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Math 8701: Complex Analysis

Problem set 5

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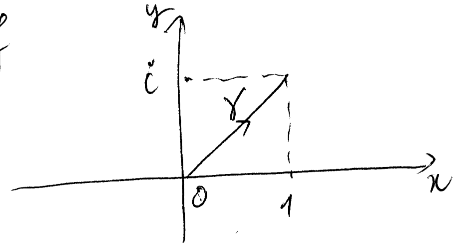
① Problem 1, Ahlfors, p. 108

Compute $\int_\gamma x\,dz$ where $\gamma$ is the directed line segment from 0 to $1+i$.

Proof




$\gamma(x) = x + xi$, $0 \le x \le 1$

Then

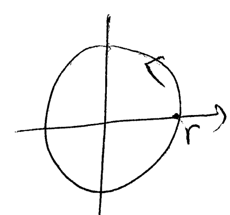
$$\int_\gamma x\,dz \overset{def}{=} \int_0^1 x\,\gamma'(x)\,dx = \int_0^1 x(1+i)\,dx$$

$$= (1+i)\frac{x^2}{2}\Big|_0^1 = \frac{1+i}{2}$$

② Problem 2, Ahlfors, p. 108

Compute $\int_{|z|=r} x\,dz$ for positive sense of circle, in two ways: firstly by use of a parameter and second by observing $x = \frac{1}{2}(z+\bar{z}) = \frac{1}{2}\left(z+\frac{r^2}{z}\right)$ on the circle.

Proof ~~+~~ The first way




$\gamma = z(t) = re^{it}$, $0 \le t \le 2\pi$

$= x(t) + iy(t)$

where $x(t) = r\cos t$, $y(t) = r\sin t$

$$\int_\gamma x\,dz \overset{def}{=} \int_0^{2\pi} r\cos t\; z'(t)\,dt$$

$$= \int_0^{2\pi} r\cos t\; ri e^{it}\,dt$$

2

$$= r^2 i\int_0^{2\pi} \cos t\, e^{it}\,dt = ir^2\int_0^{2\pi} \cos t(\cos t + i\sin t)\,dt$$

$$= ir^2\int_0^{2\pi}\cos^2 t\,dt + i\int_0^{2\pi}\cos t\sin t\,dt$$

$$= ir^2\int_0^{2\pi}\frac{1+\cos 2t}{2}\,dt + \underbrace{i\int_0^0 u\,du}_{0} \quad (u=\sin t)$$

$$= ir^2 \left.\frac{t+\frac{\sin 2t}{2}}{2}\right|_0^{2\pi}$$

$$= ir^2\,\frac{2\pi+0}{2} = i\pi r^2$$

* The second way:

For each $z \in$ the circle, $|z|^2 = r^2$. Then $z\bar{z} = r^2$. Thus $\bar{z} = \frac{r^2}{z}$.

we have $x = \frac{1}{2}(z+\bar{z}) = \frac{1}{2}\left(z+\frac{r^2}{z}\right)$.

$$\int_{|z|=r} x\,dz = \int_{|z|=r}\frac{1}{2}\left(z+\frac{r^2}{z}\right)dz = \underbrace{\int_{|z|=r}\frac{z}{2}\,dz}_{0 \text{ since } \frac{z}{2}} + \frac{r^2}{2}\underbrace{\int_{|z|=r}\frac{dz}{z}}_{2\pi i} = \frac{r^2}{2}2\pi i = i\pi r^2$$

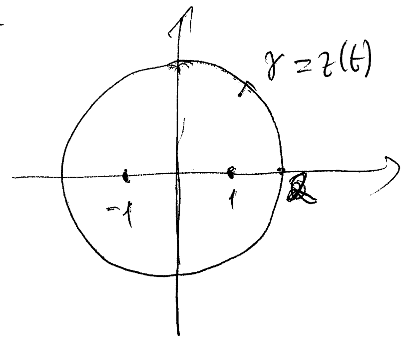
is analytic in the disc
$B(0,2r) \supset \{|z|=r\}$

The two ways gave the same answer $\int_{|z|=r} x\,dz = i\pi r^2$.

③ Problem 3, Ahlfors, p. 108

Compute $\displaystyle\int_{|z|=2} \frac{dz}{z^2-1}$ for the positive sense of the circle.

Proof




One way of parametrizing the circle is

$\gamma(t) = z(t) = 2e^{it}$ for $0 \le t \le 2\pi$.

$$\frac{1}{z^2-1} = \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z+1}$$

Thus $$\int_\gamma \frac{dz}{z^2-1} = \frac{1}{2}\int_\gamma \frac{dz}{z-1} - \frac{1}{2}\int_\gamma \frac{dz}{z+1}$$

$$= \pi i\, n(\gamma,1) - \pi i\, n(\gamma,-1)$$

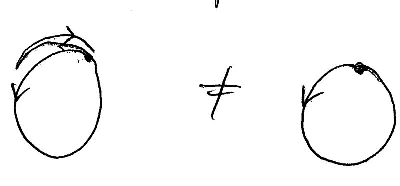
where $n(\gamma, a)$ denotes the winding number of $\gamma$ with respect to $a \notin \gamma$. Since $-1$ and $1$ belong to the same connected domain determined by $\gamma$, they have the same winding number, i.e. $n(\gamma,1) = n(\gamma,-1)$. Thus

$$\int_{|z|=2} \frac{dz}{z^2-1} = 0$$

④ Problem 4, Ahlfors, p. 108

Compute $\displaystyle\int_{|z|=1} |z-1| \cdot |dz|$

Proof Comment: the problem should say what $\gamma$ is, because



Anyway, we'll ~~con~~ assume $\gamma = e^{it}$, $0 \le t \le 2\pi$.

