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Math 8701: Complex Analysis

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Problem Set 4

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① Problem 1, Ahlfors, p. 78

Prove that the reflexion  $z \mapsto \bar{z}$  is not a linear transformation.

Proof

Suppose by contradiction that  $z \mapsto \bar{z}$  is a linear transformation. Then there exist  $a, b, c, d \in \mathbb{C}$  such that

$$\bar{z} = \frac{az + b}{cz + d} \quad \forall z \in \mathbb{C}_\infty$$

Of course we can substitute special values of  $z$  to find that there are no  $a, b, c, d$  like that. Here, however, we'll give a sort of "lazy" approach: since every  $z \in \mathbb{R}$  has  $\bar{z} = z$ , such a  $z$  is a fixed point of the linear transformation above. Hence, the LT has infinitely many fixed points. And this happens only if  $b = c = 0$  and  $a = d$ . Then

$$\bar{z} = \frac{az + b}{cz + d} = \frac{az}{d} = z \quad \forall z \in \mathbb{C}_\infty$$

This is a contradiction (since  $\bar{i} = -i \neq i$ ).

② Problem 1, Ahlfors, p. 80

5/5 Find the linear transformation which carries  $0, i, -i$  into  $1, -1, 0$ .

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Proof Let  $a, b, c, d \in \mathbb{C}$  such that  $ad - bc \neq 0$  and such that

$$S(z) = \frac{az + b}{cz + d}$$

maps  $0, i, -i$  into  $1, -1, 0$  respectively. Then we have

$$\begin{cases} S(0) = 1 \\ S(i) = -1 \\ S(-i) = 0 \end{cases} \quad \text{Equivalently,} \quad \begin{cases} \frac{b}{d} = 1 \\ \frac{ai + b}{ci + d} = -1 \\ \frac{-ai + b}{-ci + d} = 0 \end{cases}$$

$$\text{Equivalently,} \quad \begin{cases} b = d \\ ai + b = -ci - b \\ b = ai \end{cases} \quad \text{Equivalently,} \quad \begin{cases} b = d = ai \\ 2ai = -ci - ai \end{cases}$$

$$\text{Equivalently,} \quad \begin{cases} b = d = ai \\ c = -3a \end{cases} \quad \text{Equivalently,} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & ai \\ -3a & ai \end{pmatrix}$$

$$\text{Equivalently,} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & i \\ -3 & i \end{pmatrix}, \text{ or}$$

$$\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \widetilde{\begin{pmatrix} 1 & i \\ -3 & i \end{pmatrix}}$$

where  $\sim$  is the equivalence classes in  $GL(2, \mathbb{C}) / Z(GL(2, \mathbb{C}))$ , where

$Z(GL(2, \mathbb{C}))$  is the center of  $GL(2, \mathbb{C})$ . Thus,

$$S(z) = \frac{z + i}{-3z + i} \cdot \sqrt{\quad} \quad \forall z \in \mathbb{C}_\infty$$

③ Problem 4, Ahlfors, p. 80

Show that any four distinct points can be carried by a linear transformation to position  $1, -1, k, -k$ , where the value  $k$  depends on the points. How many solutions are there, and how they are related?

Proof Suppose that the four given (distinct) points are  $z_1, z_2, z_3, z_4$ . Let

$S: \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$  be a LFT such that

$$S(z) = \frac{az + b}{cz + d}$$

We can choose two  $z_i$ 's to assign  $S(z_i) = 1$ , and  $S(z_j) = -1$ . The number of choices is  $2 \times \binom{4}{2} = 12$ . Now assume that  $S(z_1) = 1$  and  $S(z_2) = -1$ .

$$\frac{az_1 + b}{cz_1 + d} = 1 \text{ implies } b - d = (c - a)z_1$$

$$\frac{az_2 + b}{cz_2 + d} = -1 \text{ implies } b + d = (-c - a)z_2$$

thus 
$$b = \frac{(c-a)z_1 + (-c-a)z_2}{2} = \frac{-z_1 - z_2}{2} a + \frac{z_1 - z_2}{2} c$$

$$d = \frac{(-c-a)z_2 - (c-a)z_1}{2} = \frac{+z_1 - z_2}{2} a + \frac{-z_1 - z_2}{2} c$$

There exists  $k$  such that  $S(z_3) = k$  and  $S(z_4) = -k$  if and only if

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$$S(z_3) = -S(z_4),$$

or equivalently 
$$\frac{az_3 + b}{cz_3 + d} = -\frac{az_4 + b}{cz_4 + d}$$

or equivalently 
$$\frac{\left(z_3 + \frac{z_1 - z_2}{2}\right)a + \frac{z_1 - z_2}{2}c}{\frac{z_1 - z_2}{2}a + \left(z_3 + \frac{z_1 - z_2}{2}\right)c} = -\frac{\left(z_4 + \frac{z_1 - z_2}{2}\right)a + \frac{z_1 - z_2}{2}c}{\frac{z_1 - z_2}{2}a + \left(z_4 + \frac{z_1 - z_2}{2}\right)c}$$

