_1(x) = x \\ T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x) \qquad \forall n \geq 1. \end{cases}$$

By induction, we have T_n is a polynomial of degree n, with leading coefficient 2^{n-1} for all $n \geq 1$. An important property of these polynomials (which is also proved by induction) is that

$$T_n(\cos t) = \cos(nt) \qquad \forall n \geq 1,\ \forall t \in \mathbb{R} \qquad (8)$$

Replacing t by $\frac{2k\pi}{n}$, for $k = 0, 1, .., n-1$, we get $T_n\left(\cos\frac{2k\pi}{n}\right) = \cos(2k\pi) = 1$.

Thus, $\cos\frac{2k\pi}{n}$, for $0 \leq k \leq n-1$, are n distinct zeros of $T_n(x) - 1$. Therefore,

$$T_n(x) - 1 = 2^{n-1} \prod_{k=0}^{n-1} \left(x - \cos\frac{2k\pi}{n}\right)$$

$$= 2^{n-1}(x-1) \prod_{k=1}^{n-1} \left(x - \cos\frac{2k\pi}{n}\right)$$

Thus, $$\prod_{k=1}^{n-1}\left(x - \cos\frac{2k\pi}{n}\right) = \frac{1}{2^{n-1}} \frac{T_n(x) - 1}{x - 1}.$$

As $x \to 1$, we get $$\prod_{k=1}^{n-1}\left(1 - \cos\frac{2k\pi}{n}\right) = \frac{T_n'(1)}{2^{n-1}} \qquad (9)$$

By comparing (9) with (7), what we need to show is that $T_n'(1) = n^2$.

For each $n \geq 0$, we denote $a_n = T_n'(1)$. Taking the derivatives of the identity

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x),$$

we get $T_{n+1}'(x) = 2T_n(x) + 2x T_n'(x) - T_{n-1}'(x)$. At $x = 1$, we have

$$a_{n+1} = 2T_n(1) + 2a_n - a_{n-1}.$$

Take $t = 0$ in (8), we have $T_n(1) = 1$. Thus, the sequence (a_n) is given by a recursive formula

$$\begin{cases} a_0 = 0, \quad a_1 = 1 \\ a_{n+1} = 2a_n - a_{n-1} + 2 \qquad \forall n \geq 1 \end{cases}$$

By induction, we get $a_n = n^2$.

4) Problem 2, Ahlfors, p. 200.

We'll show that $\Gamma\left(\frac{1}{6}\right) = 2^{-\frac{1}{3}} \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \Gamma\left(\frac{1}{3}\right)^2$.

To do so, we'll use the following identities

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1} \Gamma(z)\Gamma\left(z+\tfrac{1}{2}\right), \qquad (*)$$

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$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \qquad (**)$$

(*) is a special case of the result of the previous problem, for $n = 2$.

Applying (*) for $z = \frac{1}{6}$, we get

$$\sqrt{\pi}\,\Gamma\left(\frac{1}{3}\right) = 2^{-\frac{2}{3}} \Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{2}{3}\right).$$

Multiplying both sides by $\Gamma(\frac{1}{3})$, we get

$$\sqrt{\pi}\,\Gamma\left(\tfrac{1}{3}\right)^2 = 2^{-\frac{2}{3}}\,\Gamma\left(\tfrac{1}{6}\right)\left(\Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{2}{3}\right)\right)$$

Applying $(**)$ for $z=\frac{1}{3}$, we have $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \frac{\pi}{\sin\frac{\pi}{3}} = \frac{2\pi}{\sqrt{3}}$. Thus,

$$\sqrt{\pi}\,\Gamma\left(\tfrac{1}{3}\right)^2 = 2^{-\frac{2}{3}}\,\Gamma\left(\tfrac{1}{6}\right)\frac{2\pi}{\sqrt{3}} = \frac{2^{\frac{1}{3}}}{\sqrt{3}}\pi\,\Gamma\left(\tfrac{1}{6}\right).$$

Thus,
$$\Gamma\left(\tfrac{1}{6}\right) = \frac{\sqrt{3}}{2^{\frac{1}{3}}}\,\frac{\sqrt{\pi}}{\pi}\,\Gamma\left(\tfrac{1}{3}\right)^2 = 2^{-\frac{1}{3}}\left(\frac{3}{\pi}\right)^{\frac{1}{2}}\Gamma\left(\tfrac{1}{3}\right)^2.$$

(5) We'll find the residues of $\Gamma(z)$ at its poles. By definition, the gamma function
$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1} e^{z/n}$$
has simple poles at $z=0$ and $z=-n$, for $n\in\mathbb{N}$. As shown in problem (3) above, $\lim_{z\to 0} z\Gamma(z) = 1$. Thus the residue at 0 is 1. For $n\in\mathbb{N}$, we have $\operatorname{Res}_{z=-n}\Gamma(z) = \lim_{z\to -n}(z+n)\Gamma(z)$.