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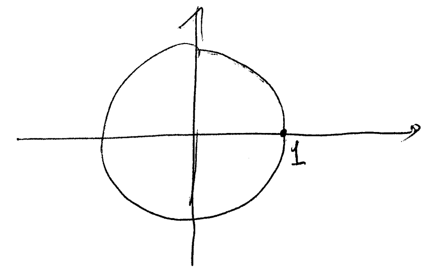
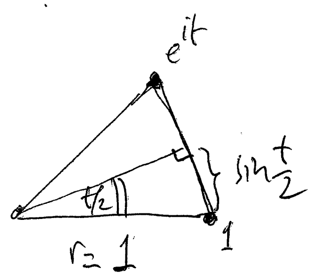



$\gamma(t) = z(t) = e^{it} \Rightarrow z'(t) = ie^{it}$

we have

$$\int_\gamma |z-1||dz| \overset{def}{=} \int_0^{2\pi} |e^{it}-1||z'(t)|\,dt = \int_0^{2\pi} |e^{it}-1|\,dt$$

-1




We have $|e^{it}-1| = 2\sin\frac{t}{2}$. Thus,

$$\int_\gamma |z-1||dz| = 2\int_0^{2\pi} \cancel{2}\sin\frac{t}{2}\,dt \overset{u=\frac{t}{2}}{=} \cancel{2}\,4\int_0^{\pi} \sin u\,du$$

$$= \cancel{2}\,4\,(-\cos u)\Big|_0^{\pi} = 4\times(1-(-1)) = \cancel{4}\,8$$

(5) Problem 7, Ahlfors, p 108

If $P(z)$ is a polynomial and $C$ denote the circle $|z-a|=R$, what is the value of $\int_C P(z)\,d\bar{z}$ ?

Proof ~~Since~~ Put $Q(z) = \overline{P(\bar{z})}$ (the same polynomial as $P(z)$ except $a_i$'s are replaced by $\bar{a}_i$'s).

By definition, $\int_C P(z)\,d\bar{z} = \overline{\int_C \overline{P(z)}\,dz} = \overline{\int_C Q(\bar{z})\,dz}$

On $C$, $|z-a|^2 = R^2$, or $R^2 = (z-a)(\bar{z}-\bar{a})$. Thus $\bar{z} = \bar{a} + \frac{R^2}{z-a}$.

~~By~~ ~~Tax~~ Taylor expansion for polynomial says $Q(u+v) = \sum_{k=0}^{n} \frac{u^k}{k!} Q^{(k)}(v)$

where $n$ is the degree of $Q$.

Thus, applying this expansion for $u = \frac{R^2}{z-a}$ and $v = \bar{a}$, we get

$$Q\left(\bar{a} + \frac{R^2}{z-a}\right) = \sum_{k=0}^{n} \frac{1}{k!}\left(\frac{R^2}{z-a}\right)^k Q^{(k)}(\bar{a})$$

Thus, $$\int_C P(z)\,d\bar{z} = \overline{\int_C Q(\bar{z})\,dz} = \overline{\sum_{k=0}^{n} \frac{R^{2k} Q^{(k)}(\bar{a})}{k!} \int_C \frac{dz}{(z-a)^k}} \quad (*)$$

For $k=0$, $\int_C \frac{dz}{(z-a)^k} = \int_C 1\,dz = 0$ since $z \mapsto 1$ is analytic everywhere.

For $k \geq 1$, $$\int_C \frac{dz}{(z-a)^k} = \int_0^{2\pi} \frac{z'(t)\,dt}{(z(t)-a)^k} \quad \text{where } z(t) = a + Re^{it}$$

$$= \int_0^{2\pi} \frac{iRe^{it}\,dt}{R^k e^{ikt}} = \int_0^{2\pi} \frac{i}{R^{k-1}} e^{i(1-k)t}\,dt$$

- If $k=1$, $\int_C \frac{dz}{z-a} = 2i\pi$

- If $k>1$, $$\int_0^{2\pi} \cancel{\frac{i}{R^{k-1}}}\, e^{i(1-k)t}\,dt \overset{u=(1-k)t}{=} \frac{1}{1-k} \underbrace{\int_0^{(1-k)2\pi} e^{iu}\,du}_{\int_0^{(1-k)2\pi} (\cos u + i \sin u)\,du = 0}$$

