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Homework 6
Math 8701: Complex Analysis

Problems 2, 3 Ahlfors p. 130 are already done and written in a previous work.

(1) Problem 1, Ahlfors p. 133

5/5 Determine explicitly the largest disk about the origin whose image under the mapping $w = z^2 + z$ is one-to-one.

Proof First, let's show that the function $f(z) = z^2 + z$ is one-to-one in the disk $B(0, \frac{1}{2})$. Suppose by contradiction that there exist $z_1 \neq z_2$ in $B(0, \frac{1}{2})$ such that $f(z_1) = f(z_2)$. We have

$$z_1^2 + z_1 = z_2^2 + z_2$$

Thus $(z_1^2 + z_1) - (z_2^2 + z_2) = 0$. Thus $(z_1 - z_2)(z_1 + z_2 + 1) = 0$. Thus $z_1 + z_2 + 1 = 0$. Thus $z_1 + z_2 = 1$. Thus

$$1 = |z_1 + z_2| \leq |z_1| + |z_2| < \frac{1}{2} + \frac{1}{2} = 1$$

This is a contradiction.

Secondly, let's show that $f(z) = z^2 + z$ is not one-to-one in the disk $B(0, \frac{1}{2} + 2\varepsilon)$ for any $\varepsilon > 0$. We ~~have~~ put $z_1 = -\frac{1}{2} - \varepsilon$ and $z_2 = -\frac{1}{2} + \varepsilon$

then $z_1 \neq z_2$ and $z_1^2 + z_1 = \left(z_1 + \frac{1}{2}\right)^2 - \frac{1}{4} = \varepsilon^2 - \frac{1}{4}$

$$= (-\varepsilon)^2 - \frac{1}{4}$$

$$= \left(z_2 + \frac{1}{2}\right)^2 - \frac{1}{4} = z_2^2 + z_2$$

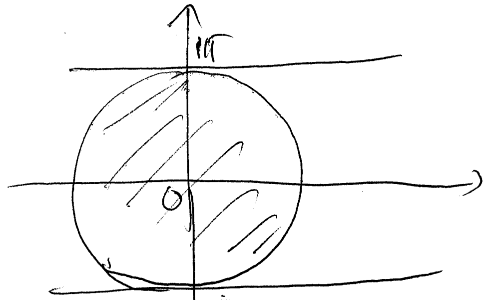
Therefore, $B(0, \frac{1}{2})$ is the largest disk about zero whose image under the mapping $w = z^2 + z$ is one-to-one.

(2) Problem 2, Ahlfors, p. 133

The same question for $w = e^z$

5/5

Proof



First we'll show that $f(z) = e^z$ is one-to-one on the disk $z \in B(0, \pi)$.

Suppose by contradiction that there exist $z_1, z_2 \in B(0, \pi)$ such that $z_1 \neq z_2$ and $e^{z_1} = e^{z_2}$. Then $e^{z_1 - z_2} = 1$. Thus $z_1 - z_2 = k 2\pi i$ where k is some integer. Since $z_1 \neq z_2$, $k \neq 0$. Thus

$$|z_1 - z_2| = |k| 2\pi \geq 2\pi$$

We have, on the other hand, $|z_1 - z_2| \leq |z_1| + |z_2| < \pi + \pi = 2\pi$

this is a contradiction.

Secondly, we'll show that $f(z)=e^z$ is not one-to-one on the disk $B(0,\pi+2\varepsilon)$ for any $\varepsilon>0$. Put $z_1=i(\pi+\varepsilon)$ and $z_2=i(-\pi+\varepsilon)$. Then $|z_1|=\pi+\varepsilon$ and $|z_2|=\pi-\varepsilon$. Thus $z_1,z_2\in B(0,\pi+2\varepsilon)$. We have $z_1-z_2=i(\pi+\varepsilon)-i(-\pi+\varepsilon)=2\pi i$. Thus $e^{z_1-z_2}=1$, or $e^{z_1}=e^{z_2}$.

Therefore $B(0,\pi)$ is the largest disk about zero whose image under the mapping $f(z)=e^z$ is one-to-one.

③ Apply the representation $f(z)=w_0+\zeta(z)^n$ to $\cos z$ with $z_0=0$. Determine $\zeta(z)$ explicitly.

Proof Put $f(z)=\cos z$ for every $z\in\mathbb{C}$. We have

$$f(0)=\cos 0=1$$

$$f'(0)=-\sin 0=0$$

$$f''(0)=-\cos 0=-1\neq 0$$

Thus 0 is a zero of order 2 of $f(z)-1$. Thus we have the presentation

$f(z)=1+\zeta(z)^2$ on a neighborhood of $z=0$. We have

$$f(z)-1=\cos z-1=-2\sin^2\frac{z}{2}=(i\sqrt{2})^2\sin^2\frac{z}{2}=\left(i\sqrt{2}\sin\frac{z}{2}\right)^2.$$

Thus we can choose $\zeta(z)=i\sqrt{2}\sin\frac{z}{2}$. This is an analytic function around

$z=0$ (and, in fact, analytic on $\mathbb{C}$).