Using  $\alpha_i, \beta_i, \gamma_i, \delta_i$  to denote the blocks, we have

$$\frac{\alpha_1 a + \beta_1 c}{\gamma_1 a + \delta_1 c} = -\frac{\alpha_2 a + \beta_2 c}{\gamma_2 a + \delta_2 c}$$

or equivalently 
$$\alpha_1 \gamma_2 a^2 + \alpha_2 \gamma_1 a^2 + \underbrace{\quad}_{\text{order } \leq 1 \text{ in } a} = 0 \quad (1)$$

the highest leading coefficient is

$$\begin{aligned} \alpha_1 \gamma_2 + \alpha_2 \gamma_1 &= \left(z_3 + \frac{z_1 - z_2}{2}\right) \left(\frac{z_1 - z_2}{2}\right) + \left(z_4 + \frac{z_1 - z_2}{2}\right) \left(\frac{z_1 - z_2}{2}\right) \\ &= (z_3 + z_4 - z_1 - z_2) \frac{z_1 - z_2}{2} \end{aligned}$$

Put  $w = a/c$ , (1) becomes a quadratic equation of  $w$  with the highest

leading coefficient  $\frac{1}{2} (z_3 + z_4 - z_1 - z_2)(z_1 - z_2)$ . If  $z_3 + z_4 - z_1 - z_2 \neq 0$  then

this equation has generally two solutions  $w$ , corresponding to 2 LFT  $S(z)$ .

If  $z_3 + z_4 - z_1 - z_2 = 0$  then this equation has generally one solution  $w$ , corresponding

to one LFT  $S(z)$ . Thus, in the most general case, we will have  $2 \times 12 = 24$  solutions  $S(z)$ . In case that the sum of two of  $z_1, z_2, z_3, z_4$  equals the sum of two others, we will have less solutions.

(4) Problem 2, Ahlfors, p. 83

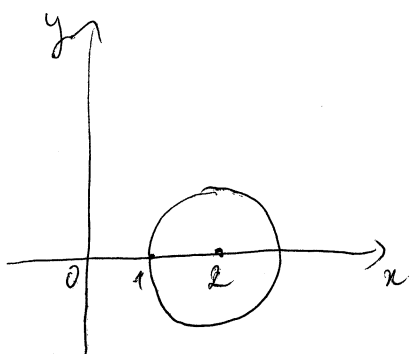
Reflect the imaginary axis, the line  $x=y$ , and the circle  $|z|=1$  in the circle  $|z-2|=1$ .

Proof we know that the reflection of the point  $z$  with respect to the circle

$$|z-a|=R \text{ is } z^* = \frac{R^2}{\bar{z}-\bar{a}} + a \quad (1)$$

In our case,  $a=2$ , and  $R=1$ .

Imaginary axis



From (1) we have

$$z = \frac{R^2}{\bar{z}^* - \bar{a}} + a$$

Now we write  $w$  instead of  $z^*$  for economical reason.

$z$  belongs to the real axis  $\Leftrightarrow z + \bar{z} = 0$ . Then we have a chain of equivalent deduction following each by each

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$$z + \bar{z} = 0 \Leftrightarrow \left( \frac{R^2}{\bar{w} - \bar{a}} + a \right) + \left( \frac{R^2}{w - a} + \bar{a} \right) = 0$$

$$\Leftrightarrow R^2 \left( \frac{w + \bar{w} - a - \bar{a}}{(\bar{w} - \bar{a})(w - a)} \right) + \cancel{2\operatorname{Re}\left(\frac{a + \bar{a}}{w - a}\right)} = 0$$

$$\Leftrightarrow R^2 \frac{w + \bar{w} - a - \bar{a}}{\bar{w}w - \bar{a}w - a\bar{w} + a\bar{a}} + a + \bar{a} = 0$$

$$\Leftrightarrow R^2(w + \bar{w} - a - \bar{a}) + (a + \bar{a})(\bar{w}w - \bar{a}w - a\bar{w} + a\bar{a}) = 0$$

$$\Leftrightarrow \cancel{R^2} (a + \bar{a})w\bar{w} + (R^2 - \bar{a}(a + \bar{a}))w + (R^2 - a(a + \bar{a}))\bar{w} - (a + \bar{a})(R^2 - a\bar{a}) = 0$$