Using the identity $\Gamma(w+1) = w\Gamma(w)$ n times, we get

$$\begin{aligned}\Gamma(z+n) &= (z+n-1)\Gamma(z+n-1) = (z+n-1)(z+n-2)\Gamma(z+n-2)\\ &= \cdots\\ &= (z+n-1)(z+n-2)\cdots(z+1)\,z\,\Gamma(z)\end{aligned}$$

Thus,
$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}$$

thus
$$\lim_{z\to -n}(z+n)\Gamma(z) = \left(\lim_{z\to -n}(z+n)\Gamma(z+n)\right)\left(\lim_{z\to -n}\frac{1}{z(z+1)\cdots(z+n-1)}\right)$$

$$= \frac{1}{(-n)(-n+1)\cdots(-n+n-1)} = \frac{1}{(-1)^n n(n-1)\cdots(n-(n-1))} = \frac{(-1)^n}{n!}.$$

Therefore,

$$\operatorname{Res}_{z=-n} \Gamma(z) = \frac{(-1)^n}{n!} \quad \forall n \geq 0.$$

(6) Problem 2, Ahlfors, p. 206

5/5 For $x>0$, we'll show that $\Gamma(x) = \sqrt{2\pi}\, x^{x-\frac{1}{2}} e^{-x} e^{\theta(x)/12}$ with $0<\theta(x)<1$.

We know by Eq. (36) and Eq. (38) in Ahlfors, p. 204 that for a complex number z,

$$\Gamma(z) = \sqrt{2\pi}\, z^{z-\frac{1}{2}} e^{-z} e^{J(z)}, \text{ where}$$

$$J(z) = \frac{1}{\pi} \int_0^\infty \frac{z}{\eta^2+z^2} \log \frac{1}{1-e^{-2\pi\eta}}\, d\eta$$

For $x>0$, we have

$$\Gamma(x) = \sqrt{2\pi}\, x^{x-\frac{1}{2}} e^{-x} e^{J(x)}, \text{ where}$$

$$J(x) = \frac{1}{\pi} \int_0^\infty \frac{x}{\eta^2+x^2} \log \frac{1}{1-e^{-2\pi\eta}}\, d\eta$$

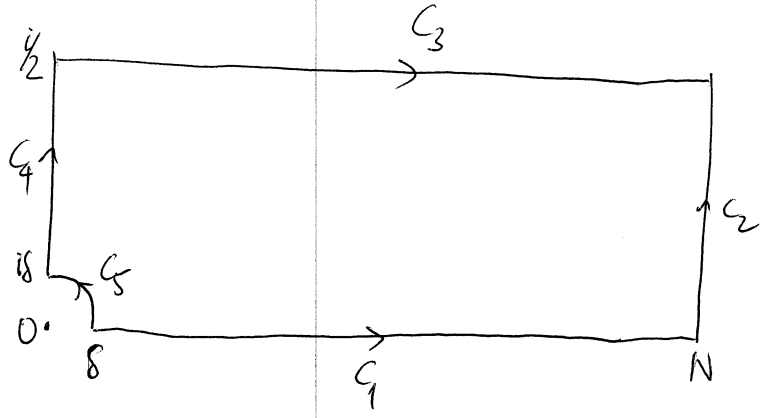
Thus, we need to show that $0 < J(x) < \frac{1}{12x}$. We have $J(x) > 0$ because the integrand is positive and continuous. By the same reason, we have

$$J(x) < \frac{1}{\pi} \int_0^\infty \frac{x}{0+x^2} \log \frac{1}{1-e^{-2\pi\eta}}\, d\eta$$

$$= \frac{-1}{\pi x} \underbrace{\int_0^\infty \log(1-e^{-2\pi\eta})\, d\eta}_{I}$$

To show that $J(x) < \frac{1}{12x}$, we'll show that $I = -\frac{\pi}{12}$. And we will do so by the method of residues.