(4) Problem 1, Ahlfors, p. 136

Show by using (36), or directly, that $|f(z)| \leq 1$ for $|z| \leq 1$ implies

$$\frac{|f'(z)|}{(1-|f(z)|^2)} \leq \frac{1}{1-|z|^2}$$

Proof We have ~~the~~ a more general statement as follow

Let ~~z_0~~ $|z_0| < R$ and $|w_0| < M$ and f be an analytic function on the disk $B(0,R)$. ~~If f~~ such that $f(z_0) = w_0$. If $|f(z)| \leq M$ $\forall |z| \leq R$ then

$$\left|\frac{M(f(z)-w_0)}{M^2-\bar{w}_0 f(z)}\right| \leq \left|\frac{R(z-z_0)}{R^2-\bar{z}_0 z}\right|$$

We apply the above result for $R=M=1$. For any z_0 in the disk $B(0,1)$ we have

$$\left|\frac{f(z)-w_0}{1-\bar{w}_0 f(z)}\right| \leq \left|\frac{z-z_0}{1-\bar{z}_0 z}\right| \quad \text{where } w_0 = f(z_0)$$

Equivalently,

$$\left|\frac{f(z)-w_0}{z-z_0}\right| \leq \left|\frac{1-\bar{w}_0 f(z)}{1-\bar{z}_0 z}\right|$$

Let z go to z_0, ~~th~~ we get $|f'(z_0)| \leq \left|\frac{1-\bar{w}_0 f(z_0)}{1-\bar{z}_0 z_0}\right| = \left|\frac{1-\bar{w}_0 w_0}{1-|z_0|^2}\right|$

Equivalently,

$$|f'(z_0)| \le \left|\frac{1-|w_0|^2}{1-|z_0|^2}\right| = \frac{1-|w_0|^2}{1-|z_0|^2} = \frac{1-|f(z_0)|^2}{1-|z_0|^2}$$

thus,

$$\frac{|f'(z_0)|}{1-|f(z_0)|^2} \le \frac{1}{1-|z_0|^2}$$

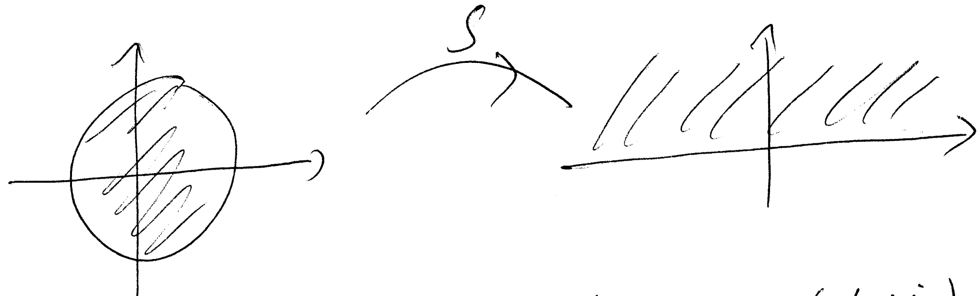
therefore,

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2} \quad \forall |z| < 1$$

5 Problem 2, Ahlfors, p. 136

Let f be analytic and $\operatorname{Im} f(z) \ge 0$ for $\operatorname{Im} z > 0$. Since f is an open mapping, the image of the domain $\operatorname{Im} z > 0$ is an open subset of $\mathbb{C}$ unless f is constant. When f is constant, the problem's statement is trivial. Thus $\operatorname{Im} f(z) > 0$ for $\operatorname{Im} z > 0$. The following linear transformation maps the unit circle to the real line

$$Sz = (z, 1, -1, i) = \frac{1-i}{2}\,\frac{z+1}{z-i}$$



We see that $\operatorname{Im} S(0) = \operatorname{Im} \frac{1-i}{-2i} = \operatorname{Im}\left(\frac{1+i}{2}\right) > 0$. Thus S map the disk $B(0,1)$ to $\Omega = \{z : \operatorname{Im} z > 0\}$ - the upper half plane. Since $f : \Omega \to \Omega$, we can put

6

Put $g: B(0,1) \longrightarrow B(0,1)$

$$g(\zeta) = S^{-1} f(S\zeta)$$

Now we can apply the ~~ineq~~ equality (36) in Ahlfors for $R=1$ and $M=1$

for $g(\zeta)$.

$$\left| \frac{g(\zeta) - g(\zeta_0)}{1 - \overline{g(\zeta_0)}\, g(\zeta)} \right| \leq \left| \frac{\zeta - \zeta_0}{1 - \bar{\zeta}_0 \zeta} \right|$$

Put $z = S\zeta$ or equivalently $\zeta = S^{-1} z$, we have

$$\left| \frac{S^{-1} f(z) - S^{-1} f(z_0)}{1 - \overline{S^{-1} f(z_0)}\, S^{-1} f(z)} \right| \leq \left| \frac{S^{-1} z - S^{-1} z_0}{1 - \overline{S^{-1} z_0}\, S^{-1} z} \right| \qquad (*)$$

Both sides of $(*)$ are of the form $\left| \frac{S^{-1} u - S^{-1} u_0}{1 - \overline{S^{-1} u_0}\, S^{-1} u} \right|$. Thus we show ~~manipulate~~

manipulate on this general form.

