Name: Tuan Pham	Math 8701: Complex Analysis
ID: 4652218	Problem Set 3

25.5/27

① Problem 1, Ahlfors p. 53

If S is a metric space with distance function $d(x,y)$, show that S with the distance functiong $\delta(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$ is also a metric space. The latter space is bounded in the sense that all distances lie under a fixed bound.

Proof: First we check the non-negativity:

$$\delta(x,y) = \frac{d(x,y)}{1+d(x,y)} \geq 0 \quad \text{since } d(x,y) \geq 0$$

5/5

Second, we check the identity of ~~non di~~ indiscernibles:

$$\delta(x,y) = 0 \Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y \ \checkmark$$

Third, we check the symmetry

$$\delta(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = \delta(y,x) \ \checkmark$$

Last, we check triangle inequality:

$$\delta(x,y) + \delta(y,z) = \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} = 2 - \left(\frac{1}{1+d(x,y)} + \frac{1}{1+d(y,z)}\right)$$

$$\delta(x,z) = \frac{d(x,z)}{1+d(x,z)} = 1 - \frac{\cancel{d(x,z)}\, 1}{1+d(x,z)}$$

2

Thus, we need to show that

$$2-\left(\frac{1}{1+d(x,y)}+\frac{1}{1+d(y,z)}\right)\geq 1-\frac{1}{1+d(x,z)}$$

or
$$1+\frac{1}{1+d(x,z)}\geq\frac{1}{1+d(x,y)}+\frac{1}{1+d(y,z)}$$

Put $c=d(x,z)$, $b=d(x,y)$, $a=d(y,z)$. We have $a+b\geq c\geq 0$. We'll show that
$$1+\frac{1}{1+c}\geq\frac{1}{1+a}+\frac{1}{1+b}$$

We have
$$\text{RHS}=\frac{2+(a+b)}{1+(a+b)+ab}\leq\frac{2+(a+b)}{1+(a+b)}=1+\frac{1}{1+(a+b)}$$
$$\leq 1+\frac{1}{1+c}=\text{LHS}.$$

Here is some comment. Whenever we have two metrics on one space, we usually ask: are they equivalent? Recall that the formal definition of equivalence between two metrics d_1 and d_2 on X is

$$\left[\, d_1 \text{ equiv } d_2 \overset{\text{def}}{\iff} \text{there exist } \alpha,\beta>0 \text{ such that } \alpha d_1(x,y)\leq d_2(x,y)\leq\beta d_1(x,y)\ \forall x,y\in X. \right]$$

The motivation underlying this definition is that if the two metrics are equivalent, they will give rise to the same topology in X. Be careful

; we

haven't made any statement about the converse. Now let check if δ and d are equivalent:

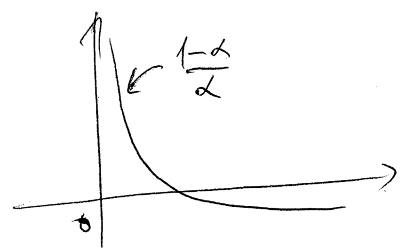
$$\delta(x,y) = \frac{d(x,y)}{1+d(x,y)} \leq d(x,y)$$

Thus we only need to find some $\alpha > 0$ such that $\delta(x,y) \geq \alpha d(x,y)$. This condition means

$$\frac{d(x,y)}{1+d(x,y)} \geq \alpha d(x,y)$$

or equivalently, $\quad d(x,y) \leq \frac{1-\alpha}{\alpha} \quad \forall x,y \in X.$

Since $\lim_{\alpha \to 0^+} \frac{1-\alpha}{\alpha} = \infty$, this inequality holds if and only if d is bounded. Thus, in case that d is bounded, d and δ are equivalent metrics and thus induce the same topology on X.