Thus, $$(*) = \overline{\frac{R^2 Q'(\bar{a})}{1!}\, 2i\pi} = -2i\pi R^2 \overline{Q'(\bar{a})} = -2i\pi R^2 P'(a)$$

$\boxed{6}$

(6) Problem 1, Ahlfors p. 120

Compute $\displaystyle\int_{|z|=1} \frac{e^z}{z}\,dz$

<u>Proof</u> Because $f(z) = e^z$ is analytic in $\mathbb{C}$, we can apply Cauchy's integral formula

$$\int_{|z|=1} \frac{e^z}{z}\,dz = \int_{|z|=1} \frac{f(z)}{z-0}\,dz = 2\pi i f(0) = 2\pi i$$

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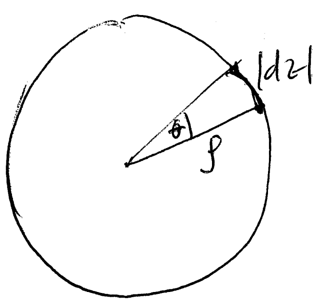
(7) Problem 3, Ahlfors, p. 120

Compute $\displaystyle\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$ where $|a| \neq \rho$.

<u>Proof</u>




we have $|dz| = \rho\, d\theta$. We know that

$\theta = \arg z$, which was defined by $-i\log\left(\frac{z}{\rho}\right)$

Thus $d\theta = d\left(-i\left(\log z - \log\rho\right)\right)$

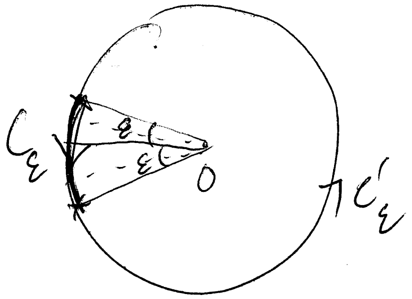
$$= -i\, d(\log z)$$

$$= -i\,\frac{dz}{z}$$

Note that here we donot consider $z$'s around $-\rho$ because the logarithm function is not analytic. For each $\varepsilon > 0$, put $C_\varepsilon$ be the path

$$z(t) = \exp\left(i(\pi + \varepsilon t)\right) \quad -1 \leq t \leq 1$$

and put $C'_\varepsilon$ to be the path $z(t) = \exp\left(i(-\pi + \varepsilon + t)\right)$, $0 \leq t \leq 2\pi - 2\varepsilon$

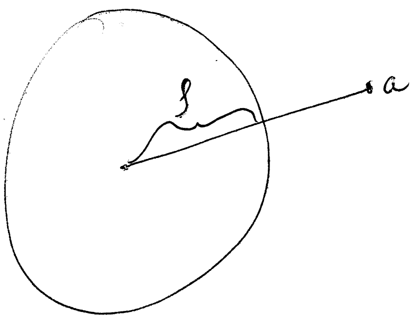



On $C_\varepsilon$, we have $|dz| = \rho\, d\theta = -i\rho \frac{dz}{z}$.

Thus,

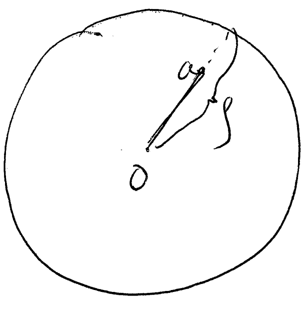
$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \int_{C_\varepsilon} \frac{|dz|}{|z-a|^2} + \int_{C'_\varepsilon} \frac{|dz|}{|z-a|^2}$$

$$= \int_{C_\varepsilon} \frac{|dz|}{|z-a|^2} \cancel{+} - i\rho \int_{C'_\varepsilon} \frac{dz}{z|z-a|^2} \qquad (*)$$




we have $|z-a| \geq |a| - \rho \quad \forall z \in \{w : |w| = \rho\}$ if $|a| > \rho$

or $|z-a| \geq \rho - |a| \quad \forall z \in \{w : |w| = \rho\}$ if $|a| < \rho$.




In both cases, $|z-a| \geq |\rho - |a||$. Thus

$$\int_{C_\varepsilon} \frac{|dz|}{|z-a|^2} \leq \int_{C_\varepsilon} \frac{|dz|}{|\rho - |a||^2} = \frac{1}{|\rho-|a||^2} \operatorname{length}(C_\varepsilon)$$

$\longrightarrow 0$ as $\varepsilon \to 0$

Let $\varepsilon \to 0$ at $(*)$, we get

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = -i\rho \lim_{\varepsilon \to 0} \int_{C'_\varepsilon} \frac{dz}{z|z-a|^2} \qquad (**)$$

We have $$\int_{C'_\varepsilon} \frac{dz}{z|z-a|^2} = \int_{|z|=\rho} \frac{dz}{z|z-a|^2} - \int_{C_\varepsilon} \frac{dz}{z|z-a|^2} \qquad (***)$$

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We have $\left| \int_{C_\varepsilon} \frac{dz}{z|z-a|^2} \right| \leq \int_{C_\varepsilon} \frac{|dz|}{|z|\,|z-a|^2} = \frac{1}{\rho} \int_{\varepsilon} \frac{|dz|}{|z-a|^2} \to 0$ as $\varepsilon \to 0$.