Put $f(z) = \log(1 - e^{-2\pi z})$, where log takes single values in the principal branch. We consider a contour composed of five parts $C_1, C_2, -C_3, -C_4, -C_5$ as follow.



C_1: $z = t$, $\delta \le t \le N$

C_2: $z = N + it$, $0 \le t \le \frac{1}{2}$

C_3: $z = t + \frac{i}{2}$, $0 \le t \le N$

C_4: $z = it$, $\delta \le t \le \frac{1}{2}$

C_5: $z = e^{it}$, $0 \le t \le \frac{\pi}{2}$

Because the equation $1 = e^{-2\pi z}$ has solutions $z = ki$ for $k \in \mathbb{Z}$, $f(z)$ is analytic in the domain determined by these ~~contours~~ curves. By Cauchy's Integral formula,
$$\left(\int_{C_1} + \int_{C_2} - \int_{C_3} - \int_{C_4} - \int_{C_5}\right) f\,dz = 0.$$

Thus, $\int_{C_1} f - \int_{C_3} f = \int_{C_4} f - \int_{C_2} f + \int_{C_5} f.$ $(*)$

We consider the limits of these integrals as $\delta \to 0$ and $N \to \infty$ independently.

- The integral on C_1

$$\int_{C_1} f = \int_{\delta}^{N} \log(1-e^{-2\pi\eta})\,d\eta \xrightarrow[N\to\infty]{\delta\to 0} \int_0^{\infty} \log(1-e^{-2\pi\eta})\,d\eta = I. \quad (1)$$

Indeed, this will be true if we can show that the limits exist.

For $\eta \geq 1$, we have

$$0 \leq \log\frac{1}{1-e^{-2\pi\eta}} \leq \frac{1}{1-e^{-2\pi\eta}} - 1 \quad \left(\text{we've used the inequality } \log x \leq x-1 \ \forall x>0\right)$$

$$= \frac{e^{-2\pi\eta}}{1-e^{-2\pi\eta}} \leq \frac{e^{-2\pi\eta}}{1-e^{-2\pi}}$$

Since the function $e^{-2\pi\eta}$ is integrable on $[1,\infty)$, so is $\log\frac{1}{1-e^{-2\pi\eta}}$. Thus, $\log(1-e^{-2\pi\eta})$ is integrable on $[1,\infty)$.

For $0<\eta\leq 1$, we have

$$\log\frac{1}{1-e^{-2\pi\eta}} = \log\frac{\eta}{1-e^{-2\pi\eta}} - \log\eta$$

We know that $\log\eta$ is integrable on $(0,1]$. Indeed,

$$\int_{\varepsilon}^{1} \log\eta\,d\eta \overset{u=\frac{1}{\eta}}{=\!=} \int_{1/\varepsilon}^{1} -(\log u)\left(-\frac{1}{u^2}\right)du = -\int_1^{\varepsilon^{-1}} \underbrace{\frac{\log u}{u^2}}_{\substack{\text{integrable on } [1,\infty) \text{ because} \\ \log u < \sqrt{u} \text{ for } u \text{ sufficiently large.}}} du$$

Then function $\dfrac{\eta}{1-e^{-2\pi\eta}}$ is positive on $(0,1]$. At $\eta=0$, we have

$$\lim_{\eta\to 0}\frac{\eta}{1-e^{-2\pi\eta}} = -\frac{1}{2\pi}\lim_{\eta\to 0}\frac{-2\pi\eta}{1-e^{-2\pi\eta}} = +\frac{1}{2\pi}\lim\frac{1}{\frac{d}{d\eta}(e^{-2\pi\eta})\Big|_{\eta=0}} = \frac{1}{2\pi}>0.$$

Thus $\dfrac{\eta}{1-e^{-2\pi\eta}}$ is positive and continuous on $[0,1]$. Thus $\log\left(\dfrac{\eta}{1-e^{-2\pi\eta}}\right)$ is integrable (in fact continuous) on $[0,1]$.

