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Math 8701: Complex Analysis

Problem Set 8

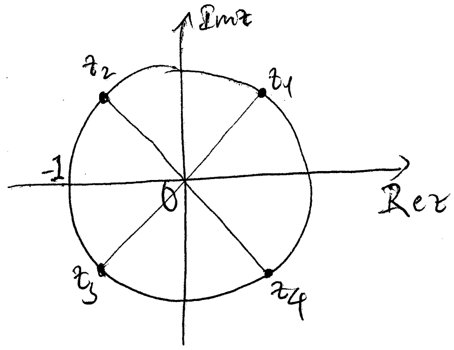
23/23

Bonus +2

(1) We'll find the residues at all singularities of the following functions:

~~4/5~~ (a) $f(z) = \dfrac{z}{z^4+1}$

5/5 The equation $z^4+1=0$ gives four distinct roots $z_1 = e^{i\frac{\pi}{4}}$, $z_2 = e^{i\frac{3\pi}{4}}$, $z_3 = e^{i\frac{-3\pi}{4}}$, $z_4 = e^{i\frac{-\pi}{4}}$ which are the four 4^{th} roots of -1.



Thus z_1, z_2, z_3, z_4 are simple poles of $f(z)$, and $f(z) = \dfrac{z}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$

the residues of f at these poles are given by

$$\operatorname{Res}_{z=z_1} f(z) = \lim_{z\to z_1} (z-z_1) f(z) = \frac{z_1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}$$

the picture above suggests $z_1 - z_2 = 2\,\mathrm{Re}(z_1) = 2\frac{\sqrt{2}}{2} = \sqrt{2}$,

$z_1 - z_3 = 2z_1$, and

$z_1 - z_4 = 2i\,\mathrm{Im}(z_1) = 2i\frac{\sqrt{2}}{2} = i\sqrt{2}$

Thus $\operatorname{Res}_{z=z_1} f(z) = \dfrac{z_1}{\sqrt{2}.\,2z_1.\,i\sqrt{2}} = \dfrac{1}{4i} = -\dfrac{i}{4}$. ✓

Similarly,

$$\operatorname{Res}_{z=z_2} f(z) = \frac{z_2}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}, \quad \text{where } z_2 - z_1 = -\sqrt{2},$$

$$z_2 - z_3 = i\sqrt{2},$$

$$z_2 - z_4 = 2z_2.$$

Thus $$\operatorname{Res}_{z=z_2} f(z) = \frac{z_2}{-\sqrt{2}(i\sqrt{2})(2z_2)} = \frac{1}{-4i} = \frac{i}{4}.$$

Similarly,

$$\operatorname{Res}_{z=z_3} f(z) = \frac{z_3}{(z_3-z_1)(z_3-z_2)(z_3-z_4)}, \quad \text{where } z_3 - z_1 = 2z_3,$$

$$z_3 - z_2 = -i\sqrt{2},$$

$$z_3 - z_4 = -\sqrt{2}.$$

Thus, $$\operatorname{Res}_{z=z_3} f(z) = \frac{z_3}{(2z_3)(-i\sqrt{2})(-\sqrt{2})} = \frac{1}{i4} = -\frac{i}{4}.$$

Similarly,

$$\operatorname{Res}_{z=z_4} f(z) = \frac{z_4}{(z_4-z_1)(z_4-z_3)(z_4-z_2)} = \frac{z_4}{(-i\sqrt{2})(+\sqrt{2})(2z_4)} = \frac{1}{-4i} = \frac{i}{4}.$$

(b)

$$f(z) = \frac{\sin z}{z^2(\pi - z)}$$

The candidates for poles are $z_1 = 0$ and $z_2 = \pi$. Because $\lim_{z\to\pi} (\pi - z) f(z) = \lim_{z\to\pi} \frac{\sin z}{z^2} = \frac{\sin\pi}{\pi^2} = 0$, z_2 is a removable singularity. We have

$$\lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{\sin z}{z(\pi - z)} = \frac{1}{\pi} \lim_{z\to 0} \frac{\sin z}{z} = \frac{1}{\pi}$$

Thus $z_1 = 0$ is a pole of order one of f. Then $\operatorname{Res}_{z=0} f(z) = \lim_{z\to 0} z f(z)$

$$= \frac{1}{\pi}.$$

(c) $$f(z) = \frac{z e^{iz}}{(z-\pi)^2}$$

The potential pole is $z=\pi$. Since $\lim\limits_{z\to\pi}(z-\pi)^2 f(z) = \lim\limits_{z\to\pi} z e^{iz} = \pi e^{i\pi} = -\pi$,