What if d is unbounded? Of course then δ and d are no longer equivalent metrics. But that doesn't mean that the induced topologies are different. By the inequality $\delta(x,y) \leq d(x,y)$, every open ball $B_d(x,r)$ in (X,d) is contained in the open ball $B_\delta(x,r)$ in (X,δ). Since any open set in (X,δ) is the union of a family of open balls in (X,δ), that set is also an union of a ~~class~~ family of open balls in (X,d). Thus an open set in

(X, δ) is also open in (X, d). Thus $\tau_\delta \subset \tau_d$ where τ_δ and τ_d respectively are the induced topologies on (X, δ) and (X, d). Another way to think of it is that the inequality $\delta(x,y) \leq d(x,y)$ says that the distance between two points x, y from the point of view of d is greater than that from the point of view of δ. That means there are more "things" in between x and y in sense of d than in sense of δ. Thus d should have more open sets than δ, and that's why $\tau_\delta \subset \tau_d$. To prove the converse $\tau_d \subset \tau_\delta$, we'll play the same game: we'll show that each ball $B_d(x, r)$ ~~is~~ contains a ball $B_\delta(x, s)$ for some $s > 0$. What we need to find is $s = s(x, r) > 0$ such that $B_\delta(x, s) \subseteq B_d(x, r)$. Choose ~~$\frac{r}{r+1} < s < 1$. we get~~

$$\cancel{d(x,y) \leq r < \frac{s}{1-s} \quad \forall y \in B_d(x,r)}$$

~~Thus $\frac{1}{1+d(x,y)} > 1-s$, or $1 - \frac{1}{1+d(x,y)} < s$~~

Choose $0 < s < \frac{r}{r+1}$. Then $\frac{s}{1-s} < r$.

For every $y \in B_\delta(x, s)$, we have $\delta(x,y) < s$. Thus

$$\frac{d(x,y)}{1+d(x,y)} < s, \quad \text{or} \quad \frac{1}{1+d(x,y)} > 1-s$$

or equivalently $d(x,y) < \frac{s}{1-s}$. Thus $d(x,y) < r$ and $y \in B_d(x,r)$.

therefore $B_\delta(x,s) \subset B_d(x,r)$, and we conclude that the topologies induced by δ and d are the same!

In fact, it is very difficult to get a different topology by "manipulating" ~~the given~~ original metric. Since we are dealing with inequalities, and continuous functions, we'll get almost always the same topology. Moreover,

$$X \times X \xrightarrow{d} \mathbb{R}^+ \xrightarrow{x \mapsto \frac{x}{x+1}} \mathbb{R}^+$$

topology has nothing to do with inequality. In some sense, the induced topology is insensitive to the continuous distorsion process of the original metric.

(2) Problem 7, Ahlfors p. 53

Show that the accumulation points of any set form a closed set.

Proof Let X be a metric space and A be the set of all accummulation points of its. We'll show that A is closed in X. By the observation that some point has "a lot" of neighbors ~~any~~ and be around neighbors in any scale, people divide point into 2 categories: accummulation points and isolated points. Isolated points are those who have "private space", that has its own open set (doesn't share with anyone else. By this notion, an isolated point is already an ~~the~~ interior point of X, and the open set containing it is the singleton of itself. Thus, the set of all isolated points ~~are~~ is just the union of open sets in X and thus open in X. The set of accummulation points is by definition the complement of the set of isolated points in X. Thus the set of accumulation points is closed in X.

③ Prove that the closure of a connected set is connected.

Proof Let X be a metric space and A be a connected set. We'll prove that $\bar{A}$, the closure of A in X, is also a connected set. What's interesting here is that if a connected space A is known to be in a larger metric space, or topological space in general, then it could give rise to another connected ~~space~~ set in that total space. In other words, a connected space A is able to enlarge itself with the help of a medium X. Thus, in some sense one could extend a connected space infinitely...

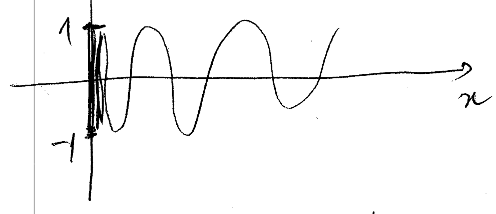
Now return to the proof. Suppose by contradiction that $\bar{A}$ is not connected. Then $\bar{A}$ can be separated into two pieces $\bar{A} = O_1 \cup O_2$ where O_1 and O_2 are disjoint, open and nonempty. ~~So~~ Thus,

$$A = \bar{A} \cap \bar{A} = A \cap (O_1 \cup O_2) = \underbrace{(A \cap O_1)}_{U_1} \cup \underbrace{(A \cap O_2)}_{U_2}$$

Since the topology on A can be viewed as the relative topology on $\bar{A}$, U_1 and U_2 are open in A. They are also disjoint. ~~Thus~~ Since A is connected, one of them must be empty. WLOG, we can assume that $A \cap O_1 = \phi$, which mean $A \subset \bar{A} \setminus O_1 = O_2$. Since O_1 and O_2 are open in $\bar{A}$, both of them are ~~open~~ closed in $\bar{A}$. Thus O_2 is ~~an open~~ a closed subset of $\bar{A}$ that contains A. By the maximality of $\bar{A}$, we must have $O_2 = \bar{A}$, which follows that $O_1 = \phi$. this is a contradiction.