Thus $\lim_{\varepsilon\to 0} \int_{C'_\varepsilon} \frac{dz}{z|z-a|^2} = \int_{|z|=\rho} \frac{dz}{z|z-a|^2}$.

Therefore, as $\varepsilon \to 0$, $(**)$ gives

$$\int_{|z|=\rho} \cancel{\frac{dz}{z|z-a|^2}} = \cancel{\lim} \int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = -i\rho \int_{|z|=\rho} \frac{dz}{z|z-a|^2} \qquad (***)$$

We have
$$\begin{aligned}|z-a|^2 &= (z-a)(\bar z - \bar a)\\ &= (z-a)\left(\frac{\rho^2}{z} - \bar a\right)\\ &= \frac{1}{z}(z-a)(\rho^2 - \bar a z)\end{aligned}$$

Thus $z|z-a|^2 = (z-a)(\rho^2 - \bar a z)$ and therefore $(***)$ gives

$$I = \int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = -i\rho \int_{|z|=\rho} \frac{dz}{(z-a)(\rho^2 - \bar a z)}$$

$$= i\rho \int_{|z|=\rho} \frac{dz}{(z-a)(\bar a z - \rho^2)} \checkmark$$

If $a = 0$, then

$$I = i\rho \int_{|z|=\rho} \frac{dz}{z(-\rho^2)} = -\frac{i}{\rho} \int_{|z|=\rho} \frac{dz}{z} = -\frac{i}{\rho} 2\pi i = \frac{2\pi}{\rho}$$

If $a \neq 0$ then

$$\frac{1}{(z-a)(\bar{a}z-\rho^2)} = \frac{1}{\bar{a}} \frac{1}{(z-a)(z-\frac{\rho^2}{\bar{a}})} = \frac{1}{\bar{a}}\left(\frac{1}{z-a} - \frac{1}{z-\frac{\rho^2}{\bar{a}}}\right)\frac{1}{a-\frac{\rho^2}{\bar{a}}}$$

$$= \left(\frac{1}{z-a} - \frac{1}{z-\frac{\rho^2}{\bar{a}}}\right)\frac{1}{|a|^2-\rho^2}$$

Thus $I = i\rho \dfrac{1}{|a|^2-\rho^2} \displaystyle\int_{|z|=\rho} \left(\frac{1}{z-a} - \frac{1}{z-\frac{\rho^2}{\bar{a}}}\right) dz$

Let $\gamma$ be the circle $|z|=\rho$, we have

$$I = i\rho \frac{1}{|a|^2-\rho^2} 2\pi i \left(n(\gamma,a) - n(\gamma, \tfrac{\rho^2}{\bar{a}})\right)$$

where $n(\gamma, z_0)$ is the winding number of $\gamma$ about $z_0$.

$$= -\frac{2\pi\rho}{|a|^2-\rho^2}\left(n(\gamma,a) - n(\gamma, \tfrac{\rho^2}{\bar{a}})\right)$$

If $\rho > |a|$ then $a$ is in the bounded domain determined by $\gamma$, while $\frac{\rho^2}{\bar{a}}$ is in the unbounded one. Thus $n(\gamma, a) = 1$ and $n(\gamma, \frac{\rho^2}{\bar{a}}) = 0$.

Thus $I = \dfrac{2\pi\rho}{\rho^2 - |a|^2}$.

If $\rho < |a|$ the $a$ is in the unbounded domain while $\frac{\rho^2}{\bar{a}}$ is in the bounded one.

Thus $n(\gamma, a) = 0$, and $n(\gamma, \frac{\rho^2}{\bar{a}}) = 1$.

Thus $I = \dfrac{2\pi\rho}{|a|^2-\rho^2}$.

Then we can combine all cases into the final result

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \frac{2\pi\rho}{\left||a|^2-\rho^2\right|} \quad \checkmark$$

(8) Problem 1, Ahlfors, p. 123.

1) Compute $I_1 = \int_{|z|=1} e^z z^{-n}\, dz$

Put $f(z) = e^z$. Then $f$ is analytic in $\mathbb{C}$. we have

$$f^{(n-1)}(0) = \frac{n!}{2\pi i} \int_{|z|=1} \frac{f(\zeta)}{(\zeta-0)^n}\, d\zeta$$

Thus

$$f^{(n-1)}(0) = \frac{n!}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^n}\, dz = \frac{n!}{2\pi i} \int_{|z|=1} e^z z^{-n}\, dz = \frac{n!}{2\pi i} I_1$$

or

$$I_1 = \frac{2\pi i}{n!} f^{(n-1)}(0)$$

Since $(e^z)' = e^z$, we have $\frac{d^n}{dz^n} e^z = e^z$ and $f^{(n-1)}(0) = e^0 = 1$.

therefore, $I_1 = \dfrac{2\pi i}{n!}$

2) Compute $\int_{|z|=2} z^n (1-z)^m\, dz$

Because the function $f(z)= z^n(1-z)^m$ is analytic in $\mathbb{C}$ (of course here we assumed $m,n \in \mathbb{N}$), $\int_{|z|=2} f(z)\,dz = 0$.

