Name: Tuan Pham
ID: 4652218

Math 8701: Complex Analysis 10/10 $\boxed{1}$
Problem set 7

$\boxed{\#9 \text{ on HW6}}$

(1) Problem 3, Ahlfors, p. 136.

The statement of Schwarz's lemma is as follow: "Let f be an analytic function in the disk $|z|<1$ such that $|f(z)| \leq 1$ and $f(0)=0$. Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. The equality holds for some $z \neq 0$ or $|f'(0)|=1$ will lead to $f(z)=cz$ for some constant c with modulus 1."

In problem 1, we applied the Schwarz's lemma for SfT^{-1} instead of f, where T is a linear transformation mapping $|z|<1$ to $|\zeta|<1$ and z_0 to 0; S is a linear transformation mapping $|w|<1$ to $|\xi|<1$ and w_0 to 0. We chose

$$T(z) = \frac{z - z_0}{1 - \bar{z}_0 z} \quad \text{and} \quad S(z) = \frac{w - w_0}{1 - \bar{w}_0 w}$$

then we used Schwarz's lemma to conclude that $|(SfT^{-1})'(0)| \leq 1$. The equality holds if $SfT^{-1}(\zeta) = c\zeta$ for some constant c with $|c| = 1$. But $z = T^{-1}\zeta$, we get $Sf(z) = cT(z)$. Thus $f(z) = S^{-1}(cT(z))$.

Since a composition of two linear transformation is also a linear transformation, f is a linear tranformation.

2

In problem 2, we applied the result of problem 1 for $TfT^{-1} = g$, instead of f, where T is a linear transformation mapping the upper half plane onto the unit disk $T: H \to B(0,1)$. We choose $T = \phi^{-1}$ where

$$\phi: B(0,1) \to H$$

$$\phi(z) = (z, 1, -1, i) = \frac{1-i}{2}\,\frac{z+1}{z-i}$$

the equality holds if ~~and only if~~ g is a linear transformation. We have $f = T^{-1} g T$. Thus f will be a linear transformation.

(2) Problem 4, Ahlfors p. 136

Let $f: B(0,1) \to H$ be an analytic function from the unit disk to the upper half plane, and $f(z_0) = w_0$. We want a version of Schwarz's Lemma for f.

We need the following composition

$$B(0,1) \xrightarrow{T^{-1}} B(0,1) \xrightarrow{f} H \xrightarrow{S} B(0,1)$$

$$0 \longmapsto z_0 \longmapsto f(z_0) = w_0 \longmapsto 0$$

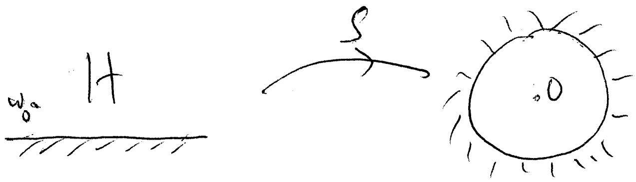
Then we can apply the Schwarz's lemma for $g = S f T^{-1}$. As in problem 1, T can be chosen as $T(z) = \dfrac{z - z_0}{1 - \bar{z}_0 z}$.

Now we will find S. We want a map that maps $\mathbb{R}$ to the circle S^1 and w_0 to 0. Since the symmetric of w_0 with respect to $\mathbb{R}$ is $\bar{w}_0$, and the symmetry of 0 with respect to S^1 is ∞, the map S should do the following:

$$\overline{w_0} \mapsto \infty$$

$$w_0 \mapsto 0$$

$$1 \mapsto 1$$



Put $z = S(w)$. We have $(w, \overline{w_0}, w_0, 1) = (z, \infty, 0, 1)$, which is equivalent to

$$\frac{w - w_0}{w - 1} \Big/ \frac{\overline{w_0} - w_0}{\overline{w_0} - 1} = \frac{z - 0}{z - 1}.$$

Thus
$$\frac{z}{z-1} = \frac{\overline{w_0} - 1}{\overline{w_0} - w_0} \, \frac{w - w_0}{w - 1}$$

thus
$$\frac{1}{z-1} = \frac{\overline{w_0} - 1}{\overline{w_0} - w_0} \, \frac{w - w_0}{w - 1} - 1 = \frac{(\overline{w_0} - w)(1 - w_0)}{(\overline{w_0} - w_0)(w - 1)}$$

thus
$$z = \cancel{\frac{\overline{w_0} - w_0}{\overline{w_0} - 1}} \, \cancel{\frac{w-1}{w-w_0}} \, \frac{(\overline{w_0} - w_0)(w - 1)}{(\overline{w_0} - w)(1 - w_0)} + 1 = \frac{(\overline{w_0} - 1)(w - w_0)}{(1 - w_0)(\overline{w_0} - w)}$$

therefore,
$$Sw = \frac{\overline{w_0} - 1}{w_0 - 1} \, \frac{w - w_0}{w - \overline{w_0}}.$$

Then Schwarz's lemma gives us $|g(\zeta)| \le |\zeta|$, i.e. $|S f T^{-1} \zeta| \le |\zeta|$.