In case  $\operatorname{Re}(a) > 0$ , the following is equivalent to

$$w\bar{w} + \frac{R^2 - \bar{a}(a + \bar{a})}{a + \bar{a}}w + \frac{R^2 - a(a + \bar{a})}{a + \bar{a}}\bar{w} = R^2 - a\bar{a}$$

which is equiv. to

$$\left( w + \frac{R^2 - a(a + \bar{a})}{a + \bar{a}} \right) \left( \bar{w} + \frac{R^2 - \bar{a}(a + \bar{a})}{a + \bar{a}} \right) = \frac{R^2 - a(a + \bar{a})}{a + \bar{a}} \frac{R^2 - \bar{a}(a + \bar{a})}{a + \bar{a}} + R^2 - a\bar{a}$$

which is equiv. to

$$\left| w + \frac{R^2 - a(a + \bar{a})}{a + \bar{a}} \right| = \sqrt{\left( \frac{R^2 - a(a + \bar{a})}{a + \bar{a}} \right)^2 + R^2 - a\bar{a}}$$

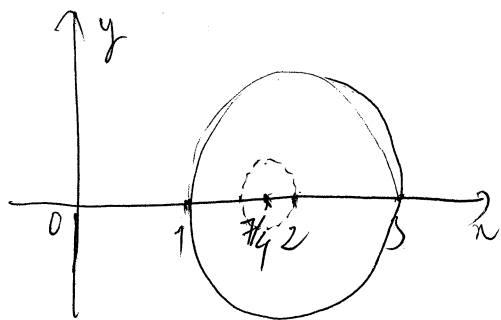
Now substitute  $a=2$ , and  $R=1$ , we get

$$\left| w + \frac{1^2 - 2(2+2)}{2+2} \right| = \sqrt{\left( \frac{1^2 - 2(2+2)}{2+2} \right)^2 + 1^2 - 2^2}$$

which is equiv. to  $\left| w - \frac{7}{4} \right| = \frac{1}{4}$ .

Therefore, the reflection of the imaginary line in  $|z-2|=1$  is

$$\left| z - \frac{7}{4} \right| = \frac{1}{4}$$



The circle  $|z|=1$

We have  $z = \frac{R^2}{\bar{w} - \bar{a}} + a$  ( $w$  stands for  $z^*$ )

And we have the following chain of deduction

$$|z|=1 \Leftrightarrow \left| \frac{R^2}{\bar{w} - \bar{a}} + a \right| = 1$$

$$\Leftrightarrow \left( \frac{R^2}{\bar{w} - \bar{a}} + a \right) \left( \frac{R^2}{w - a} + \bar{a} \right) = 1$$

$$\Leftrightarrow \frac{R^4}{(\bar{w} - \bar{a})(w - a)} + \frac{aR^2}{w - a} + \frac{\bar{a}R^2}{\bar{w} - \bar{a}} + a\bar{a} = 1$$

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$$\Leftrightarrow R^4 + aR^2(\bar{w}-\bar{a}) + \bar{a}R^2(w-a) + (a\bar{a}-1)(\bar{w}-\bar{a})(w-a) = 0$$

Now replacing  $R=1$  and  $a=2$ , we get

$$1 + 2(\bar{w}-2) + 2(w-2) + 3(\bar{w}-2)(w-2) = 0$$

$$\Leftrightarrow 1 + 2\bar{w} + 2w - 8 + 3\bar{w}w - 6w - 6\bar{w} + 12 + 3w\bar{w} = 0$$

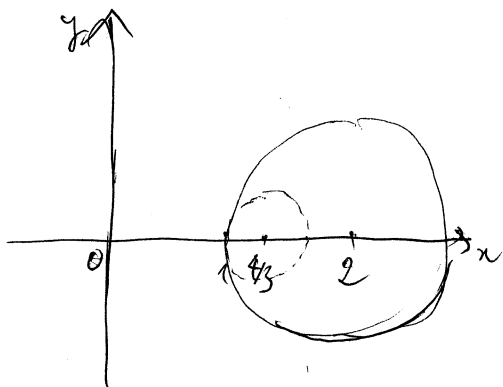
$$\Leftrightarrow -4\bar{w} - 4w + \frac{5}{3} + 6w\bar{w} = 0$$

$$\Leftrightarrow w\bar{w} - \frac{4}{3}w - \frac{4}{3}\bar{w} + \frac{5}{3} = 0$$

$$\Leftrightarrow \left(w - \frac{4}{3}\right)\left(\bar{w} - \frac{4}{3}\right) - \frac{1}{9} = 0$$

$$\Leftrightarrow \left|w - \frac{4}{3}\right| = \frac{1}{3}$$

Therefore, the reflection of the circle  $|z|=1$  is the circle  $\left|z - \frac{4}{3}\right| = \frac{1}{3}$

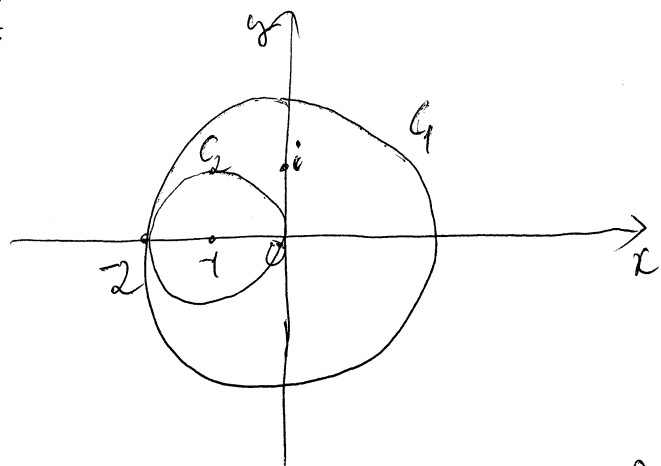


⑤ Problem 4, Ahlfors, p. 83

Find the linear transformation which carries the circle  $|z|=2$  into  $|z+1|=1$ , the point  $-2$  into the origin, and the origin into  $i$ .