- <u>The integral on C_3</u>

$$\int_{C_3} f = \int_0^N \log\left(1-e^{-2\pi\left(t+\frac{i}{2}\right)}\right)dt$$

$$= \int_0^N \log\left(1+e^{-2\pi t}\right)dt \quad \text{because } e^{-2\pi\left(t+\frac{i}{2}\right)} = e^{-2\pi t}e^{-i\pi} = -e^{-2\pi t}.$$

Then

$$\int_{C_3} f+\int_{C_1} f = \int_0^N \log\left(1+e^{-2\pi t}\right)dt + \int_\delta^N \log\left(1-e^{-2\pi t}\right)dt$$

$$= \int_0^\delta \log\left(1+e^{-2\pi t}\right)dt + \underbrace{\int_\delta^N \left[\log\left(1+e^{-2\pi t}\right)+\log\left(1-e^{-2\pi t}\right)\right]dt}_{(**)}$$

$$(**) = \int_\delta^N \log\left(1-e^{-4\pi t}\right)dt \;\overset{u=2t}{=\!=}\; \frac{1}{2}\int_{2\delta}^{2N}\log\left(1-e^{-2\pi u}\right)du$$

Thus,

$$\underbrace{\int_{C_3} f+\int_{C_1} f}_{\to I} = \underbrace{\int_0^\delta \log\left(1+e^{-2\pi t}\right)dt}_{\to 0} + \frac{1}{2}\underbrace{\int_{2\delta}^{2N}\log\left(1-e^{-2\pi u}\right)du}_{\to I} \quad \text{as } \delta\to 0,\ N\to\infty$$

Thus, $\lim_{\substack{\delta\to 0\\ N\to\infty}} \int_{C_3} f = -\frac{I}{2}$ (2)

- The integral on C_5

$$\int_{C_5} f = \int_{C_5} \log(1-e^{-2\pi z})\,dz$$

Since $\log w := \log|w| + i\,Arg(w)$, where $-\pi < Arg(w) < \pi$, we have

$$|\log w| \leq |\log|w|| + \pi.$$

Thus, $\left|\int_{C_5} f\right| \leq \int_{C_5} |\log(1-e^{-2\pi z})|\,|dz| \leqslant \frac{\delta\pi}{2}\cdot\max_{C_5} |\log(1-e^{-2\pi z})|$ (***)

we have, $|\log(1-e^{-2\pi z})| \leq |\log|1-e^{-2\pi z}|| + \pi$

$$\leq \log(1+|e^{-2\pi z}|) + \pi$$

$$\leq \log(1+e^{2\pi|z|}) + \pi$$

$$= \log(1+e^{2\pi\delta}) + \pi$$

Thus, (***) gives $\left|\int_{C_5} f\right| \leq \frac{\delta\pi}{2}\left[\log(1+e^{2\pi\delta}) + \pi\right] \longrightarrow 0$ as $\delta\to 0$.

Thus $\int_{C_5} f \longrightarrow 0$ as $\delta\to 0$ (3).

- The integral on C_2

$$\int_{C_2} f = i\int_0^{1/2} \log\left(1-e^{-2\pi(N+it)}\right)dt.$$

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Then $\left|\int_{C_2} f\right| \leq \int_0^{1/2} \underbrace{\left|\log\left(1-e^{-2\pi(N+ti)}\right)\right|}_{f_N(t)} dt$

Using the inequality $|\log w| \leq |\log|w|| + \pi$ again, we have

$$0 \leq f_N(t) \leq \log\left|1-e^{-2\pi(N+ti)}\right| + \pi$$

$$\leq \log\left(1+\left|e^{-2\pi(N+ti)}\right|\right) + \pi$$

$$= \log\left(1+e^{-2\pi N}\right) + \pi$$

$$\leq (\log 2) + \pi$$

Thus $f_N(t)$ is ~~bounded~~ ~~and~~ continuous on $[0, \frac{1}{2}]$ and bounded by $(\log 2)+\pi$ from the above ~~by~~ for all N. By Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{N\to\infty} \int_0^{1/2} f_N(t)\, dt = \int_0^{1/2} \left(\lim_{N\to\infty} f_N(t)\right) dt = \int_0^{1/2} 0\, dt = 0.$$

(because $1-e^{-2\pi(N+ti)} = 1-e^{-2\pi N}e^{-2\pi ti} \to 1$)

Thus, $\int_{C_2} f \longrightarrow 0$ as $N \to \infty$. (4)