3) Compute $I_\rho = \int_{|z|=\rho} |z-a|^{-4}\,|dz|$

Like in Problem ⑦, $|dz| = -i\rho \frac{dz}{z}$ and $|z-a|^2 = \frac{1}{z}(z-a)(\rho^2-\bar{a} z)$

Thus $|z-a|^4 = \frac{1}{z^2}(z-a)^2(\rho^2-\bar{a}z)^2$, and

$$\frac{|dz|}{|z-a|^4} = \frac{z^2}{(z-a)^2(\rho^2-\bar{a}z)^2}\,\frac{-i\rho}{z}\,dz = -i\rho\,\frac{z}{(z-a)^2(\rho^2-\bar{a}z)^2}\,dz$$

Thus
$$I_\rho = -i\rho \int_{|z|=\rho} \frac{z}{(z-a)^2(\bar{a}z-\rho^2)^2}\,dz$$

- If $a=0$ then
$$I_\rho = \int_{|z|=\rho} \frac{|dz|}{|z|^4} = \frac{1}{\rho^4}\int_{|z|=\rho}|dz| = \frac{1}{\rho^4}2\pi\rho = \frac{2\pi}{\rho^3}$$

- If $a \neq 0$ then
$$I_\rho = \frac{-i\rho}{\bar{a}^2}\int_{|z|=\rho}\frac{z}{(z-a)^2(z-b)^2}\,dz \qquad \text{where } b=\frac{\rho^2}{\bar{a}}.$$

We have
$$\int_{\underbrace{|z|=\rho}_{\gamma}} \frac{z}{(z-a)^2(z-b)^2}\,dz = \frac{-a-b}{(a-b)^3}\int_\gamma \frac{dz}{z-a} + \frac{a}{(a-b)^2}\int\frac{dz}{(z-a)^2} + \frac{a+b}{(a-b)^3}\int\frac{dz}{z-b} + \frac{b}{(a-b)^2}\int\frac{dz}{(z-b)^2}$$

We use the formula $f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\cancel{(z-\zeta)^n}(\zeta - z)^{n+1}}$

Applying this for $f(z)=1$ and $n=1$, we get

$$\int_\gamma \frac{d\zeta}{(\zeta - z)^2} = 0$$

Thus,
$$\int_\gamma \frac{z}{(z-a)^2(z-b)^2}\,dz = \frac{-a-b}{(a-b)^3}\int_\gamma \frac{dz}{z-a} + \frac{a+b}{(a-b)^3}\int_\gamma \frac{dz}{z-b}$$

$$= \frac{a+b}{(a-b)^3}\left(\int_\gamma \frac{dz}{z-b} - \int_\gamma \frac{dz}{z-a}\right) = \frac{a+b}{(a-b)^3}\,2\pi i\,\big(n(\gamma,b) - n(\gamma,a)\big)$$

- If $|a|<\rho$ then $n(\gamma,b)=0$, $n(\gamma,a)=1$. Thus

$$I_3 = \frac{-i\rho}{\cancel{4}|a|^2}\int_\gamma \frac{z}{(z-a)^2(z-b)^2}\,dz = \frac{+\rho}{\cancel{4}|a|^2}\,\frac{a+b}{(a-b)^3}\,2\pi$$

- If $|a|>\rho$ then $n(\gamma,b)=1$ and $n(\gamma,a)=0$. Thus

$$I_3 = \frac{-i\rho}{\cancel{4}|a|^2}\left(+\frac{a+b}{(a-b)^3}\right)2\pi i = -\frac{\rho}{\cancel{4}|a|^2}\,\frac{a+b}{(a-b)^3}\,2\pi$$

we have
$$a+b = a + \frac{\rho^2}{\bar a} = \frac{|a|^2+\rho^2}{\bar a}$$

$$(a-b)^3 = \left(a - \frac{\rho^2}{\bar a}\right)^3 = \frac{(|a|^2-\rho^2)^3}{\bar a^3}$$

Thus
$$\frac{a+b}{(a-b)^3} = \frac{|a|^2+\rho^2}{\bar a}\,\frac{\bar a^3}{(|a|^2-\rho^2)^3} = \bar a^2\,\frac{|a|^2+\rho^2}{(|a|^2-\rho^2)^3}$$

Thus $\dfrac{\rho}{\bar{a}^2}\dfrac{a+b}{(a-b)^3} = \dfrac{\rho(|a|^2+\rho^2)}{(|a|^2-\rho^2)^3}$

Therefore, we can combine all case into

$$I_3 = \frac{2\pi\rho(|a|^2+\rho^2)}{(|a|^2-\rho^2)^3}$$

(9) Problem 2, Ahlfors, p. 123

5/5 Prove that a function which is analytic in the whole ~~domain~~ plane and satisfies an inequality $|f(z)| < |z|^n$ for some $n$ and all sufficiently large $|z|$ reduces to a polynomial.