Proof



Let  $C_1$  be the circle  $|z|=2$ , and  $C_2$  the circle  $|z+1|=1$ . Let  $S$  be the LFT that maps  $C_1$  to  $C_2$  in which  $-2$  is mapped to  $0$ , and  $0$  is mapped to  $i$ . Since  $\infty$  is symmetric to  $0$  with respect to  $C_1$ , it is mapped to the reflection point of  $i$  with respect to  $C_2$ . We have the formula

$$z^* = \frac{R^2}{\bar{z} - \bar{a}} + a$$

Apply this for  $R=1$ ,  $a=-1$  and  $z=i$ , we get

$$z^* = \frac{1}{-i+1} - 1$$

$$\text{thus } z^* = \frac{i}{-i+1} = \frac{i(1+i)}{2} = \frac{-1+i}{2}$$

thus  $Sz$  must be such that

$$\begin{aligned} -2 &\mapsto 0 \\ 0 &\mapsto i \\ \infty &\mapsto \frac{-1+i}{2} \end{aligned}$$

We write  $Sz = \frac{az+b}{cz+d}$ . The first and the third conditions gives us

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$$S_z = \frac{-1+i}{2} \frac{z+2}{z+d}$$

The condition  $S(0)=i$  gives  $i = \frac{-1+i}{2} \frac{2}{d} = \frac{-1+i}{d}$ . Thus

$$d = \frac{-1+i}{i} = \frac{i^2+i}{i} = 1+i$$

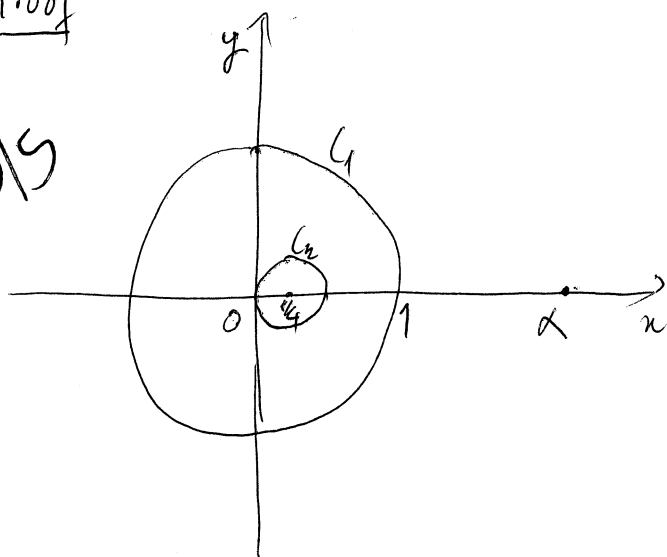
Therefore,  $S_z = \frac{-1+i}{2} \frac{z+2}{z+1+i}$

⑥ Problem 7, Ahlfors, p. 83

Find a linear transformation which carries  $|z|=1$  and  $|z-\frac{1}{4}|=\frac{1}{4}$  to concentric circles. What is the ratio of radii?

Proof

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We will choose the linear transformation to be of the form  $Sz = \frac{1}{z-\alpha}$

where  $\alpha \in \mathbb{R}$  to be chosen such that  $C_1: |z|=1$  and  $C_2: |z-\frac{1}{4}|=\frac{1}{4}$

are mapped into concentric circles.

we will find a general result: we'll find the image of a circle

$C: |z-a|=r$  under  $S$ . Put  $w = \frac{1}{z-\alpha}$ , we have  $z = \frac{1}{w} + \alpha$ .

Then  $|z-a|=r \Leftrightarrow \left| \frac{1}{w} + a - a \right| = r$

Put  $b = a - \alpha$ , we have the following chain of equivalent propositions

$$\left| \frac{1}{w} - b \right| = r \Leftrightarrow |1 - bw| = r|w|$$

$$\Leftrightarrow (1 - bw)(1 - \bar{b}\bar{w}) = r^2 w \bar{w}$$

$$\Leftrightarrow (r^2 - |b|^2) w \bar{w} + b \bar{w} + \bar{b} w - 1 = 0$$

In case that  $r^2 \neq |b|^2$ , the above is equivalent to

$$w \bar{w} + \frac{b}{r^2 - |b|^2} \bar{w} + \frac{\bar{b}}{r^2 - |b|^2} w - \frac{1}{r^2 - |b|^2} = 0$$

$$\text{or} \quad \left( w + \frac{\bar{b}}{r^2 - |b|^2} \right) \left( \bar{w} + \frac{b}{r^2 - |b|^2} \right) = \frac{r^2}{(r^2 - |b|^2)^2}$$

$$\text{or} \quad \left| w + \frac{\bar{b}}{r^2 - |b|^2} \right| = \frac{r}{|r^2 - |b|^2|}$$

Thus  $C: |z-a|=r$  is mapped into a circle centered at  $-\frac{\bar{b}}{r^2 - |b|^2}$  with

radius  $\frac{r}{|r^2 - |b|^2|}$ .