- The integral on C_4

$$\int_{C_4} f = i\int_\delta^{1/2} \log\left(1-e^{-2\pi i t}\right) dt$$

Using the identity $1-e^{-2\pi i t} = (1-\cos 2\pi t) + i(\sin 2\pi t)$

$$= 2(\sin\pi t)(\sin \pi t + i\cos\pi t)$$

$$= 2e^{i\pi\left(\frac{1}{2}-t\right)}\sin\pi t,$$ we have

$$\log\left(1-e^{-2\pi i t}\right) = \log\left(2e^{i\pi\left(\frac{1}{2}-t\right)}\sin\pi t\right)$$
$$= i\pi\left(\tfrac{1}{2}-t\right) + \log(2\sin\pi t).$$

Thus,
$$i\int_0^{1/2}\log(1-e^{-2\pi i t})\,dt = i\left[\int_0^{1/2} i\pi\left(\tfrac{1}{2}-t\right)dt + \int_0^{1/2}\log(2\sin\pi t)\,dt\right.$$
$$= \underbrace{+\frac{\pi}{2}\left(\frac{1}{2}-t\right)^2\Big|_0^{1/2}}_{-\pi/8} + i\int_0^{1/2}\log(2\sin\pi t)\,dt$$

Therefore,
$$\int_{C_4} f(z)\,dz \xrightarrow{\delta\to 0} -\frac{\pi}{8} + i\int_0^{1/2}\log(2\sin\pi t)\,dt \qquad (5).$$

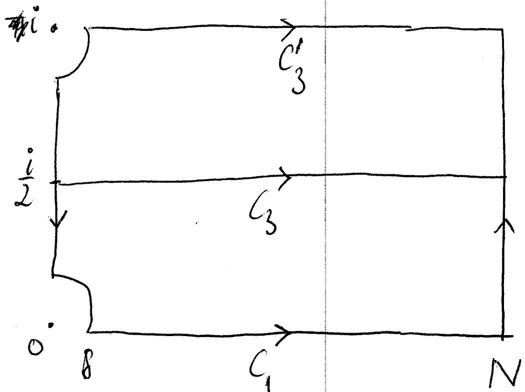
We substitute the limits found in (1)–(5) into (*) and get
$$I - \left(-\frac{I}{2}\right) = \left(-\frac{\pi}{8} + i\int_0^{1/2}\log(2\sin\pi t)\,dt\right) - 0 + 0$$

Thus,
$$I = -\frac{\pi}{12} + \frac{2i}{3}\int_0^{1/2}\log(2\sin\pi t)\,dt.$$

Because I, by its definition, is a real number, we get two identities:
$$I = -\frac{\pi}{12} \quad \text{and} \quad \int_0^{1/2}\log(2\sin\pi t)\,dt = 0.$$

The first identity helps us finish this problem. The second identity is equivalent to the identity $\int_0^{\pi}\log(\sin x)\,dx = -\pi\log 2$, which was derived in Ahlfors, page 160. He also used the method of residue. His

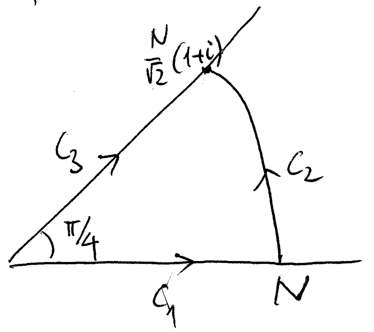
Contour, however, doesn't work for our problem. Up to a change of variable, our contour is just a half of Ahlfors's contour. This large contour doesn't work for our problem because $\int_{C_1} f = \int_{C_3'} f$ by the periodicity of the



exponential function. Thus the term $\int_{C_1} f - \int_{C_3'} f$ ~~become~~ would become zero in equation (*), and wouldn't give us any information about $\int_{C_1} f$. For our chosen contour, $\int_{C_3} f = -\frac{1}{2}\int_{C_1} f$. Thus $\int_{C_1} f$ remains meaningful when we take the difference $\int_{C_1} f - \int_{C_3} f$.

(7) Using the identity $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$ together with Cauchy's Integral formula, we'll compute the Fresnel integrals $\int_0^\infty \cos(x^2)\,dx$ and $\int_0^\infty \sin(x^2)\,dx$.

Put $f(z) = e^{-z^2}$ for all $z \in \mathbb{C}$. Then $f(z)$ is an entire function.



Consider a contour composed of $C_1, C_2, -C_3$.