(4) Problem 4, Ahlfors p. 58

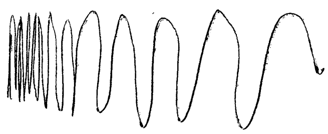
Let A be the set of points $(x,y) \in \mathbb{R}^2$ with $x = 0$, $|y| \le 1$, and let B be the set with $x > 0$, $y = \sin 1/x$. Is $A \cup B$ connected?



$$A = \{(x,y) : x = 0, |y| \le 1\}$$

$$B = \{(x,y) : x > 0, y = \sin \tfrac{1}{x}\}$$

Proof The set $A \cup B$ is called the Topologist's Sine Curve. It illustrates vividly that a set which ~~compo~~ consists of two components seemingly to be separated may be still connected. First, since A is just an interval in $\mathbb{R}$, it is connected. We'll show that B is also connected. Suppose by



contradiction that $B = O_1 \cup O_2$ where O_1 and O_2 are open, disjoint and nonempty subsets in B. For each $x>0$, we denote the point $M_x = (x, \sin\frac{1}{x})$. Then

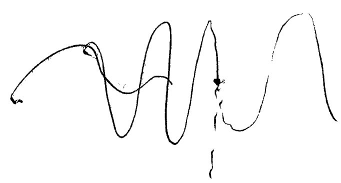
$$B = O_1 \cup O_2 = \{M_x : x>0\}$$

Since $O_1, O_2 \neq \emptyset$, there exist $x_1>0$ and $x_2>0$ such that $M_{x_1} \in O_1$ and $M_{x_2} \in O_2$. WLOG we can assume $x_1 < x_2$. Note that for each $x>0$, M_x belongs to exactly one of O_2 and O_2. We define

$$x_0 = \sup\{x < x_2 : M_x \in O_1\}$$

Then $0 < x_1 \le x_0 \overset{x_2}{\le}$. Thus $M_{x_0} \in O_1$ or $M_{x_0} \in O_2$. If $M_{x_0} \in O_1$,

then there exists an open ball in $\mathbb{R}^2$, say $B(M_{x_0}, \delta)$, which ~~lies completely in~~ such that



x_0

$$B \cap B(M_{x_0}, \delta) \subset O_1.$$

Since $\lim_{x \to x_0} \sin\frac{1}{x} = \sin\frac{1}{x}$, we have $\lim_{x\to x_0} M_x = M_{x_0}$ in $\mathbb{R}^2$. Thus There exists $\delta' > 0$ such that $M_x \in B(M_{x_0}, \delta)$ $\forall |x - x_0| < \delta$. Thus $M_x \in O_1$ $\forall |x-x_0| < \delta'$. Particularly, $M_{x_0 + \delta'/2} \in O_1$. This contradicts the definition of x_0.

Therefore $M_{x_0} \in O_2$. Similarly, there exists $\delta' > 0$ such that $M_x \in O_2 \; \forall |x - x_0| < \delta'$. Thus $M_x \notin O_1 \; \forall x_0 - \frac{\delta'}{2} < x < x_0$. Thus x_0 is no longer the least upper bound of the set $\{x < x_2 : M_x \in O_1\}$, and this is a contradiction. Therefore, both A and B are connected.

