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Math 8701  
Complex Analysis

32/33

1

### Problem Set 2

(1) Problem 2, Ahlfors, p. 32

If  $Q$  is a polynomial with distinct roots  $\alpha_1, \dots, \alpha_n$ , and if  $P$  is a polynomial of degree less than  $n$ , show that

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^n \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)} \quad (1)$$

5/6

Proof ~~There is~~ a much simpler proof. See the solutions.

Let  $a_n$  be <sup>the</sup> coefficient corresponding to the highest order monomial of  $Q$ . By replacing  $\frac{1}{a_n}Q(z)$  by  $\frac{1}{a_n}Q(z)$ , we can always assume that  $a_n = 1$ . Since  $\alpha_1, \dots, \alpha_n$  are (distinct) roots of  $Q(z)$ , we write

$$Q(z) = (z-\alpha_1) \dots (z-\alpha_n) \quad (2)$$

By product rule of differentiation, we get

$$Q'(z) = (z-\alpha_2) \dots (z-\alpha_n) + (z-\alpha_1)(z-\alpha_3) \dots (z-\alpha_n) + \dots + (z-\alpha_1) \dots (z-\alpha_{n-1}) \quad (3)$$

Now we put

$$R(z) = \frac{P(z)}{Q(z)} - \sum_{k=1}^n \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)} \quad (4)$$

and we are trying to prove that  $R(z)$  is identically 0.

2

From (3), we substitute  $z = \alpha_1$  and see that

$$Q'(\alpha_1) = (\alpha_1 - \alpha_2) \cdots (\alpha_1 - \alpha_n) \neq 0 \quad (5)$$

With similar role as  $\alpha_1$ , every other  $\alpha_k$  satisfies  $Q'(\alpha_k) \neq 0$ . Thus the definition of rational function  $R(z)$  is legitimate. To show that  $R(z)$  is identically zero, we first show that  $R(z)$  has no poles. By its definition, all possible poles of  $R(z)$  are  $\alpha_1, \dots, \alpha_n$  and  $\infty$ . We'll prove that none of these can be poles of  $R(z)$ . Because  $P$  is of degree less

than that of  $Q$ ,

$$\lim_{z \rightarrow \infty} \frac{P(z)}{Q(z)} = 0$$

Not to mention,

$$\lim_{z \rightarrow \infty} \frac{1}{z - \alpha_k} = 0 \quad \text{for every } k = 1, 2, \dots, n$$

Thus

$$R(z) = \frac{P(z)}{Q(z)} - \sum_{k=1}^n \frac{P(\alpha_k)}{Q'(\alpha_k)} \frac{1}{z - \alpha_k} \rightarrow 0 \text{ as } z \rightarrow \infty$$

Therefore  $\infty$  is not a pole of  $R(z)$ . Due to the similar role of  $\alpha_1, \dots, \alpha_n$ , we only need to show that  $\alpha_1$  is not a pole of  $R(z)$ , i.e. to show that the limit  $\lim_{z \rightarrow \alpha_1} R(z)$  exists (and not  $\infty$ ). We have

$$R(z) = \left( \frac{P(z)}{Q(z)} - \frac{P(\alpha_1)}{Q'(\alpha_1)} \frac{1}{z - \alpha_1} \right) - \sum_{k=2}^n \frac{P(\alpha_k)}{Q'(\alpha_k)} \frac{1}{z - \alpha_k}$$

All terms in the  $\sum$  symbol have finite limits as  $z \rightarrow \alpha_1$  because  $\boxed{3}$   
 $\alpha_2, \dots, \alpha_n \neq \alpha_1$ . Thus we only need to prove that the following limit exists

$$\lim_{z \rightarrow \alpha_1} \left( \frac{P(z)}{Q(z)} - \frac{P(\alpha_1)}{Q'(\alpha_1)} \frac{1}{z - \alpha_1} \right)$$

Denote the rational function in the parentheses by  $K(z)$ . We have

$$\begin{aligned} K(z) &= \frac{P(z)}{(z - \alpha_1) \dots (z - \alpha_n)} - \frac{P(\alpha_1)}{Q'(\alpha_1)} \frac{1}{z - \alpha_1} \\ &= \frac{P(z) Q'(\alpha_1) - P(\alpha_1) (z - \alpha_2) \dots (z - \alpha_n)}{Q'(\alpha_1) (z - \alpha_1) \dots (z - \alpha_n)} \end{aligned}$$