Now we apply this result for  $a = 0$  and  $r = 1$  (then  $b = -\alpha$ ): the circle  $\gamma: |z|=1$

is mapped to a circle centered at  $-\frac{\bar{b}}{r^2 - |b|^2} = -\frac{-\alpha}{1 - \alpha^2} = \frac{\alpha}{1 - \alpha^2}$

Next, we apply the above result for  $a = \frac{1}{4}$  and  $r = \frac{1}{4}$  (then  $b = \frac{1}{4} - \alpha$ ):

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the circle  $G_2: \left| z - \frac{1}{4} \right| = \frac{1}{4}$  is mapped to a circle centered at

$$\begin{aligned} -\frac{\bar{b}}{r^2 - |b|^2} &= -\frac{\frac{1}{4} - \alpha}{\frac{1}{16} - \left(\frac{1}{4} - \alpha\right)^2} = -\frac{4 - 16\alpha}{1 - (1 - 4\alpha)^2} = \frac{4(4\alpha - 1)}{4\alpha(2 - 4\alpha)} \\ &= \frac{4\alpha - 1}{2\alpha(1 - 2\alpha)} \end{aligned}$$

then the circles  $G_1$  and  $G_2$  is mapped into concentric circles of

$$\frac{\alpha}{1 - \alpha^2} = \frac{4\alpha - 1}{2\alpha(1 - 2\alpha)},$$

which is equivalent to  $2\alpha^2(1 - 2\alpha) = (1 - \alpha^2)(4\alpha - 1)$

$$\Leftrightarrow 2\alpha^2 - 4\alpha^3 = 4\alpha - 4\alpha^3 - 1 + \alpha^2$$

$$\Leftrightarrow \alpha^2 = 4\alpha - 1$$

$$\Leftrightarrow \alpha^2 - 4\alpha + 1 = 0$$

$$\Leftrightarrow \alpha = 2 \pm \sqrt{3}$$

thus we can find at least two LFLs  $S_1$  and  $S_2$  that maps  $G_1$  and  $G_2$  into concentric circles:

$$S_1 z = \frac{1}{z - (2 + \sqrt{3})}$$

$$\text{and } S_2 z = \frac{1}{z - (2 - \sqrt{3})}$$

The radius of the image of  $G_1$  is

$$r_1 = \frac{r}{|r^2 - |b|^2|} = \frac{1}{|1 - \alpha^2|}$$

The radius of the image of  $G_2$  is

$$r_2 = \frac{r}{|r^2 - |b|^2|} = \frac{1/4}{\left| \frac{1}{16} - \left(\frac{1}{4} - \alpha\right)^2 \right|}$$

$$\begin{aligned}
 \text{Thus, } \frac{r_1}{r_2} &= \frac{4}{|1-\alpha^2|} \left| \frac{1}{16} - \left(\frac{1}{4} - \alpha\right)^2 \right| \\
 &= \frac{4}{|1-\alpha^2|} \frac{1}{16} |1 - (1-4\alpha)^2| \\
 &= \frac{1}{4} \frac{1}{|1-\alpha^2|} |4\alpha(2-4\alpha)| \\
 &= \left| \frac{2\alpha(1-2\alpha)}{1-\alpha^2} \right| = \alpha \left| \frac{2-4\alpha}{1-\alpha^2} \right| = \alpha \\
 &= 1 \text{ since } \alpha^2 - 4\alpha + 1 = 0
 \end{aligned}$$

Thus  $\frac{r_1}{r_2} = 2 - \sqrt{3}$  for the case of  $S_1 z = \frac{1}{z - (2 - \sqrt{3})}$

$\frac{r_1}{r_2} = 2 + \sqrt{3}$  for the case of  $S_2 z = \frac{1}{z - (2 + \sqrt{3})}$

We see that the radius ratio is not preserved under inversion.

Just a little comment here: why did we think of inversion? because other basic linear transformation (translation, rotation, dilation) map center to center, and thus will never give concentric circles. The inversion can be consider eccentric.

⑦ Problem 1, Ahlfors, p. 84

If  $z_1, z_2, z_3, z_4$  are points on a circle, show that  $z_1, z_3, z_4$  and  $z_2, z_3, z_4$  determine the same orientation if and only if  $(z_1, z_2, z_3, z_4) > 0$ .