Put $z = T^{-1}\zeta$, we get $|Sf(z)| \le |Tz|$. Thus we get

$$\left| \frac{\overline{w_0} - 1}{w_0 - 1} \, \frac{f(z) - w_0}{f(z) - \overline{w_0}} \right| \le \left| \frac{z - z_0}{1 - \overline{z_0} z} \right|$$

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Since $\left|\dfrac{\overline{w}_0 - 1}{w_0 - 1}\right| = 1$, we have the following inequality

$$\left|\frac{f(z) - w_0}{f(z) - \overline{w}_0}\right| \le \left|\frac{z - z_0}{1 - \overline{z}_0 z}\right| \qquad \forall |z| < 1$$

5/5 ③ Problem 5, Ahlfors, p. 136

HW6

As in the ~~two previous~~ problems, an analytic function $f: B(a, r_1) \to B(b, r_2)$ can be composed with linear transformations to become a map from the disk to itself. Specifically,

- If $f: B(a, r_1) \longrightarrow B(b, r_2)$ we have the following chain

$$B(0,1) \xrightarrow{S^{-1}} B(0,1) \xrightarrow{\phi} B(a,r_1) \xrightarrow{f} B(b,r_2) \xrightarrow{\psi} B(0,1) \xrightarrow{T} B(0,1)$$

$$0 \longmapsto \zeta_0 = \frac{z_0 - a}{r_1} \longmapsto z_0 \longmapsto w_0 \longmapsto \frac{w_0 - b}{r_2} = \eta_0 \longmapsto 0$$

where $S(\zeta) = \dfrac{\zeta - \zeta_0}{1 - \overline{\zeta}_0 \zeta}$, $\phi(\zeta) = r_1\zeta + a$, $\psi(w) = \dfrac{w - b}{r_2}$, and

$$T(w) = \frac{w - \eta_0}{1 - \overline{\eta}_0 w}$$

Then $g = T\psi f \phi S^{-1}$ is analytic from $B(0,1)$ to $B(0,1)$ and $g(0) = 0$. The one-to-one of f is equivalent to the one-to-one of g.

- If $f: B(a, r) \longrightarrow H$, the upper half plane, then we have the

$$B(0,1) \xrightarrow{S^{-1}} B(0,1) \xrightarrow{\phi} B(a,r) \xrightarrow{f} H \xrightarrow{T} B(0,1)$$

$$0 \longmapsto \zeta_0 = \frac{z_0 - a}{r} \longmapsto z_0 \longmapsto w_0 \longmapsto 0$$

where S and ϕ are as above and $$Tw = \frac{\overline{w_0} - 1}{w_0 - 1}\,\frac{w - w_0}{w - \overline{w_0}}.$$

then $g = Tf\phi S^{-1}$ is analytic from $B(0,1)$ to $B(0,1)$ and $g(0) = 0$. Then g is bijective if and only f is bijective.

Therefore, our problem (showing that f is one-to-one then it is a linear transformation) becomes the following:

"Let g be an analytic bijective function from $B(0,1)$ to $B(0,1)$ such that ~~$|g(z)| < 1$ for a~~ $g(0) = 0$. Then there exists a constant c with $|c| = 1$ such that $g(z) \equiv cz$".

We'll solve the above problem. By Schwarz's lemma, we have $|g(z)| \le |z|$ for all $|z| \le 1$. ~~By there~~ We put $h : B(0,1) \to B(0,1)$ to be the inverse function of g. Then h is analytic and $h(0) = 0$. By Schwarz's lemma, we have $|h(w)| \le |w|$ for all $|w| < 1$. Thus

$$|z| \le |h(g(z))| \le |g(z)| \quad \forall z \in B(0,1)$$

Thus $|g(z)| = |z|$ $\forall z \in B(0,1)$. By Schwarz's lemma, There exists a constant c with modulus 1 such that $g(z) = cz$ for all $|z| < 1$.