Denote the ~~denom~~ numerator by  $S(z)$ . Then  $S(z)$  is a polynomial and

$$S(z) = P(z) Q'(\alpha_1) - P(\alpha_1) (z - \alpha_2) \dots (z - \alpha_n)$$

$$\begin{aligned} \text{Then } S(\alpha_1) &= P(\alpha_1) Q'(\alpha_1) - P(\alpha_1) (\alpha_1 - \alpha_2) \dots (\alpha_1 - \alpha_n) \\ &= P(\alpha_1) \underbrace{[Q'(\alpha_1) - (\alpha_1 - \alpha_2) \dots (\alpha_1 - \alpha_n)]}_{= 0 \text{ by (5)}} \end{aligned}$$

Thus  $S(z)$  has a factor  $(z - \alpha_1)$ , and we can write  $S(z) = (z - \alpha_1) \tilde{S}(z)$  where  $\tilde{S}(z)$  is also a polynomial. Then

$$\begin{aligned} K(z) &= \frac{S(z)}{Q'(\alpha_1) (z - \alpha_1) \dots (z - \alpha_n)} = \frac{(z - \alpha_1) \tilde{S}(z)}{Q'(\alpha_1) (z - \alpha_1) \dots (z - \alpha_n)} \\ &= \frac{\tilde{S}(z)}{Q'(\alpha_1) (z - \alpha_2) \dots (z - \alpha_n)} \end{aligned}$$

4

$$\text{Thus } \lim_{z \rightarrow \alpha_1} K(z) = \frac{f(\alpha_1)}{Q'(\alpha_1)(\alpha_1 - \alpha_2) \dots (\alpha_1 - \alpha_n)}$$

Hence  $\alpha_1$  is not a pole of  $R(z)$ . We have proved that  $R(z)$  has no poles. Thus  $R(z)$  is either identically constant or identically zero. However, we showed that  $\lim_{z \rightarrow \infty} R(z) = 0$ . Thus  $R(z)$  is identically zero, whence

$$\text{we get the identity } \frac{P(z)}{Q(z)} = \sum_{k=1}^n \frac{P(\alpha_k)}{Q'(\alpha_k)(z - \alpha_k)}$$

Remark: Given the set of all zeros in  $\mathbb{C}$ , and the set of all poles in  $\mathbb{C}$ , then the rational function is uniquely determined up to a constant multiple.

Indeed, if a rational function  $R(z)$  in a reduced form has the following representation

$$R(z) = \frac{P(z)}{Q(z)}$$

where  $P(z)$  and  $Q(z)$  have no common factor and the highest coefficient of  $Q(z)$  is 1. If we are given  $\{\alpha_1, \dots, \alpha_k\}$  to be the set of all complex zeros, and  $\{\alpha'_1, \dots, \alpha'_s\}$  to be the set of all complex poles of  $R(z)$  then

$$Q(z) = (z - \alpha'_1) \dots (z - \alpha'_s), \text{ and}$$

$$P(z) = C(z - \alpha_1) \dots (z - \alpha_k)$$

$$\text{And } R(z) = C \frac{(z - \alpha_1) \dots (z - \alpha_k)}{(z - \alpha'_1) \dots (z - \alpha'_s)}$$



This is amazing because it's not true in  $\mathbb{R}$ ! For example

$$R_1(z) = \frac{1}{z^2+1}, \quad R_2(z) = \frac{1}{z^2+z}$$

Both have no real zero and no real pole, but they have no relations to each other.

(2) Problem 3, Ahlfors, p. 32

Use the formula in the preceding exercise to prove that there exists a unique polynomial  $P$  of degree  $< n$  with given values  $c_k$  at the points  $\alpha_k$ . ~~This is~~

Proof Suppose that we are given  $P(\alpha_k) = c_k \in \mathbb{C}$ . We put

$$Q(z) = (z - \alpha_1) \dots (z - \alpha_n)$$

Because we know that the degree of  $P$  is less than  $n$ , we can apply the previous result

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^n \frac{P(\alpha_k)}{Q'(\alpha_k)(z - \alpha_k)} = \sum_{k=1}^n \frac{c_k}{Q'(\alpha_k)(z - \alpha_k)}$$

Thus

$$P(z) = Q(z) \sum_{k=1}^n \frac{c_k}{Q'(\alpha_k)(z - \alpha_k)} = \sum_{k=1}^n \frac{c_k}{Q'(\alpha_k)} \frac{Q(z)}{z - \alpha_k} \quad (6)$$