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Proof

A point  $z$  not on the circle is said to be on the left of it if and only if  $\text{Im}(z, z_1, z_3, z_4) < 0$ . ~~And~~  $z$  is with respect to  $(z_1, z_3, z_4)$  said to be on the left of the circle w.r.t  $(z_2, z_3, z_4)$  if and only if  $\text{Im}(z, z_2, z_3, z_4) < 0$ . Thus,  $z_1, z_3, z_4$  and  $z_2, z_3, z_4$  determine the same orientation of and only if

$$\frac{\text{Im}(z, z_1, z_3, z_4)}{\text{Im}(z, z_2, z_3, z_4)} > 0 \quad \forall z \text{ not on the circle.}$$

We have

$$\begin{aligned} \frac{\text{Im}(z, z_1, z_3, z_4)}{\text{Im}(z, z_2, z_3, z_4)} &= \left( \frac{z-z_3}{z-z_4}, \frac{z_1-z_3}{z_1-z_4} \right) / \left( \frac{z-z_3}{z-z_4}, \frac{z_2-z_3}{z_2-z_4} \right) \\ &= \frac{z_2-z_3}{z_2-z_4} : \frac{z_1-z_3}{z_1-z_4} \\ &= \frac{1}{(z_1, z_2, z_3, z_4)} \in \mathbb{R} \quad \text{since these four points lie on a circle} \end{aligned}$$

Thus

$$\frac{\text{Im}(z, z_1, z_3, z_4)}{\text{Im}(z, z_2, z_3, z_4)} = \frac{1}{(z_1, z_2, z_3, z_4)}$$

And the condition is equiv. to  $(z_1, z_2, z_3, z_4) > 0$ .

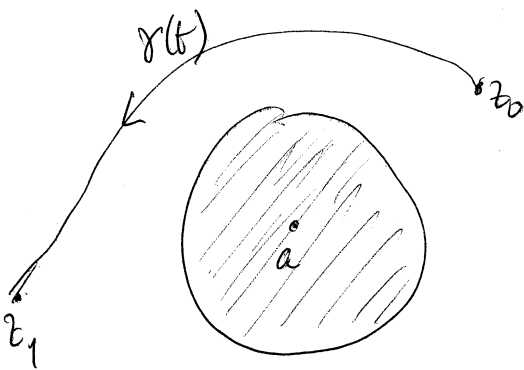
(8) Problem 3, Ahlfors, p. 84

Verify that the inside of the circle  $|z-a|=R$  is formed by all points  $z$  with  $|z-a| < R$ .

Proof Let  $z_2, z_3, z_4$  be on a circle  $C$  such that the orientation of  $C$  determined by this triple agree that  $\infty$  is on the right of  $C$ . That is,  $\text{Im}(\infty, z_2, z_3, z_4) > 0$ . With the topology induced from that of the Riemann sphere, the function  $S_z = (z, z_2, z_3, z_4)$  is continuous on  $\mathbb{C} \cup \infty$ . Thus there

exists  $z_0 \in \mathbb{C}$  such that  $|z_0-a| > R$  and  $S_{z_0} > 0$ . The map  $z \mapsto \text{Im} S_z$  is continuous. Thus  $\lim_{z \rightarrow \infty} \text{Im}(S_z) = \text{Im}(S_\infty) > 0$ . Thus there exists

$z_0 \in \mathbb{C}$  such that  $|z_0-a| > R$  and  $\text{Im}(S_{z_0}) > 0$ .



We'll show that  $\text{Im}(S_z) > 0$  for every

$z \in \mathbb{C}$  such that  $|z-a| > R$ . Suppose

by contradiction that there exists

$z_1 \in \Omega_1 = \{z : |z-a| > R\}$  such that

$\text{Im}(S_{z_1}) \leq 0$ . Since  $\Omega_1$  is path-connected,

there is a continuous map  $\gamma: [0,1] \rightarrow \Omega_1$  such that  $\gamma(0) = z_0, \gamma(1) = z_1$ .

Then  $\text{Im}(S \circ \gamma)$  is a continuous map from  $[0,1]$  to  $\mathbb{R}$  such that  $\phi =$

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$\varphi(0) > 0$  and  $\varphi(1) \leq 0$ . Thus there exists  $t_0 \in [0, 1]$  such that  $\varphi(t_0) = 0$ . Thus there exists  $z'_2 = \gamma(t_0) \in \Omega_1$  such that  $\text{Im} S(z'_2) = 0$ . We know that  $C = \{z \in \mathbb{C}; \text{Im} S(z) = 0\}$ . Thus  $z_2 \in C$ , which is a contradiction. Therefore,  $\text{Im}(S z) > 0 \quad \forall z \in \Omega_2$ . It implies that all  $z \in \mathbb{C}$  such that  $|z - a| > R$  are on the right of  $C$ .