Suppose by contradiction that $C = A \cup B$ is not connected. Then $C = U_1 \cup U_2$ where U_1 and U_2 are open, disjoint and nonempty subsets of C. We have

$$A = A \cap C = \underbrace{(A \cap U_1) \cup (A \cap U_2)}_{\text{open, disjoint subsets of } A}.$$

Since A is connected, either $A \cap U_1$ or $A \cap U_2$ is empty. We can assume $A \cap U_2 = \phi$. Thus $A \subset U_1$. Similarly $B \subset U_1$ or $B \subset U_2$. If $B \subset U_1$ then $A \cup B = U_1$ and thus $U_2 = \phi$, which is contradictory. Thus $B \subset U_2$. Since $A \cup B = U_1 \cup U_2$, we have $A = U_1$ and $B = U_2$. In other words, A and B are open and disjoint in C. Since $O = (0,0) \in A$, there exists an open disc $B(0,\delta)$ in $\mathbb{R}^2$ such that $(B(0,\delta) \cap C) \subset A$. Thus $(B(0,\delta) \cap C) \cap B = \phi$. Hence,

$$B(0,\delta) \cap B = \phi$$

For each $n \in \mathbb{N}$ we see that $\left(x_n, \sin\frac{1}{x_n}\right) \in B$ where $x_n = \frac{1}{2n\pi}$.

thus $\left(\frac{1}{2n\pi}, 0\right) \in B \ \forall n \in \mathbb{N}$. If n is suff. large, then

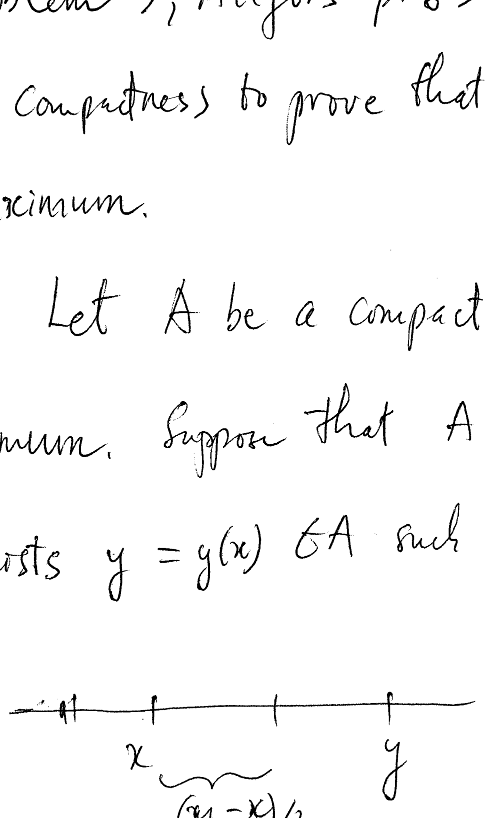
$\left(\frac{1}{2n\pi}, 0\right) \in B \cap B(0,\delta)$. This is a contradiction.

In conclusion, $A \cup B$ is connected.

⑤ Problem 3, Ahlfors p. 63

Use compactness to prove that a closed bounded set of real numbers has a maximum.

Proof Let A be a compact set in $\mathbb{R}$. We are to show that A has a maximum. Suppose that A has no maximum. Then for each $x \in A$, there exists $y = y(x) \in A$ such that $x < y$. Put $B(x) = \left(x - \frac{y-x}{2}, x + \frac{y-x}{2}\right) \cap A$



Then $\bigcup_{x \in A} B(x) = A$. Thus $\{B(x)\}$ is an open covering of A. Thus there exists a finite subcovering since A is compact. Let's call them $B(x_1), B(x_2), \dots B(x_n)$, or just briefly $B_1, B_2, \dots, B_n$. Also let's denote ~~$B(y_1) \dots B(y_n)$ say~~ $y(x_1), \dots, y(x_n)$ by $y_1, \dots, y_n$. We have

$$A = \bigcup_{i=1}^{n} B_i$$

and $x < y_i \quad \forall x \in B_i$. Thus $x < \max\{y_i\} \ \forall x \in \bigcup_{i=1}^{n} B_i$.

Thus $x < \max\{y_i\} = y_j \ \forall x \in A$. Thus $y_j < y_j$, and this is a contradiction.

(6) Problem 4, Ahlfors p. 63

If $E_1 \supset E_2 \supset E_3 \supset \dots$ is a decreasing sequence of nonempty compact sets, then the intersection $\bigcap_{n=1}^{\infty} E_n$ is not empty (Cantor's lemma). Show by example that this need not to be true if the sets are merely closed.