Conversely, if  $P(z)$  is given explicitly as (6),  $P(\alpha_k) = c_k$

□

$$P(\alpha_1) = \frac{c_1 (\alpha_1 - \alpha_2) \dots (\alpha_1 - \alpha_n)}{(\alpha_1 - \alpha_2) \dots (\alpha_1 - \alpha_n)} = c_1$$

Similarly,  $P(\alpha_k) = c_k$  for every  $k = 1, 2, \dots, n$ . Thus  $P(z)$  is uniquely determined

by (6):

$$P(z) = \sum_{k=1}^n c_k \prod_{\substack{j=1 \\ j \neq k}}^n \frac{z - \alpha_j}{\alpha_k - \alpha_j}$$

This is called Lagrange's interpolation polynomial.

③ Problem 6, Ahlfors, p. 33

If  $R(z)$  is a rational function of order  $n$ , how large and how small can the order of  $R'(z)$  be?

Proof If  $n=0$  then  $R(z)$  has no poles nor zeros. Thus  $R(z)$  is just a constant and  $R'(z)$  is identically 0. ~~The~~ The order of  $R'(z)$  is not defined.

Now we consider the case  $n \geq 1$ . We will divide rational functions of order  $n$  into 2 categories and consider each category in detail.

Category 1:  $R(z)$  has no <sup>complex</sup> pole.

In this case,  $R(z)$  is just a polynomial and  $n$  is the degree of this polynomial.

Thus  $R'(z)$  <sup>always</sup> ~~also~~ has order  $n-1$ . Note that if  $n=1$ ,  $R'(z)$  is identical to a nonzero constant.

Category 2:  $R(z)$  has at least one complex pole.

Then we can write  $R(z)$  in the reduced form  $R(z) = \frac{P(z)}{Q(z)}$ , where

$Q$  is a polynomial of degree greater than or equal to 1,  $Q$  has the highest coefficient 1 (since it can be absorbed in  $P(z)$ ), and  $P(z)$  and  $Q(z)$  has no common zeros. By the fundamental theorem of algebra, we can write

$$Q(z) = (z - \alpha_1)^{h_1} \dots (z - \alpha_m)^{h_m}$$

where  $m \geq 1$ , each  $h_i \geq 1$  and  $\alpha_i$ 's are distinct. Note that  $P(\alpha_i) \neq 0$  for every  $i = 1, \dots, m$ . We have

$$\begin{aligned} Q'(z) &= \left[ (z - \alpha_1)^{h_1} \right]' (z - \alpha_2)^{h_2} \dots (z - \alpha_m)^{h_m} + \dots + (z - \alpha_1)^{h_1} \dots (z - \alpha_{m-1})^{h_{m-1}} \left[ (z - \alpha_m)^{h_m} \right]' \\ &= h_1 (z - \alpha_1)^{h_1 - 1} (z - \alpha_2)^{h_2} \dots (z - \alpha_m)^{h_m} + \dots + h_m (z - \alpha_1)^{h_1} \dots (z - \alpha_{m-1})^{h_{m-1}} (z - \alpha_m)^{h_m - 1} \\ &= (z - \alpha_1)^{h_1 - 1} \dots (z - \alpha_m)^{h_m - 1} \left[ h_1 (z - \alpha_2) \dots (z - \alpha_m) + \dots + h_m (z - \alpha_1) \dots (z - \alpha_{m-1}) \right] \end{aligned}$$

We have

$$\begin{aligned} R'(z) &= \left( \frac{P(z)}{Q(z)} \right)' = \frac{P'(z)Q(z) - P(z)Q'(z)}{Q(z)^2} \\ &= \frac{(z - \alpha_1)^{h_1 - 1} \dots (z - \alpha_m)^{h_m - 1} \left\{ (z - \alpha_1) \dots (z - \alpha_m) P'(z) - P(z) \left[ h_1 (z - \alpha_2) \dots (z - \alpha_m) + \dots + h_m (z - \alpha_1) \dots (z - \alpha_{m-1}) \right] \right\}}{(z - \alpha_1)^{2h_1} \dots (z - \alpha_m)^{2h_m}} \end{aligned}$$

8

$$= \frac{(z-\alpha_1)\dots(z-\alpha_m)P'(z) - P(z)[h_1(z-\alpha_2)\dots(z-\alpha_m) + \dots + h_m(z-\alpha_1)\dots(z-\alpha_{m-1})]}{(z-\alpha_1)^{h_1+1}\dots(z-\alpha_m)^{h_m+1}} \quad (7)$$

We put  $k$  to be the degree of  $P(z)$  and  $C$  to be its highest coefficient. Also we put  $s = \sum_{i=1}^m h_i$ . Then  $s$  is the degree of  $Q(z)$ .