If we can show that there exists  $z_1 \in \mathbb{C}$  such that  $|z_1 - a| < R$  and  $\text{Im}(S z_1) < 0$ , then we'll follow the same approach above (note that  $\Omega_2 = \{z: |z - a| < R\}$  is also path-connected) to conclude that all points of  $\Omega_2$  are on the left of  $C$ . That means we only need to check if the center  $z = a$  is on the left of  $C$ . Since the cross-ratio is LFT-invariant, we have

$$\begin{aligned} \overline{\text{Im}(a, z_2, z_3, z_4)} &= \overline{\text{Im}(\infty, \bar{z}_2, \bar{z}_3, \bar{z}_4)} \xrightarrow{z \mapsto z - \bar{a}} (\infty, \bar{z}_2 - \bar{a}, \bar{z}_3 - \bar{a}, \bar{z}_4 - \bar{a}) \\ &\xrightarrow{z \mapsto \frac{R^2}{z}} \left(0, \frac{R^2}{\bar{z}_2 - \bar{a}}, \frac{R^2}{\bar{z}_3 - \bar{a}}, \frac{R^2}{\bar{z}_4 - \bar{a}}\right) \\ &= (0, z_2 - a, z_3 - a, z_4 - a) \\ &\xrightarrow{z \mapsto z + a} \text{Im}(a, z_2, z_3, z_4) \end{aligned}$$

Thus  $\text{Im}(a, z_2, z_3, z_4) = -\text{Im}(\infty, \bar{z}_2, \bar{z}_3, \bar{z}_4) < 0$ . This concludes the proof.



9) Problem 2, Ahlfors, p. 88

Suppose that the coefficients of the transform

$$S_z = \frac{az+b}{cz+d}$$

are normalized by  $ad-bc=1$ . Show that  $S$  is elliptic if and only if  $-2 < a+d < 2$ , parabolic if  $a+d = \pm 2$ , hyperbolic if  $a+d < -2$  or  $a+d > 2$ .

Proof

The equation of fixed points of  $S$  is

$$z = \frac{az+b}{cz+d}$$

which is equiv to  $cz^2 + (d-a)z - b = 0$ . The discriminant is

$$\Delta = (d-a)^2 + 4cb$$

By using the condition  $ad-bc=1$ , we get  $\Delta = (a+d)^2 - 4$ . Thus if  $a+d = \pm 2$ ,  $S_z$  has only one fixed point. Therefore  $S_z$  is parabolic.

In case  $a+d \neq \pm 2$  then  $S_z$  has at least two distinct fixed points

$\alpha$  and  $\beta$ . The equation of  $S_z$  can be ~~be~~ rewritten as

$$\frac{S_z - \alpha}{S_z - \beta} = k \frac{z - \alpha}{z - \beta} \quad \forall z \in \mathbb{C}_\infty$$

where  $k$  is a constant. Thus,

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$$k = \frac{S_z - \alpha}{S_z - \beta} \frac{z - \beta}{z - \alpha} = \frac{\frac{az+b}{cz+d} - \alpha}{\frac{az+b}{cz+d} - \beta} \frac{z - \beta}{z - \alpha}$$

$$= \frac{(a - c\alpha)z + (b - \alpha d)}{(a - c\beta)z + (b - \beta d)} \frac{z - \beta}{z - \alpha}$$

Since this is true for all  $z \in \mathbb{C}_\infty$ , we can substitute  $z = \infty$  and get

$$k = \frac{a - c\alpha}{a - c\beta}$$

Here by definition of ellipticity and hyperbolicity,

$$S_z \text{ is elliptic} \Leftrightarrow |k| = 1$$

$$S_z \text{ is hyperbolic} \Leftrightarrow k \in \mathbb{R}$$

Put  $u = a - c\alpha$  and  $v = a - c\beta$ . Then

$$u + v = 2a - c(\alpha + \beta) = 2a - c \frac{a+d}{c} = a+d$$

$$uv = (a - c\alpha)(a - c\beta) = a^2 - ac(\alpha + \beta) + c^2\alpha\beta = 1$$

$$k = \frac{u}{v}$$

Thus the condition for ellipticity is

$$S_z \text{ is elliptic} \Leftrightarrow \begin{cases} \text{the system } u+v=a+d, uv=1 \text{ has two distinct solutions} \\ |k| = \left| \frac{u}{v} \right| = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} (a+d)^2 \neq 4 \\ u = \bar{v} \\ |u| = 1 \\ a+d = u + \bar{u} \end{cases} \Leftrightarrow \begin{cases} (a+d)^2 \neq 4 \\ a+d = 2\operatorname{Re}(u) \in \mathbb{R} \\ u = \bar{v} \\ |u| = 1 \end{cases}$$

And this is equiv to  $|a+d| < 2$ , or  $-2 < a+d < 2$ .