$C_1 : z = t, \ 0 \le t \le N$

$C_2 : z = N e^{it}, \ 0 \le t \le \frac{\pi}{4}$

$C_3 : z = \frac{t}{\sqrt{2}}(1+i), \ 0 \le t \le N$

By Cauchy's Integral formula, $\int_{C_1+C_2-C_3} f(z)\,dz = 0$. Thus,

$$\int_{C_3} f = \int_{C_1} f + \int_{C_2} f. \qquad (*)$$

We'll consider the limits of these integrals as $N \to \infty$.

- <u>The integral over C_1</u>

$$\int_{C_1} f = \int_0^N e^{-t^2} dt \xrightarrow{N\to\infty} \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2} \qquad (1)$$

- <u>The integral over C_3</u>

$$\int_{C_3} f = \int_0^N e^{-\frac{t^2}{2}(1+i)^2} \frac{1+i}{\sqrt{2}} dt = \frac{1+i}{\sqrt{2}} \int_0^N e^{-it^2} dt$$

$$= \frac{1+i}{\sqrt{2}} \int_0^N [\cos(t^2) - i\sin(t^2)]\,dt$$

Then $\int_{C_3} f = \frac{1}{\sqrt{2}}(a_N + b_N) + \frac{i}{\sqrt{2}}(a_N - b_N)$, where $a_N = \int_0^N \cos(t^2)\,dt$, $\quad (2)$

$b_N = \int_0^N \sin(t^2)\,dt$.

- <u>The integral over C_2</u>

On C_2, $z = Ne^{it}$. Thus $dz = Nie^{it}$ and

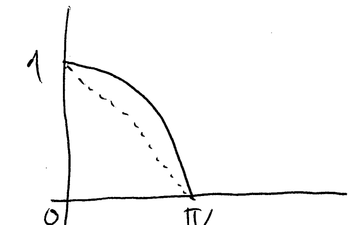
$$\int_{C_2} f = \int_0^{\pi/4} e^{-N^2 e^{2it}} Nie^{it}\,dt$$

Thus, $\left|\int_{C_2} f\right| \leq \int_0^{\pi/4} \left|e^{-N^2 e^{2it}}\right| \left|Nie^{it}\right| dt = \int_0^{\pi/4} e^{-N^2\cos(2t)} N\,dt \qquad (**)$

We should find a linear lower bound for $\cos(2t)$ in order to estimate this

Completion: 8/8

integral from the above. The graph of $\cos x$ on $0 \le x \le \frac{\pi}{2}$ gives us a hint, namely $\cos x \geqslant 1 - \frac{2x}{\pi}$ for all $0 \le x \le \frac{\pi}{2}$.



This can be verified by taking derivatives twice.

Now we apply this inequality for $x = 2t$. Then

$$\cos 2t \geqslant 1 - \frac{4t}{\pi}, \ \forall\, 0 \le t \le \frac{\pi}{4}$$

Then $$(**) \le \int_0^{\pi/4} N e^{-N^2 + \frac{4N^2}{\pi} t}\, dt = N e^{-N^2} \int_0^{\pi/4} e^{\frac{4N^2}{\pi} t}\, dt$$

$$= N e^{-N^2} \frac{\pi}{4N^2} \left(e^{\frac{4N^2}{\pi} t} \right) \Bigg|_{t=0}^{t=\frac{\pi}{4}}$$

$$= \frac{\pi e^{-N^2}}{4N} \left(e^{N^2} - 1 \right) = \frac{\pi}{4N} \left(1 - e^{-N^2} \right)$$

Thus, $$\left| \int_{C_3} f \right| \le \frac{\pi}{4N} \left(1 - e^{-N^2} \right) \longrightarrow 0 \quad \text{as } N \to \infty \qquad (3)$$

Now we substitute (1), (2), (3) into (*) and interested in $N \to \infty$. We get

$$\lim_{N \to \infty} \left[\frac{1}{\sqrt{2}} (a_N + b_N) + \frac{i}{\sqrt{2}} (a_N - b_N) \right] = \frac{\sqrt{\pi}}{2}$$

Thus, $\lim (a_N + b_N) = \frac{\sqrt{\pi}}{\sqrt{2}}$ and $\lim (a_N - b_N) = 0$. In particular, the limits of (a_N) and (b_N) exist. Thus, the Fresnel integrals converges in sense of improper integral and

$$\int_0^\infty \cos(t^2)\, dt = \int_0^\infty \sin(t^2)\, dt = \lim_{N \to \infty} a_N = \lim_{N \to \infty} b_N = \frac{1}{2} \sqrt{\frac{\pi}{2}}.$$