By definition, the order of  $R(z)$  is  $n = \max\{k, s\}$ . Now we will show that the expression (7) of  $R'(z)$  is the reduced form. To see that, we check whether the numerator vanishes as  $z = \alpha_1$ . The numerator of (\*) at  $z = \alpha_1$  is  $-P(\alpha_1)h_1(\alpha_1 - \alpha_2)\dots(\alpha_1 - \alpha_m) \neq 0$ . Thus similarly, the numerator of (\*) at any  $z = \alpha_k$  is nonzero. Thus (7) is the reduced form of  $R'(z)$ . The (formal) highest order of the numerator is  $m+k-1$  whose ~~coeff~~ associating coefficient is  $k - h_1 - \dots - h_m = k - s$ . We name the ~~den~~ numerator of (\*) by  $A(z)$  and the denominator  $B(z)$ . Then

$$A(z) = (k-s)z^{k+m-1} + \underbrace{\text{remainder}}_{\text{of order } \leq (k+m-1)}$$

$$B(z) = (z-\alpha_1)^{h_1+1}\dots(z-\alpha_m)^{h_m+1} \text{ is of order } \sum_{i=1}^m (h_i+1) = s+m$$

Thus the order of  $R'(z)$  is ~~deg R'~~  $\deg R' = \max\{\deg A, \deg B\}$

$$= \max\{\deg A, s+m\} \quad (8)$$

$$\begin{aligned}
&\leq \max\{k+m-1, s+m\} \\
&= \max\{k-1, s\} + m \\
&\leq \max\{k, s\} + m \\
&= n + m \\
&\leq 2n
\end{aligned}$$

Thus the ~~degree~~<sup>order</sup> of  $R'(z)$  is at most  $2n$ . The equality is obtain if and only if

$$\begin{cases}
k-s \neq 0 & (\text{the first sign } \leq) \\
k-1 < s & (\text{the second sign } \leq) \\
m = n & (\text{the third sign } \leq)
\end{cases}$$

which is equivalent to  $k < s$  and  $m = n$ , ~~this happen if and only if~~  $Q(z)$  ~~has~~ which is again equivalent to  $k < s = n = m$ . This happens if and only if  $Q(z)$  has  $n$  distinct roots, and  $P(z)$  ~~has~~ is of the degree less than  $n$ , and  $P(z)$  and  $Q(z)$  have no common factor. For example

$$\begin{aligned}
P(z) &= z^2 \\
Q(z) &= (z-1)(z+2i)(z-i)
\end{aligned}$$

then  $R'(z)$  is of order  $6$ .

Now we look for ~~the~~ a lower bound of order of  $R'(z)$ . If  $k = s$  then  $n = \max\{k, s\} = s$ . Then from (8) we have the order of  $R'(z)$  is

10

$$d = \max \{ \deg A, s+m \} \geq s+m = k+m \geq n+1.$$

If  $k \neq s$ , then  $\deg A = k+m-1$ . Then (8) gives

$$\begin{aligned} d &= \max \{ k+m-1, s+m \} \stackrel{\text{II}}{=} \max \{ k, s+1 \} + (m-1) \\ &\geq \max \{ k, s \} + (m-1) \\ &\geq n \end{aligned}$$

Thus the degree of  $R'(z)$  is at least  $n$ . The equality holds if and only

if

$$\begin{cases} k \neq s \\ s+1 \leq k & (\text{the first sign } \geq) \\ m = 1 & (\text{the second sign } \geq) \end{cases}$$

or equivalently  $s < k = n$  and  $m = 1$ ; This is the case that

$Q(z) = (z-a)^s$  with  $s < n$ , and  $P(z)$  is a polynomial of order  $n$  and

does not vanish at  $z = a$ . For example,

$$P(z) = z^7 + 1$$

$$Q(z) = z^6$$

then  $R'(z)$  will be of order 7.

To sum up

Let  $R(z) = \frac{P(z)}{Q(z)}$  be the reduced form of a

rational function  $R(z)$  of order  $n$ . Then

\* If  $n = 0$ , we do not have definition for order of  $R'(z)$ .