<u>Proof</u> Put $f_1(z) = \dfrac{f(z)-f(0)}{2z}$. Then $f_1(z)$ is analytic on $\mathbb{C}\setminus\{0\}$.

Moreover, since $\lim_{z\to 0} f_1(z)z = \lim_{z\to 0}\dfrac{f(z)-f(0)}{2} = 0$, $f_1$ has a unique continuous extension, which also implies the analytic extension of $f_1$ at $z=0$. In particular,

$$f_1(0) := \frac{1}{2\pi i}\int_C \frac{f_1(z)}{z}\,dz$$

We have $|f_1(z)| = \left|\dfrac{f(z)-f(0)}{2z}\right| \leq \dfrac{1}{2}\dfrac{|f(z)|}{|z|} + \dfrac{1}{2}\dfrac{|f(0)|}{|z|}$

For $z$ sufficiently large, we have $\dfrac{|f(z)|}{|z|} < |z|^{n-1}$ and $\dfrac{|f(0)|}{|z|} < |z|^{n-2}$

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Thus $|f_1(z)| < |z|^{n-1}$. Once we have an analytic $f_k$ such that $|f_k(z)| < |z|^{n-k}$ for all $|z|$ suff. large, we define $f_{k+1}$ by

$$f_{k+1}(z) = \frac{f_k(z) - f_k(0)}{2z}$$

then $f_{k+1}$ can be extended analytically at $z=0$ because $\lim_{z\to 0} z f_{k+1}(z) = 0$.

We have $$|f_{k+1}(z)| = \left|\frac{f_k(z)-f_k(0)}{2z}\right| \le \frac{1}{2}\left|\frac{f_k(z)}{z}\right| + \frac{1}{2}\left|\frac{f_k(0)}{z}\right|$$

For $|z|$ suff. large, $\left|\frac{f_k(z)}{z}\right| < |z|^{n-k-1}$ and $\left|\frac{f_k(0)}{z}\right| < |z|^{n-k-1}$.

Thus $|f_{k+1}(z)| < |z|^{n-k-1}$. Then we have constructed $f_1, f_2, \dots, f_n$. We have

$|f_n(z)| < |z|^{n-n} = 1$ for $|z|$ suff. large. Thus $f_n$ is bounded. By

Liouville's theorem, $f_n(z)$ must be identically a constant, say $f_n(z) = C, \forall z \in \mathbb{C}$.

We have
$$f(z) = 2z f_1(z) + f(0)$$
$$f_1(z) = 2z f_2(z) + f_1(0)$$
$$\vdots$$
$$f_{n-1}(z) = 2z f_n(z) + f_{n-1}(0)$$

thus
$$f(z) = 2z(2z f_2(z) + f_1(0)) + f(0)$$
$$= f(0) + 2z f_1(0) + 2^2 z^2 f_2(z)$$
$$= f(0) + 2z f_1(0) + 2^2 z^2 f_2(0) + 2^3 z^3 f_3(z)$$

$= \cdots$

$= f(0) + 2z f_1(0) + 2^2 z^2 f_2(0) + \cdots + 2^{n-1} z^{n-1} f_{N-1}(0) + 2^n C z^n$

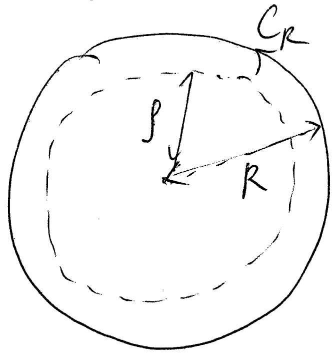
Therefore $f$ is a polynomial of degree at most $n$. ✓ interesting proof!

(10) Problem 3, Ahlfors, p. 123

If $f(z)$ is analytic and $|f(z)| \le M$ for $|z| \le R$, find an upper bound for $|f^{(n)}(z)|$ in $|z| \le \rho < R$.

Proof We have two ways of evaluating $|f^{n}(z)|$.

First way




We apply Cauchy's Integral Formula to $f$ on the curve $C_R = \{z : |z| = R\}$. Then

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{for every } z \text{ inside } C_R$$

Then
$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

For every $z$ such that $|z| \le \rho < R$, we have $|\zeta - z| \ge R - \rho$ for every $\zeta \in C_R$. Thus,

$$|f^{(n)}(z)| = \frac{n!}{2\pi} \left| \int_{C_R} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \right| \le \frac{n!}{2\pi} \int_{C_R} \frac{|f(\zeta)|}{|\zeta - z|^{n+1}}\, |d\zeta|$$

$$\le \frac{n!}{2\pi} \int_{C_R} \frac{M}{(R - \rho)^{n+1}}\, |d\zeta|$$

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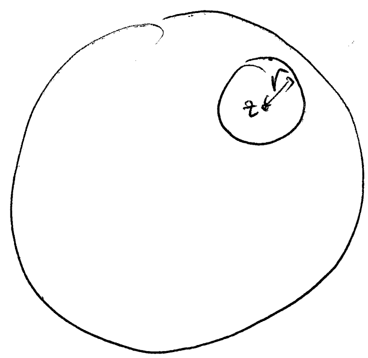
$$= \frac{n!}{2\pi} \frac{M}{(R-\rho)^{n+1}} 2\pi R = \frac{RMn!}{(R-\rho)^{n+1}}$$

therefore, $\left|f^{(n)}(z)\right| \leq \frac{RMn!}{(R-\rho)^{n+1}} \quad \forall 0 \leq |z| \leq \rho < R.$