Put $\alpha = \frac{1-i}{2}$. Then $S^{-1} u = \frac{\alpha + iu}{u - \alpha}$.

Thus,

$$\left| \frac{S^{-1} u - S^{-1} u_0}{1 - \overline{S^{-1} u_0}\, S^{-1} u} \right| = \left| \frac{\frac{\alpha + iu}{u - \alpha} - \frac{\alpha + iu}{u_0 - \alpha}}{1 - \frac{\bar{\alpha} - i\bar{u}_0}{\bar{u}_0 - \bar{\alpha}} \frac{\alpha + iu}{u - \alpha}} \right|$$

$$= \left| \frac{(\alpha + iu)(u_0 - \alpha) - (u - \alpha)(\alpha + iu)}{(\bar{u}_0 - \bar{\alpha})(u - \alpha) - (\bar{\alpha} - i\bar{u}_0)(\alpha + iu)} \right|$$

$$= \left| \frac{\alpha u_0 + i u u_0 - \alpha^2 - i\alpha u - (\alpha u - \alpha^2 + i u_0 u - i\alpha u_0)}{(\bar{u}_0 u - \bar{\alpha} u - \alpha \bar{u}_0 + \bar{\alpha}\alpha) - (\bar{\alpha}\alpha - i\alpha\bar{u}_0 + i\bar{\alpha} u + \bar{u}_0 u)} \right|$$

$$= \left| \frac{\alpha u_0 - i\alpha u - \alpha u + i\alpha u_0}{-\bar{\alpha} u - \alpha \bar{u}_0 + i\alpha \bar{u}_0 - i\bar{\alpha} u} \right|$$

$$= \left| \frac{(\alpha + i\alpha)(u_0 - u)}{-\bar{\alpha}(1+i) u + \alpha(i-1)\bar{u}_0} \right| \qquad (**)$$

we have $\displaystyle \bar{\alpha}(1+i) = \frac{1+i}{2}(1+i) = \frac{1-1+2i}{2} = i$

$$\alpha(i-1) = \frac{1-i}{2}(i-1) = \frac{2i+1-1}{2} = i$$

Thus $\displaystyle (**) = \left| \frac{\alpha(1+i)(u_0 - u)}{i(\bar{u}_0 - u)} \right| = \frac{|\alpha|\,|1+i|}{|i|} \frac{|u_0 - u|}{|\bar{u}_0 - u|} = \frac{|u_0 - u|}{|\bar{u}_0 - u|}$

Therefore, $\displaystyle \left| \frac{S^{-1}u - S^{-1}u_0}{1 - \overline{S^{-1}u_0}\, S^{-1}u} \right| = \frac{|u_0 - u|}{|\bar{u}_0 - u|}$

Apply this for $u = f(z)$, and then $u = z$, we get

$$\left| \frac{S^{-1}f(z) - S^{-1}f(z_0)}{1 - \overline{S^{-1}f(z_0)}\, S^{-1}f(z)} \right| = \frac{|f(z_0) - f(z)|}{|\overline{f(z_0)} - f(z)|}$$

and $\displaystyle \left| \frac{S^{-1}z - S^{-1}z_0}{1 - \overline{S^{-1}z_0}\, S^{-1}z} \right| = \left| \frac{z_0 - z}{\bar{z}_0 - z} \right|$

Thus, the inequality (*) becomes

$$\frac{|f(z)-f(z_0)|}{|f(z)-\overline{f(z_0)}|} \le \frac{|z-z_0|}{|z-\overline{z_0}|} \qquad (1)$$

From (1), we have

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \le \left|\frac{f(z)-\overline{f(z_0)}}{z-\overline{z_0}}\right| \qquad (2)$$

We have $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)$

thus ~~for~~ the LHS of (2) goes to $|f'(z_0)|$ as z goes to z_0.

$$\lim_{z\to z_0} \frac{f(z)-\overline{f(z_0)}}{z-\overline{z_0}} = \frac{f(z_0)-\overline{f(z_0)}}{z_0-\overline{z_0}} = \frac{2i\,\mathrm{Im}\,f(z_0)}{2i\,\mathrm{Im}\,z_0} = \frac{\mathrm{Im}\,f(z_0)}{\mathrm{Im}\,z_0}$$

Since $\mathrm{Im}\,f(z_0) > 0$ and $\mathrm{Im}\,z_0 > 0$, ~~we~~ the RHS of (2) goes to $\frac{\mathrm{Im}\,f(z_0)}{\mathrm{Im}\,z_0}$

as z goes to z_0. Thus from (2) we get

$$|f'(z_0)| \le \frac{\mathrm{Im}\,f(z_0)}{\mathrm{Im}\,z_0}$$

therefore $|f'(z)| \le \frac{\mathrm{Im}\,f(z)}{\mathrm{Im}\,z}$ $\forall z \in \Omega$

for now...
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completion: 6/10