5/5 (4) Problem 1, Prof. Brubaker

Let $\gamma:[0,1]\to\Omega$ be a closed curve in an open connected domain Ω. Suppose that γ is homotopic to the constant curve $\gamma_a \equiv a$ for some $a\in\gamma([0,1])$. We take $b\in\Omega$ arbitrarily and will show that γ is also homotopic to the constant path $\gamma_b \equiv b$.

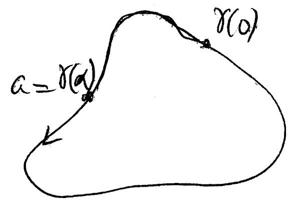
We notice that this kind of claim is different from the strict (usual) sense: when we refer to a curve γ, it is a specific map from $[0,1]$ to Ω. That means $\gamma(0)$ is already specified as soon as we speak of γ. In that sense, we couldn't choose arbitrarily any constant curve and say that γ is homotopic to it; the constant curve <u>must</u> be $\gamma(0)$. Here we encounter a casual claim. We'll find a way to understand what Professor means later. interesting point!

In the set-up of computing line-integral, two curves that are a reparametrization of each other are considered the same. Thus γ is considered the same as $\gamma^{\alpha}:[0,1]\to\Omega$

$$\gamma'(s)=\begin{cases}\gamma(\alpha+s), & 0\le s\le 1-\alpha\\ \gamma(s-(1-\alpha)), & 1-\alpha\le s\le 1\end{cases}$$

(Here $0<\alpha<1$, $\gamma(\alpha)=a$)

This is a reparametrization of γ that has $\gamma(\alpha)=a$ as the starting and ending point. To say that γ is homotopic to γ_a, we means there exists

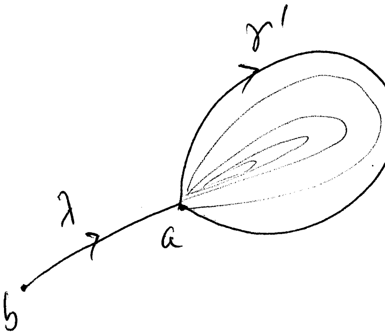


a continuous function $\Gamma : [0,1]\times[0,1]\times \Omega$ such that

$$\begin{cases} \Gamma(s,0) = \gamma(s) \\ \Gamma(s,1) = a \\ \Gamma(0,t) = a \\ \Gamma(1,t) = a \end{cases}$$

Then γ' is homotopic to the constant curve $\gamma'(0)$ by the "reparametrization" of the above homotopy, i.e. $\Gamma' : [0,1]\times[0,1] \longrightarrow \Omega$ such that

$$\Gamma'(s,t) = \begin{cases} \Gamma(\alpha+s,t), & 0\le s\le 1-\alpha,\ 0\le t\le 1 \\ \Gamma(s-(1-\alpha),t), & 1-\alpha\le s\le 1,\ 0\le t\le 1 \end{cases}$$



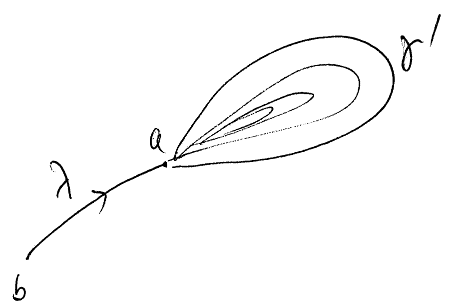
Let b be any point in Ω. Since Ω is connected, there exists a path $\lambda : [0,1]\to\Omega$ from a to b in Ω. In the set-up of computing integral, γ' is considered the same as the chain $\gamma'' = \lambda + \gamma' + (-\lambda)$. Actually

γ'' is a path defined as

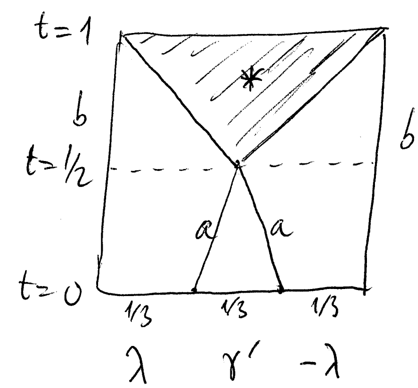
$$\gamma''(s) = \begin{cases} \lambda(3s), & 0\le s\le 1/3 \\ \gamma'(3s-1), & 1/3\le s\le 2/3 \\ \lambda(3-3s), & 2/3\le s\le 1 \end{cases}$$

Now we probably understand what Professor Brabecher means: by the fact that γ' is homotopic to the constant ~~map~~ curve $\gamma(0)$, show that γ'' is homotopic to the constant curve at b.