Second way For each $|z| \leq \rho < R$, we have a ~~circle~~ disk centered at $z$ contained in the disk $\{w: |w| < R\}$. To make sure that this disk is inside $C_R$, we need $r \leq R-\rho$. Because $f$ is analytic in $C_r' = \{w: |w-z| = r\}$, we have



$$f(z) = \frac{1}{2\pi i} \int_{C_r'} \frac{f(\zeta)}{\zeta - z} d\zeta$$

Thus $$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_r'} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

Thus $$\left|f^{(n)}(z)\right| \leq \frac{n!}{2\pi} \int_{C_r'} \frac{|f(\zeta)|}{|\zeta - z|} |d\zeta| \leq \frac{n! M r^{-n-1} 2\pi r}{2\pi}$$

$$= n! M r^{-n}$$

To get a least upper bound as possible, we choose $r$ to be as large as possible. Thus we choose $r = R-\rho$ and get

$$\left|f^{(n)}(z)\right| \leq \frac{Mn!}{(R-\rho)^n} \quad \forall 0 \leq |z| \leq \rho < R$$

this bound is better than the first one.

(11) Problem 2, Ahlfors, p. 130

Show that a function which is analytic in the whole plane and has a nonessential singular at $z=\infty$ reduces to a polynomial.

<u>Proof</u>

Let $f$ be an analytic function in the whole plane. ~~Becau~~ Suppose that $f$ has a nonessential singular at $z=\infty$. Then by definition, there exists $\alpha \in \mathbb{R}$ such that $\lim_{z\to\infty} |z|^\alpha |f(z)| = 0$ or $\lim_{z\to\infty} |z|^\alpha |f(z)| = \infty$.

We will show that ~~f~~ there are only 3 possibilities

(i) $\lim_{z\to\infty} |f(z)| = 0$

(ii) there exists $n \in \mathbb{N}$ such that $\lim_{z\to\infty} \frac{|f(z)|}{|z|^n} = 0$

(iii) ~~$\lim_{z\to\infty} |f(z)| = \infty$~~ there exists $n \in \mathbb{N}$ such that $\lim_{z\to\infty} |z^n f(z)| = \infty$.

Indeed, first we suppose that there exists $\alpha \in \mathbb{R}$ such that $\lim_{z\to\infty} |z|^\alpha |f(z)| = 0$.

• If $\alpha > 0$, then $\lim_{z\to\infty} |z|^\alpha = \infty$ and $\lim_{z\to\infty} \frac{1}{|z|^\alpha} = 0$. Thus

$$\lim_{z\to\infty} |f(z)| = \lim_{z\to\infty} (|z|^\alpha |f(z)|) \lim_{z\to\infty} \left(\frac{1}{|z|^\alpha}\right) = 0$$

then ~~f~~ ~~is~~ we have (i).

• If $\alpha = 0$ then of course we have (i).

- If $\alpha < 0$ then we put $\beta = -\alpha > 0$. We have

$$\lim_{z\to\infty} \frac{|f(z)|}{|z|^\beta} = 0$$

Let $n \in \mathbb{N}$ be such that $n > \beta$. Then

$$\lim_{z\to\infty} \frac{|f(z)|}{|z|^n} = \underbrace{\left(\lim_{z\to\infty} \frac{|f(z)|}{|z|^\beta}\right)}_{0} \underbrace{\left(\lim_{z\to\infty} |z|^{\beta-n}\right)}_{0} = 0$$

Then we get (ii). Secondly, we assume that there exists $\alpha \in \mathbb{R}$ such that $\lim_{z\to\infty} ||z|^\alpha f(z)| = \infty$.

- If $\alpha < 0$ then $\lim_{z\to\infty} |z|^\alpha = 0$. thus $\lim_{z\to\infty} |f(z)| = \infty$. Thus we get (iii).
- If $\alpha = 0$ then of course we have (iii).
- If $\alpha > 0$ then we can take $n \in \mathbb{N}$ such that $n > \alpha$. Then we will still have $\lim_{z\to\infty} |z^n f(z)| = \infty$. Then we have (iii).

So it is sufficient to show that if $f$ satisfies either (i), (ii) or (iii), together with the analyticity over the whole domain, then $f$ reduces to a polynomial.

- If we have (i): ~~then there exists $R > 0$ such that $|f(z)| \leq 1$ for any~~ in case there exists $z_0 \in \mathbb{C}$ such that $f(z_0) \neq 0$, there exists

$R > |z_0|$ such that $|f(z)| < |f(z_0)|$ for any $|z| > R$. Thus the maximum of $|f(z)|$ over $\mathbb{C}$ if exists is obtained on the closed disk $\{z : |z| \le R\}$. This disk is compact, Thus $|f(z)|$ attains maximum value on it. Thus the maximum of $|f(z)|$ over $\mathbb{C}$ exists. Thus $f$ is bounded. By Louiville's theorem, $f$ must be a constant function, and thus a polynomial.