The condition for hyperbolicity is

$$S_z \text{ is hyperbolic} \Leftrightarrow \begin{cases} \text{the system } u+v = a+d, uv = 1 \text{ has two distinct solutions} \\ k = \frac{u}{v} \in \mathbb{R} \end{cases}$$

If  $a+d \in \mathbb{R}$  and either  $a+d > 2$  or  $a+d < -2$ , we have two real solutions  $u$  and  $v$ . Then  $k = \frac{u}{v} \in \mathbb{R}$ , whence  $S_z$  is hyperbolic. Note that the converse is not true, for example

$$S_z = \frac{z - \frac{8}{3}}{z + \frac{4}{3}i} \text{ has two fixed points } \alpha = -2i, \beta = \frac{4}{3}i$$

and  $k = -9$ , while  $a+d = \frac{8}{3}i \notin \mathbb{R}$ .

(10) Problem 4, Ahlfors, p. 88

If  $S$  is hyperbolic or loxodromic, show that  $S^n z$  converges to a ~~fixed~~ fixed point as  $n \rightarrow \infty$ , the same for all  $z$ , except when  $z$  coincides the other fixed point. (The limit is the attractor, the

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other the repellent fixed point. What happens when  $n \rightarrow -\infty$ ? What happens in the parabolic case?).

Proof In case that  $S_z$  is hyperbolic or loxodromic, we can write

$$\frac{S_z - \alpha}{S_z - \beta} = k \frac{z - \alpha}{z - \beta} \quad \forall z \in \mathbb{C}_\infty \quad (*)$$

where  $\alpha$  and  $\beta$  are two distinct fixed points of  $S_z$ , and  $|k| \neq 1$ .

\* If  $|k| < 1$ , then we replace  $z$  by  $S_z$  to get

$$\frac{S^2 z - \alpha}{S^2 z - \beta} = k^2 \frac{S z - \alpha}{S z - \beta} = k^2 \frac{z - \alpha}{z - \beta}$$

Similarly, for  $n \in \mathbb{N}$ ,

$$\frac{S^n z - \alpha}{S^n z - \beta} = k^n \frac{z - \alpha}{z - \beta} \quad \forall z \in \mathbb{C}_\infty.$$

If  $z \neq \beta$  then the right hand side goes to zero as  $n \rightarrow \infty$ . Thus

$$\lim_{n \rightarrow \infty} \frac{S^n z - \alpha}{S^n z - \beta} = 0$$

Thus  $\lim_{n \rightarrow \infty} \frac{\beta - \alpha}{\beta - S^n z} = 1$ , and therefore  $\lim_{n \rightarrow \infty} S^n z = \alpha$ .

\* If  $|k| > 1$ , then we'll take the reciprocal

$$\frac{S z - \beta}{S z - \alpha} = \frac{1}{k} \frac{z - \beta}{z - \alpha}$$

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and very since  $|\frac{1}{k}| < 1$ , we return to the previous case:

$$\lim_{n \rightarrow \infty} S^n z = \beta \text{ provided that } z \neq \alpha.$$

In case  $n \rightarrow -\infty$  (this is a very deliberate question!), we should

Substitute  $z$  by  $S^{-1}z$  in (\*) to get

$$\frac{z - \alpha}{z - \beta} = k \frac{S^{-1}z - \alpha}{S^{-1}z - \beta}, \text{ or } \frac{S^{-1}z - \alpha}{S^{-1}z - \beta} = \frac{1}{k} \frac{z - \alpha}{z - \beta}$$

$$\text{Thus } \frac{S^{-n}z - \alpha}{S^{-n}z - \beta} = \frac{1}{k^n} \frac{z - \alpha}{z - \beta}$$

\* If  $|k| > 1$  then  $\lim_{n \rightarrow -\infty} S^{-n}z = \alpha.$

\* If  $|k| < 1$  then  $\lim_{n \rightarrow -\infty} S^{-n}z = \beta.$

That is, the attractive fixed point now becomes repellent, and the repellent becomes attractive.

In parabolic case,  $Sz$  can be written as the form

$$\frac{w}{Sz - a} = \frac{w}{z - a} + c, \text{ where } w, c \in \mathbb{C} \text{ and } a \text{ is the only fixed point of } Sz.$$

Then for  $n \in \mathbb{N}$ , we get

$$\frac{\omega}{S^n z - a} = \frac{\omega}{S^{n-1} z - a} + C = \dots = \frac{\omega}{S z - a} + (n-1)C = \frac{\omega}{z - a} + nC$$

Note that the same is true for  $n < 0$  since we will substitute  $z$  by

$S^{-1}z$  instead of  $Sz$ . Then

$$\lim_{n \rightarrow \infty} \frac{\omega}{S^n z - a} = \infty$$

thus  $\lim_{n \rightarrow \infty} S^n z = a$  provided that  $z \neq a$ . If  $z = a$ , since

$Sa = a$ , it is clear that  $S^n a = a \quad \forall n \in \mathbb{Z}$ . Therefore,

$$\lim_{n \rightarrow \infty} S^n z = a \quad \forall z \in \mathbb{C}_\infty$$

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