\* If  $n \geq 1$ , then  $n \leq \text{ord}(R'(z)) \leq 2n$

•  $\text{ord}(R'(z)) = n$  if and only if

$$\begin{cases} Q(z) = (z-a)^s \quad \text{where } s < n \\ P(z) \text{ is a polynomial of degree } n, \text{ and does not vanish} \\ \text{at } z = a \end{cases}$$

•  $\text{ord}(R'(z)) = 2n$  if and only if

$$\begin{cases} Q(z) \text{ has } n \text{ distinct roots} \\ P(z) \text{ is of order } < n \text{ and has no common } \cancel{\text{root}} \text{ with } Q(z) \end{cases}$$

(4) Problem 3, Ahlfors, p. 37.

Show that the sum of an absolutely convergent series does not change if the terms are rearranged.

Proof Let  $(a_n)$  be a complex sequence of complex numbers that is absolutely convergent. Let  $\sigma: \mathbb{N} \rightarrow \mathbb{N}$  be a permutation of  $\mathbb{N}$ , i.e. a bijection. We

will show that  $\sum_{n=1}^{\infty} a_{\sigma(n)}$  exists and equals  $\sum_{n=1}^{\infty} a_n$ .

For short, we can denote  $b_n = a_{\sigma(n)}$  for every  $n \in \mathbb{N}$ . We need to show that

$\sum_{n=1}^{\infty} b_n$  exists and  $\overset{\text{equal}}{\sum_{n=1}^{\infty} a_n}$ . First, the absolute convergence of  $(a_n)$

means that for each  $\varepsilon > 0$ , there exists  $N(\varepsilon) \in \mathbb{N}$  such that

12

$$\sum_{k=n}^m |a_k| < \varepsilon \quad \text{for every } m \geq n \geq N(\varepsilon).$$

Now we wish to prove that the series of  $(b_n)$  is also absolutely convergent.

For each  $\varepsilon' > 0$ , we need to find  $M(\varepsilon') \in \mathbb{N}$  such that

$$\sum_{j=l}^s |b_j| < \varepsilon' \quad \text{for every } s \geq l \geq M(\varepsilon')$$

We have 
$$\sum_{j=l}^s |b_j| = \sum_{j=l}^s |a_{\sigma(j)}|$$

Because  $\sigma$  is a bijection, for each  $k \in \{1, 2, \dots, N(\varepsilon')\}$ , there exists  $n_k \in \mathbb{N}$  such that  $k = \sigma(n_k)$ . We put  $M(\varepsilon') = \max\{n_1, n_2, \dots, n_{N(\varepsilon')}\} + 1$  and we see that

if  $j \geq M(\varepsilon')$  then  $j \notin \{n_1, n_2, \dots, n_{N(\varepsilon')}\}$  and thus  $\sigma(j) \notin \{\sigma(n_1), \dots, \sigma(n_{N(\varepsilon')})\}$ ,

and thus  $\sigma(j) \notin \{1, 2, \dots, N(\varepsilon')\}$ , and thus  $\sigma(j) \geq N(\varepsilon')$ . Therefore, for

every  $l \leq s$   ~~$s \geq$~~   $l \geq M(\varepsilon')$ , the sum  $\sum_{j=l}^s |a_{\sigma(j)}|$  is a ~~finite~~ sum

of ~~only~~ finitely many entries of equal  $|a_k|$ 's whose indices are ~~not~~ greater

than  $N(\varepsilon)$ . Thus 
$$\sum_{j=l}^s |a_{\sigma(j)}| < \varepsilon'$$

Now we have obtained the absolute convergence of  $\sum_{n=1}^{\infty} b_n$  (note that we can

start from  $|\sum_{k=n}^m a_k| < \varepsilon$  and also obtain the convergence of  $\sum_{n=1}^{\infty} b_n$  without using the

absolute convergence of  $\sum_{n=1}^{\infty} a_n$  ) , we'll show that 
$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$$



13

For each  $n \in \mathbb{N}$ , we denote the partial sums by

$$s_n = a_1 + \dots + a_n$$

$$s'_n = b_1 + \dots + b_n$$

Therefore we get  $\lim_{n \rightarrow \infty} s_n = L = \sum_{k=1}^{\infty} a_k$ . We want to show that  $\lim_{n \rightarrow \infty} s'_n = L$ . By the previous part, we have the result: for each  $\varepsilon > 0$ , there exist  $N(\varepsilon), M(\varepsilon) \in \mathbb{N}$  such that  $\sum_{k=n}^m |a_k| < \varepsilon$  for every  $n > n > N(\varepsilon)$ , and  $\sum_{k=1}^j |a_k| > N(\varepsilon)$  for every  $j \geq M(\varepsilon)$ .