Proof This is actually a special case of a more general statement, which is called "finite intersection property": In a ~~compact~~ topological space, if a family of compact sets has the finite intersection property (the intersection of any finitely many sets is finite) then it will have non-empty intersection. However, we will prove Cantor's lemma without using this result. Suppose by contradicting that $\bigcap_{n=1}^{\infty} E_n = \phi$. Then

$$E_1 = E_1 \setminus \bigcap_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (E_1 \setminus E_n)$$

Since each E_n is compact, it is closed in E_1, and hence $E_1 \setminus E_n$ is open in E_1. Thus $\{E_1 \setminus E_n : n \in \mathbb{N}\}$ is an open covering of E_1. ~~This~~ Since E_1 is compact, it has a finite subcovering. Thus

12

$$E_1 = \bigcup_{k=1}^{m} (E_1 \setminus E_{n_k}) = E_1 \setminus \left(\bigcap_{k=1}^{m} E_{n_k} \right)$$

thus $\bigcap_{k=1}^{m} E_{n_k} = \phi$, ~~which~~. We know that

$$E_{n_1} \supset E_{n_2} \supset \cdots \supset E_{n_m}$$

Thus $\bigcap_{k=1}^{m} E_{n_k} = E_{n_m}$. Thus $E_{n_m} = \phi$, which contradicts the hypothesis.

If we remove the compactness in the hypothesis and replace it by closedness, the conclusion may not be true. For instance, for each $n \in \mathbb{N}$, we put $E_n = \mathbb{R} \setminus (-n, n)$. Then $E_1 \supset E_2 \supset E_3 \supset \cdots$ and each set is closed in the previous ones. However,

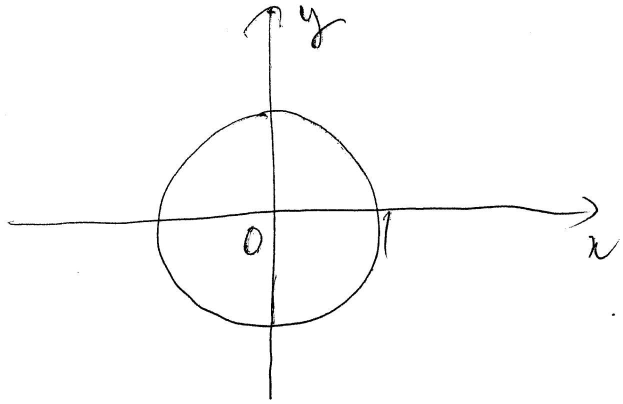
$$\bigcap_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty} \left(\mathbb{R} \setminus (-n, n) \right) = \mathbb{R} \setminus \left(\bigcup_{n=1}^{\infty} (-n, n) \right) = \mathbb{R} \setminus \mathbb{R} = \phi.$$

One of the reason why this is not true when we replace compactness by closedness is that the closedness is not an <u>absolute</u> property. It is a relative property of one space with respect to another (larger) space. thus the choice of E_n's is much more arbitrary and may not satisfy the condition $\bigcap_{n=1}^{\infty} E_n \neq \phi$

(7) Problem 1, Ahlfors p. 66

4.5/5 Construct a topological mapping of the open disk $|z| < 1$ onto the whole plane.

Proof



Let B_2 be the disk $|z|<1$. Consider the map

$$f : B_2 \longrightarrow \mathbb{C}$$
$$z \longmapsto \frac{z}{1-|z|} \checkmark$$

Check f is ~~B~~injective (one-to-one)

$f(z_1)=f(z_2)$ implies $\frac{z_1}{1-|z_1|} = \frac{z_2}{1-|z_2|}$ (*)

Taking the modulus both side gives $\frac{|z_1|}{1-|z_1|} = \frac{|z_2|}{1-|z_2|}$, or $|z_1| = |z_2|$.

Then substituting back to (*) gives $z_1 = z_2$. $\checkmark$

Check f is surjective (onto)

$f(z)=w$ is equivalent to $\frac{z}{1-|z|} = w$. Taking the modulus both side gives

$\frac{|z|}{1-|z|} = |w|$, or $|z| = |w| - |z||w|$, or $|z| = \frac{|w|}{1+|w|}$. Thus