"We'll construct a homotopy from γ'' to the constant curve at b from the homotopy Γ' and the path γ'.



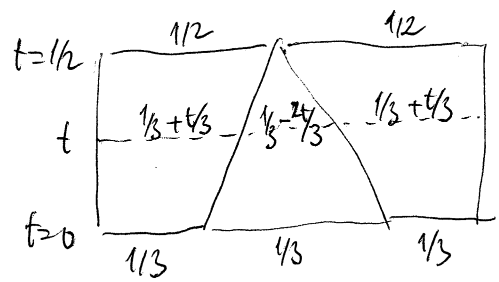
The intuitive plan



The plan on the map.

We define a function $\Gamma'' : [0,1] \times [0,1] \to \Omega$ as follow

- For $0 \leq t < 1/2$:



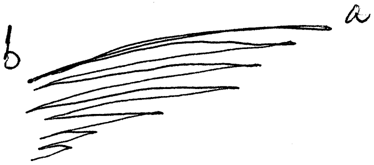
$$\Gamma''(s,t) = \begin{cases} \lambda\left(\frac{3}{1+t}s\right), & 0 \leq s \leq \frac{1+t}{3} \\ \Gamma'(\,\cdot\,,2t), & \frac{1+t}{3} \leq s \leq \frac{2-t}{3} \\ (-\lambda)\left(\frac{3}{1+t}s - \frac{2-t}{1+t}\right), & \frac{2-t}{3} \leq s \leq 1 \end{cases}$$

Note that $(-\lambda)$ is defined as $(-\lambda)(t) = \lambda(1-t)$ for every $0 \leq t \leq 1$.

- For $1/2 \leq t \leq 1$:

$$\frac{3}{1-2t}s - \frac{1+t}{1-2t}$$

$$\Gamma''(s,t)=\begin{cases}\lambda(2s), & 0\le s\le 1-t\\ \lambda(2(1-t)), & 1-t\le s\le t\\ (-\lambda)(2s-1), & t\le s\le 1\end{cases}$$



We check if Γ'' is continuous at $t=\frac{1}{2}$:

The ~~first formulas~~ give

$$\Gamma''\cancel{(s,\tfrac{1}{2})=}\quad \lambda\cancel{(2s),\ 0\le s\le 1/2}$$

$$\cancel{\lim_{s\to 1/2}\Gamma''(s,\tfrac{1}{2})=}$$

Around $t=\frac{1}{2}$, $\Gamma''(s,t)$ is around a when s is near $\frac{1}{2}$. otherwise, when s is not going to $\frac{1}{2}$, $\Gamma''(s,\frac{1}{2})=\begin{cases}\lambda(2s), & 0\le s<\frac{1}{2}\\ (-\lambda)(2s-1), & \frac{1}{2}<s\le 1\end{cases}$

thus Γ'' is continuous. Moreover,

$$\Gamma''(s,0)=\begin{cases}\lambda(3s), & 0\le s\le \frac{1}{3}\\ \Gamma'(3s-1,0), & \frac{1}{3}\le s\le \frac{2}{3}\\ (-\lambda)(3s-2), & \frac{2}{3}\le s\le 1\end{cases}=\begin{cases}\lambda(3s), & 0\le s\le \frac{1}{3}\\ \gamma'(3s-1), & \frac{1}{3}\le s\le \frac{2}{3}\\ (-\lambda)(3s-2), & \frac{2}{3}\le s\le 1\end{cases}$$

$$=\gamma''(s).$$

$$\Gamma''(s,1)=\begin{cases}\lambda(2s), & 0\le s\le 0\\ \lambda(0), & 0\le s\le 1\\ (-\lambda)(2s-1), & 1\le s\le 1\end{cases}=\lambda(0)=b$$

$$\Gamma''(0,t) = \begin{cases} \lambda(0), & 0 \le t < \frac{1}{2} \\ \lambda(0), & \frac{1}{2} \le t \le 1 \end{cases} = \lambda(0) = b$$

$$\Gamma''(1,t) = \begin{cases} (-\lambda)\left(\frac{3}{1+t} - \frac{2-t}{1+t}\right), & 0 \le t < \frac{1}{2} \\ (-\lambda)(1), & \frac{1}{2} \le t \le 1 \end{cases}$$

$$= (-\lambda)(1) = \lambda(0) = b \qquad \checkmark$$

Therefore Γ'' is a homotopy from γ'' to the constant path b.