- If we have (ii): then there exists $R > 0$ such that $\frac{|f(z)|}{|z|^n} < 1$ for $|z| > R$. Thus $|f(z)| < |z|^n$ for any $|z| > R$. Therefore, by Prob. 9, $f$ is a polynomial.
- If we have (iii): Put $g(z) = z^n f(z)$. Then $g(z)$ is analytic in $\mathbb{C}$ and $\lim_{z \to \infty} |g(z)| = \infty$. It suffices to show that $g(z)$ is a polynomial. Indeed, if $g(z)$ is a polynomial then $z = 0$ must be a zero of order at least $n$ of $g(z)$. Thus $f(z) = \frac{g(z)}{z^n}$ is also a polynomial. ~~by the Fundamental~~

First, we'll show that $g(z)$ has only finitely many distinct zeros. Suppose by contradiction that there are $z_1, z_2, z_3, \ldots$ distinct zeros of $g(z)$. If the sequence $(z_m)$ is unbounded then there exists a subsequence $(z_{m_k})$ going to $\infty$, ~~Then~~ and $g(z_{m_k}) = 0$ $\forall k$. This contradict the fact that $\lim_{z \to \infty} g(z) = \infty$.

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If the sequence $(z_n)$ is bounded, then it has an ~~convergence subsequence~~ ~~$z_{n_k} \to z_0 \in \mathbb{C}$. Thus~~ accummulation point. Thus $g(z)$ must be identically zero. This is also a contradiction. Therefore, $g$ has only finitely many distinct zeros $z_1, \dots, z_m$. Since $g$ is not identically zero, each $z_i$ is a zero of finite order $n_i$. Thus the function

$$h(z) = \frac{g(z)}{(z-z_1)^{n_1} \cdots (z-z_m)^{n_m}}$$

is analytic and nonzero. Put $k(z) = \frac{1}{h(z)} = \frac{(z-z_1)^{n_1} \cdots (z-z_m)^{n_m}}{g(z)}$.

Then $k(z)$ is analytic on $\mathbb{C}$. Put $q = n_1 + \dots + n_m + 1$, we have

$$\frac{k(z)}{z^q} = \frac{(z-z_1)^{n_1} \cdots (z-z_m)^{n_m}}{z^q} \frac{1}{g(z)}$$

thus

$$\frac{|k(z)|}{|z|^q} = \frac{|(z-z_1)^{n_1} \cdots (z-z_m)^{n_m}|}{|z|^q} \frac{1}{|g(z)|} \to 0 \text{ as } z \to \infty$$

Thus we return the case (ii) but here $f$ is $k$. Thus $k$ is a polynomial.

If $k$ has degree ~~grt~~ at least one then it must have a zero point by the Fundamental Theorem of Algebra. This contradict the definition of $k$. Thus $k(z)$ must be identically a nonzero constant $C$. Thus

$$g(z) = \frac{1}{C}(z-z_1)^{n_1} \cdots (z-z_m)^{n_m}, \text{ which is a polynomial.}$$

(12) Problem 3, Ahlfors, p. 130

Show that the functions $e^z$, $\sin z$ and $\cos z$ have essential singularities at $\infty$.

<u>Proof</u> There are, in fact, many ways to solve this problem. We can use the result lately to say that if they ($e^z$, $\sin z$, $\cos z$) ~~are~~ have nonessential singularities at $\infty$ then they must be a polynomial. Thus, their n-fold derivative must be identically zero for some $n \in \mathbb{N}$. This is impossible because $(e^z)^{(n)} = e^z$, $(\sin z)^{(n)}$ is either $\pm \sin z$ or $\pm \cos z$, $(\cos z)^{(n)}$ is either $\pm \cos z$ or $\pm \sin z$. We can also interprete this situation as follows: $e^z$, $\sin z$ and $\cos z$ are nothing but power series that ~~converge~~ have $\infty$-radius of convergence. This will roughly leads to the contradiction because a polynomial is of "finite" series.

Here, however, we choose an intuitive approach: $f(z)$ has a nonessential singularity at $\infty$ if and only if there is a polynomial that "fits" $f(z)$ in sense that it helps $f(z)$ ~~do~~ go to 0 or $\infty$ as $z \to \infty$. In other words, their product or quotient should go to 0 or $\infty$. Since $e^{-i2\pi n} = \cos\left(\frac{\pi}{2} + n\pi\right) = \sin(n\pi) = 0$ for any $n \in \mathbb{N}$, the limit of their product or quotient must be 0. However, $e^n$, $\cos(in) = \frac{e^n + e^{-n}}{2}$ and $\sin(in) = \frac{e^n - e^{-n}}{2i}$ go to $\infty$ at the exponential rate:

Thus there is no polynomial that could help $e^z$, $\cos z$, $\sin z$ go to 0 as $z$ goes to infinity.

completion: 18/18