$\frac{z}{1-|z|} = w \iff \frac{z}{1-\frac{|w|}{1+|w|}} = w$, which is equiv. to $\frac{z}{\frac{1}{1+|w|}} = w$, which

which is in the open disk. -.5

is equiv. to $z = \dfrac{w}{1+|w|}$. Thus f is surjective and

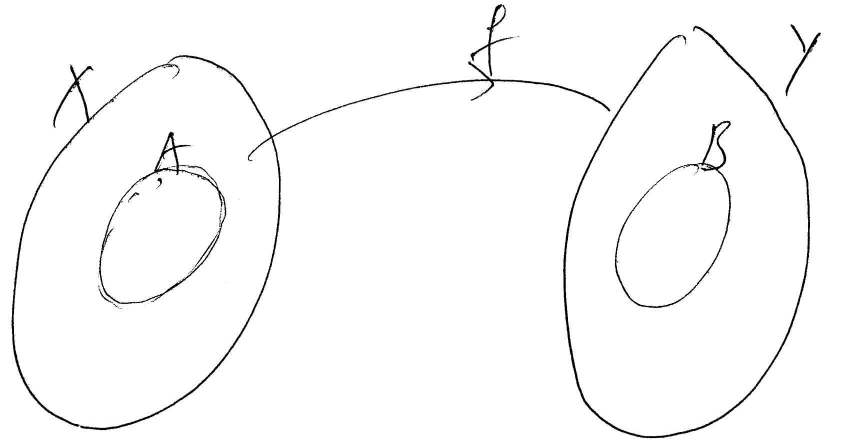
$$f^{-1}(w) = \frac{w}{1+|w|} \quad \forall w \in \mathbb{C}$$ ✓

Check the bi-continuity

Since the denominators of $f(z)$ and $f^{-1}(w)$ are never zero, they are both continuous. Thus f is a topological mapping from B_2 to $\mathbb{C}$. ✓

(8) Prove that every continuous one-to-one mapping of a compact space is topological.

Proof



Let X and Y be metric spaces such that X is compact. Let $f: X \to Y$ be continuous and one-to-one. We'll show that $f: X \to \mathrm{Im} f$ is topological. WLOG, we can assume $\mathrm{Im} f = Y$ to say that f is surjective. Thus f is bijective and continuous from X to Y. All what we need to prove is that f^{-1} is continuous. To do so, we need to prove $(f^{-1})^{-1}$(closed set) = closed set.

In other words, we need to show that $f(A)$ is closed in Y for every closed set A in X. Since A is closed in X and X is compact, A is also compact. Since f is continuous, $f(A) = B$ is compact. Since B is a compact subspace of Y, it is closed in Y.

(9) Problem 1, Ahlfors p. 72

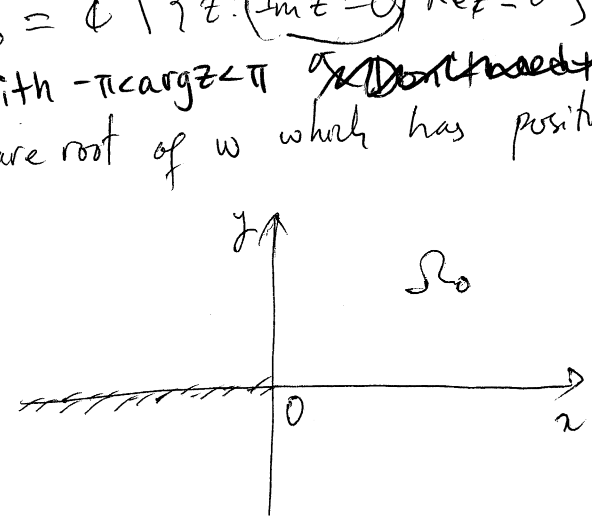
~~4/5~~ Give a precise definition of a single-valued branch of $\sqrt{1+z} + \sqrt{1-z}$ in a suitable region, and prove that it is analytic.

~~3/5~~

4/5

Proof. We know that the function $w \mapsto \sqrt{w}$ is ~~sg~~ single-valued in $\Omega_0 = \mathbb{C} \setminus \{z : \operatorname{Im} z = 0, \operatorname{Re} z \leq 0\}$, in sense ~~of~~ that there is exactly only one square root of w which has positive real part. And the function

(-1) with $-\pi < \arg z < \pi$ or ~~Don't need this~~ sorry



$\Omega_0 : w \mapsto \sqrt{w}$ takes thus unique value. Moreover the function

$$g : \Omega_0 \to \mathbb{C}$$
$$w \mapsto \sqrt{w}$$

is analytic (Ahlfors p. 70). We know that the function (single-valued)

sorry! ↑ ~~this isn't proved in Ahlfors~~ (-1)