For each  $n > \max\{N(\varepsilon), M(\varepsilon)\}$  we have

$$s'_n - s_n = (b_1 + \dots + b_n) - (a_1 + \dots + a_n)$$

$$= (a_{\sigma(1)} + \dots + a_{\sigma(n)}) - (a_1 + \dots + a_n)$$

The first  $N(\varepsilon)$  terms of the sum  $a_1 + \dots + a_n$  are canceled in the subtraction.

Thus the only terms left are  $a_k$ 's or  $-a_k$ 's whose indices are greater than  $N(\varepsilon)$ . Thus

$$|s'_n - s_n| = \left| \sum_{\substack{k > N(\varepsilon) \\ \text{finite} \\ \delta \in \{-1, 1\}}} \delta a_k \right| \leq 2 \sum_{k=N(\varepsilon)}^{N(\varepsilon)+p} |a_k| < 2\varepsilon$$

where  $p$  is the number of terms left

Thus  $(s'_n)$  and  $(s_n)$  must have the same limit, which completes the proof.

14

(5) Problem 5, Ahlfors p. 37.

Discuss the uniform convergence of the series  $\sum_{n=1}^{\infty} \frac{x}{n(1+n^2x^2)}$

for real values of  $x$ .

~~Proof~~ If  $x=0$ , ~~the~~ each term in the series is 0, and the series is absolutely convergent.

~~If  $x \neq 0$ , we put  $a_n = \frac{x}{n(1+n^2x^2)}$~~

For each  $n \in \mathbb{N}$ , we define the function  $f_n(x) = \frac{x}{n(1+n^2x^2)}$  for every  $x \in \mathbb{R}$ .

Then we have  $|f_n(x)| = \frac{2\sqrt{n}x}{2n^{3/2}(1+n^2x^2)} \leq \frac{1+n^2x^2}{2n^{3/2}(1+n^2x^2)} = \frac{1}{2n^{3/2}}$

We know that the sequence  $\sum_{n=1}^{\infty} \frac{1}{2n^{3/2}}$  is convergent. Thus the sequence

$\sum_{n=1}^{\infty} f_n(x)$  is absolutely convergent on  $\mathbb{R}$ .

(6) Problem 2, Ahlfors p. 41

Expand  $\frac{2z+3}{z+1}$  in powers of  $z-1$ . What is the radius of convergence?

Proof We have  $\frac{2z+3}{z+1} = 2 + \frac{1}{z+1} = 2 + \frac{1}{2+(z-1)} = 2 + \frac{1}{2} \frac{1}{1 - \frac{z-1}{2}}$

Put  $u = \frac{1-z}{2}$ , we have  $\frac{1}{1-u} = 1+u+u^2+\dots$

$$= \sum_{n=0}^{\infty} u^n = \sum_{n=0}^{\infty} \left(\frac{1-z}{2}\right)^n$$

Thus,  $\frac{2z+3}{z+1} = 2 + \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1-z}{2}\right)^n = 2 + \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (z-1)^n$

$$= 2 + \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{2} \left(-\frac{1}{2}\right)^n (z-1)^n$$

Therefore,

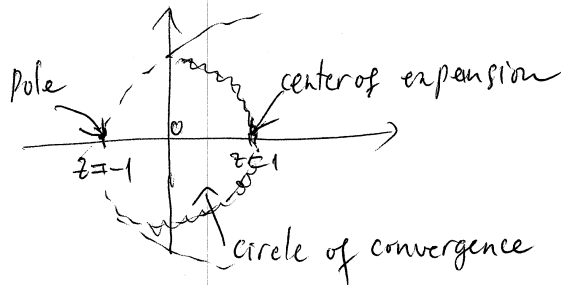
$$\frac{2z+3}{z+1} = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{1}{2} \left(-\frac{1}{2}\right)^n (z-1)^n$$

This is a power series with general term  $a_n = \frac{1}{2} \left(-\frac{1}{2}\right)^n \quad \forall n \geq 1$ .