Note: We can prove regorously that Γ'' is continuous at $t = \frac{1}{2}$ as follow:

- If $s_0 < \frac{1}{2}$ ~~then~~ or $s_0 > \frac{1}{2}$ then Γ'' is continuous at $(s_0, \frac{1}{2})$.
- If $s_0 = \frac{1}{2}$, then we'll show that $\lim_{(s,t)\to(\frac{1}{2},\frac{1}{2}^-)} \Gamma''(s,t) = \lim_{(s,t)\to(\frac{1}{2},\frac{1}{2}^+)} \Gamma''(s,t) = a$.

Put $u = \frac{3s}{1-2t} - \frac{1+t}{1-2t} \in [0,1]$. Then we know that Γ' is uniformly continuous in $[0,1] \times [0,1]$ because this space is compact. Thus $|\Gamma'(u,2t) - \Gamma'(u,1)| < \varepsilon$ for any $u \in [0,1]$ and $|t - \frac{1}{2}| < \delta(\varepsilon)$. Moreover, $\Gamma'(u,1) = a$. Therefore,

$$\lim_{(s,t)\to(\frac{1}{2},\frac{1}{2}^-)} \Gamma''(s,t) = a$$

the formula of Γ'' when $t \geq \frac{1}{2}$ gives us $\lim_{(s,t)\to(\frac{1}{2},\frac{1}{2}^+)} \Gamma''(s,t) = a$. Therefore Γ'' is continuous at $(\frac{1}{2}, \frac{1}{2})$.

5 Problem 2, Prof. Brubaker

5/5

We'll point out an example ~~that~~ to demonstrate that if we say two curves γ and γ' are homotopic if there exists a continuous function $\Gamma: [0,1]\times[0,1]\to\Omega$ such that $\begin{cases}\Gamma(s,0)=\gamma(s)\\ \Gamma(s,1)=\gamma(s')\end{cases}$ then there exists a choice of function f such that $\int_\gamma f\,dz \neq \int_{\gamma'} f\,dz$.

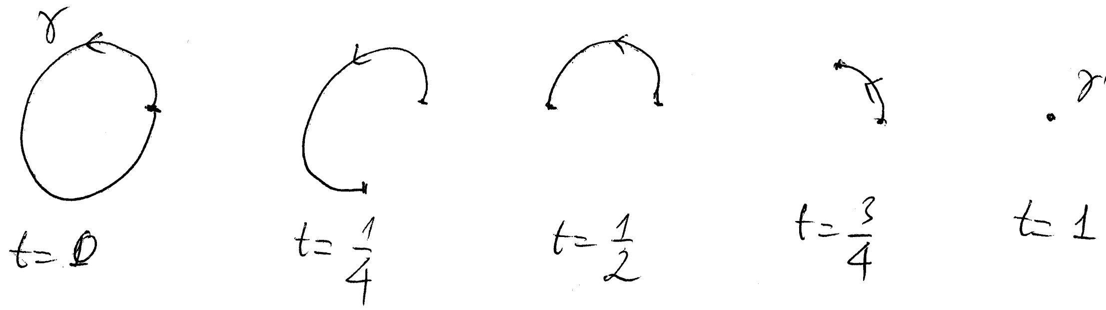
We choose $f(z)=\frac{1}{z}$. Then f is analytic in $\mathbb{C}\setminus\{0\}$. We choose $\gamma(s)=e^{2\pi i s}$ for all $s\in[0,1]$ and $\gamma'(s)=1$ for all $s\in[0,1]$. Then

$$\int_\gamma f(z)\,dz = \int_\gamma \frac{dz}{z} = 2\pi i$$

$$\int_{\gamma'} f(z)\,dz = \int_{\gamma'} \frac{dz}{z} = 0$$

Thus $\int_\gamma f(z)\,dz = \int_{\gamma'} f(z)\,dz$. We define $\Gamma: [0,1]\times[0,1]\to\mathbb{C}\setminus\{0\}$ as follow

$\Gamma(s,t)=e^{2\pi i(1-t)s}$ for all $0\le s,t\le 1$. Then Γ is a continuous function and $\Gamma(s,0)=e^{2\pi i s}=\gamma(s)$ and $\Gamma(s,1)=1=\gamma'(s)$. Therefore γ and γ' are homotopic in the altered sense. This is a counterexample to show that the base point is important for homotopy. The picture of the deformation from γ to γ' is as follow:


γ
γ'
t = 0
t = 1/4
t = 1/2
t = 3/4
t = 1