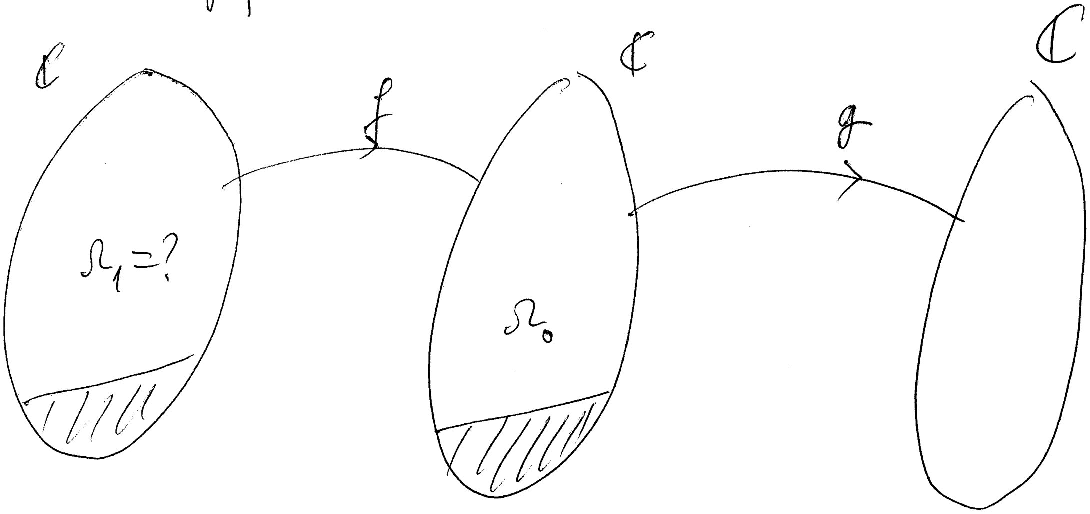
$$f : \mathbb{C} \to \mathbb{C}$$
$$z \mapsto 1+z$$

and

$$h : \mathbb{C} \to \mathbb{C}$$
$$z \mapsto 1-z$$

are both ~~an~~ analytic on $\mathbb{C}$. Thus we have the single-valued function

$z \mapsto g\circ f(z)$ and $z \mapsto g\circ h(z)$



The domain of analyticity of $g\circ f$ is $\Omega_1 = \{z\in\mathbb{C} : f(z)\in\Omega_0\}$, that is

$$\Omega_1 = \{z\in\mathbb{C} : 1+z\in\Omega_0\}$$
$$= \mathbb{C}\setminus\{z : \operatorname{Im}(1+z)=0,\ \operatorname{Re}(1+z)\le 0\}$$
$$= \mathbb{C}\setminus\{z : \operatorname{Im}(z)=0,\ \operatorname{Re}(z)\le -1\}$$

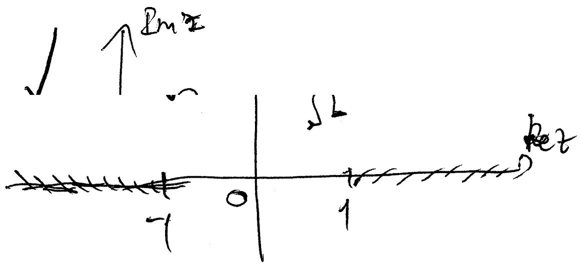
Similarly, the domain of analyticity of ~~hof~~ $g\circ h$ is

$$\Omega_2 = \{z : 1-z\in\Omega_0\}$$
$$= \mathbb{C}\setminus\{z : \operatorname{Im}(1-z)=0,\ \operatorname{Re}(\cancel{1-z})\le 0\}$$
$$= \mathbb{C}\setminus\{z : \operatorname{Im} z=0,\ \operatorname{Re} z\ge 1\}$$

Thus, the domain of anlyticity of $z\mapsto g\circ f(z)+g\circ h(z)=\sqrt{1+z}+\sqrt{1-z}$ is

$$\Omega = \Omega_1\cap\Omega_2 = \mathbb{C}\setminus\{z : \operatorname{Im} z=0;\ \operatorname{Re} z\le -1 \text{ or } \operatorname{Re} z\ge 1\}$$

which



And $\sqrt{1+z}$ is the square root of $1+z$ which has positive real part,

$\sqrt{1-z}$ is the square root of $1-z$ has positive real part.

completion: 12/12