Then  $\sqrt[n]{|a_n|} = \sqrt[n]{\frac{1}{2^{n+1}}} = \frac{1}{\sqrt{2}} \frac{1}{2} \rightarrow \frac{1}{2}$  as  $n \rightarrow \infty$ . Thus

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \frac{1}{2}$$

and the radius of convergence is  $R = \frac{1}{\limsup \sqrt[n]{|a_n|}} = 2$ .



16

(7) Problem 4, Ahlfors, p. 41

5/5 If  $\sum a_n z^n$  has radius of convergence  $R$ , what is the radius of convergence of  $\sum a_n z^{2n}$ ? of  $\sum a_n^2 z^n$ ?

Proof

Let  $R$  be the radius of convergence of  $\sum a_n z^n$ .

$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|}$$

The radius of convergence of  $\sum a_n z^{2n}$  is  $R'$  where

$$\begin{aligned} \frac{1}{R'} &= \limsup \sqrt[2n]{|a_n|} = \limsup \left( \sqrt[n]{|a_n|} \right)^{1/2} \\ &= \left( \limsup \sqrt[n]{|a_n|} \right)^{1/2} \quad (\text{by the continuity of root function}) \\ &= \left( \frac{1}{R} \right)^{1/2} = \frac{1}{\sqrt{R}} \quad \checkmark \end{aligned}$$

Thus  $R' = \sqrt{R}$ .

The radius of convergence of  $\sum a_n^2 z^n$  is  $R''$  where

$$\begin{aligned} \frac{1}{R''} &= \limsup \sqrt[n]{|a_n|^2} = \limsup \left( \sqrt[n]{|a_n|} \right)^2 = \left( \limsup \sqrt[n]{|a_n|} \right)^2 \\ &= \left( \frac{1}{R} \right)^2 = \frac{1}{R^2} \quad (\text{by the continuity of } x \mapsto x^2) \\ &= \frac{1}{R^2} \quad \checkmark \end{aligned}$$

Thus  $R'' = R^2$ .

⑧ Problem 8, Ahlfors, p. 41

For what values of  $z$  is  $\sum_0^{\infty} \left(\frac{z}{1+z}\right)^n$  convergent?

Proof This geometric series is convergent if and only if

$$\left|\frac{z}{1+z}\right| < 1$$

which is ~~equivalent~~ equivalent to  $|z| < |1+z|$ . We have

$$|z| < |1+z|$$

$$\Leftrightarrow |z|^2 < |1+z|^2$$

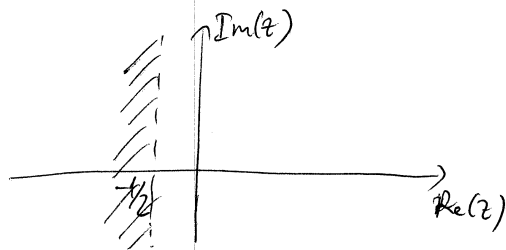
$$\Leftrightarrow z\bar{z} < (1+z)(1+\bar{z})$$

$$\Leftrightarrow z\bar{z} < 1+z+\bar{z}+z\bar{z}$$

$$\Leftrightarrow 0 < 1+2\operatorname{Re}(z)$$

$$\Leftrightarrow \operatorname{Re}(z) > -\frac{1}{2}$$

Thus the series is convergent if and only if  $\operatorname{Re}(z) > -\frac{1}{2}$



⑨ Problem 9, Ahlfors, p. 41

For what values of  $z$  is  $\sum_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n \frac{z^n}{1+z^{2n}}$  convergent?

18

Proof we consider 4 cases for  $z$ .

Case 1:  $|z| > 1$

In this case, we will show that the series is absolutely convergent. Indeed,

$$\left| \frac{z^n}{1+z^{2n}} \right| = \frac{|z|^n}{|1+z^{2n}|} \leq \frac{|z|^n}{|z|^{2n}-1} < \frac{|z|^n-1}{(|z|^n-1)(|z|^n+1)} \leq \frac{1}{|z|^n+1} \leq \frac{z}{|z|^n}$$

Since  $|z| > 1$ ,  $\frac{1}{|z|} < 1$  and the series  $\sum_{n=0}^{\infty} \frac{z}{|z|^n}$  converges. Thus the

series  $\sum_{n=0}^{\infty} \frac{z^n}{1+z^{2n}}$  absolutely converges by comparison test.

Case 2:  $|z| < 1$

Put  $w = \frac{1}{z}$ . Then  $|w| > 1$  and  $\frac{z^n}{1+z^{2n}} = \frac{\left(\frac{1}{w}\right)^n}{1+\left(\frac{1}{w}\right)^{2n}} = \frac{w^n}{1+w^{2n}}$

Then we return to case 1, and conclude that the series converges absolutely.

Case 3:  $z = 0$

The series is obviously convergent

Case 4:  $|z| = 1$

We can write  $z = \cos \theta + i \sin \theta$  for some  $\theta \in \mathbb{R}$ . Then

$$z^n = \cos n\theta + i \sin n\theta \quad (\text{the binomial formula})$$

and  $1 + z^{2n} = 1 + (\cos 2n\theta + i \sin 2n\theta)$

$$= (1 + \cos 2n\theta) + i \sin 2n\theta$$

$$= 2 \cos^2 n\theta + 2i \cos n\theta \sin n\theta$$

$$= 2 \cos n\theta (\cos n\theta + i \sin n\theta)$$

$$= 2(\cos n\theta) z^n$$

20

Using a calculator, we get  $\tan(1+ci) \approx 1.0224 - 0.4511i$

(11) Problem 5, Ahlfors, p. 47

Find the real and imaginary part of  $\exp(e^z)$

Proof Let  $z = x+iy$ . Then by the additivity of  $e^z$ , we have

$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y) = (e^x \cos y) + i(e^x \sin y)$$

Put  $\alpha = e^x \cos y \in \mathbb{R}$  and  $\beta = e^x \sin y \in \mathbb{R}$ . We have  $e^z = \alpha + i\beta$  and

$$\text{thus } \exp(e^z) = e^{\alpha + i\beta} = (e^\alpha \cos \beta) + i(e^\alpha \sin \beta)$$

Therefore,

$$\operatorname{Re}(\exp(e^z)) = e^\alpha \cos \beta = \exp(e^x \cos y) \cos(e^x \sin y)$$

$$\operatorname{Im}(\exp(e^z)) = e^\alpha \sin \beta = \exp(e^x \cos y) \sin(e^x \sin y)$$

(12) Problem 6, Ahlfors, p. 47

4/5 Determine the value of  $2^i, i^i, (-1)^{2i}$

\* Determine  $2^i$

$$\begin{aligned} \text{By definition, } 2^i &= \exp(i \log 2) = \cos(\log 2) + i \sin(\log 2) \\ &\approx 0.9550 + 0.2965i \end{aligned}$$

\* Determine  $i^i$

By definition,  $i^i = \exp(i \log i)$ . We have

$$\begin{aligned} \log i &= \log|i| + i \arg(i) = \log 1 + i\left(\frac{\pi}{2} + 2k\pi\right) \\ &= i\left(\frac{\pi}{2} + 2k\pi\right) \text{ where } k \in \mathbb{Z} \end{aligned}$$

20

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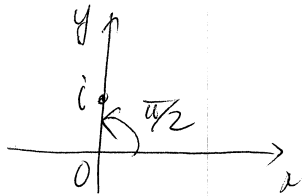
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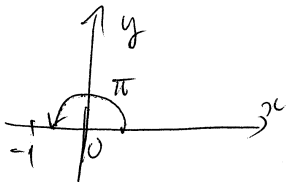
$$\text{Thus } i^i = \exp\left(i \cdot i \left(\frac{\pi}{2} + 2k\pi\right)\right) = \exp\left(-\frac{\pi}{2} - 2k\pi\right).$$

$$\text{Thus } i^i = \exp\left(-\frac{\pi}{2} + 2l\pi\right) \text{ where } l \in \mathbb{Z} \quad \checkmark$$

Determine  $(-1)^{2i}$

By definition,  $(-1)^{2i} = \exp(2i \log(-1))$ . We have

$$\log(-1) = \log|-1| + i \arg(-1) = 0 + i(\pi + 2k\pi) = \pi(2k+1)i \quad \text{where } k \in \mathbb{Z}. \quad \textcircled{-1}$$



$$\begin{aligned} \text{Thus } (-1)^{2i} &= \exp(2i\pi(2k+1)i) = \exp(-2\pi(2k+1)) \\ &= \cos(2\pi(2k+1)) + i \sin(2\pi(2k+1)) \\ &= \cos 2\pi + i \sin 2\pi \\ &= \mathbf{X} \end{aligned}$$

Therefore  $(-1)^{2i} = 1$ .

Completion: 18/18