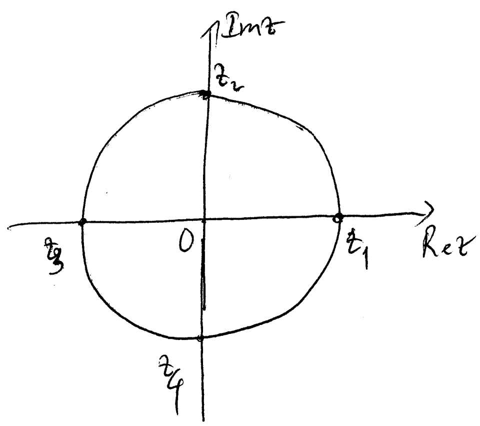
π is a pole of order two of $f(z)$. Thus,

$$\operatorname{Res}_{z=\pi} f(z) = \frac{d}{dz}\left((z-\pi)^2 f(z)\right)\Big|_{z=\pi} = \frac{d}{dz}\left(z e^{iz}\right)\Big|_{z=\pi} = \left(e^{iz} + iz e^{iz}\right)\Big|_{z=\pi}$$
$$= e^{i\pi} + i\pi e^{i\pi}$$
$$= -1 - i\pi.$$

(d) $$f(z) = \frac{z^3+5}{(z^4-1)}(z+1)$$ okay!

We have $$f(z) = \frac{z^4+5z}{z^4-1} + 1 = \frac{(z^4-1)+(5z+1)}{z^4-1} + 1 = \frac{5z+1}{z^4-1} + 2$$

The equation $z^4-1=0$ gives four distinct roots $z_1 = 1$, $z_2 = i$, $z_3 = -1$, $z_4 = -i$, which are the 4^{th} roots of $+1$. Since all of them are not zeros of $5z+1$,



they are simple poles of f. We have

$$f(z) = \frac{5z+1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} + 2$$

As the picture suggests, we have the differences

$z_1 - z_2 = 1-i$; $z_1 - z_3 = 2$; $z_1 - z_4 = 1+i$;

$z_2 - z_1 = -1+i$; $z_2 - z_3 = 1+i$; $z_2 - z_4 = 2i$;

$z_3 - z_1 = -2$; $z_3 - z_2 = -1-i$; $z_3 - z_4 = -1+i$;

$z_4 - z_1 = -1-i$; $z_4 - z_2 = -2i$; $z_4 - z_3 = 1-i$.

Thus, $(z_1-z_2)(z_1-z_3)(z_1-z_4) = (1-i)2(1+i) = (1^2-i^2)2 = 4$,

$(z_2-z_1)(z_2-z_3)(z_2-z_4) = (-1+i)(1+i)2i = (i^2-1)2i = -4i$,

$(z_3-z_1)(z_3-z_2)(z_3-z_4) = -2(-1-i)(-1+i) = -2(1^2-i^2) = -4$,

$(z_4-z_1)(z_4-z_2)(z_4-z_3) = (-1-i)(-2i)(1-i) = -2i(i^2-1^2) = 4i$.

Then we get

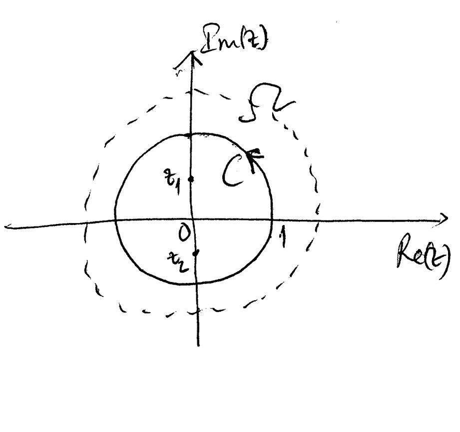
$$\operatorname{Res}_{z=z_1} f(z) = \lim_{z\to z_1}(z-z_1)f(z) = \frac{5z_1+1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} = \frac{6}{4} = \frac{3}{2},$$

$$\operatorname{Res}_{z=z_2} f(z) = \frac{5z_2+1}{(z_2-z_1)(z_2-z_3)(z_2-z_4)} = \frac{5i+1}{-4i} = \frac{1}{4}(-5+i),$$

$$\operatorname{Res}_{z=z_3} f(z) = \frac{5z_3+1}{(z_3-z_1)(z_3-z_2)(z_3-z_4)} = \frac{-4}{-4} = 1,$$

$$\operatorname{Res}_{z=z_4} f(z) = \frac{5z_4+1}{(z_4-z_1)(z_4-z_2)(z_4-z_3)} = \frac{-5i+1}{4i} = \frac{1}{4}(-5-i).$$

(2) (a) We'll compute the integral over C, the unitcircle traversed counter-clockwise, of $f(z) = \dfrac{e^{\pi}z}{4z^2+1}$.



The (simple) poles of $f(z)$ are the roots of the equation $4z^2+1=0$., ~~and the~~ namely $z_1 = \frac{i}{2}$ and $z_2 = -\frac{i}{2}$. Since z_1 and z_2 lie in the same connected component on the plane determined by C, we have $n(C,z_1)=n(C,z_2)=n(C,0)=1$. Let Ω is a disk of radius a bit larger than 1. Then C is homologaous with respect to Ω. Thus,

$$(2\pi i)^{-1}\int_C f(z)\,dz = \operatorname{Res}_{z=z_1} f(z) + \operatorname{Res}_{z=z_2} f(z).$$

Since z_1 and z_2 are simple poles, we have

$$\operatorname{Res}_{z=z_1} f(z) = \lim_{z\to z_1}(z-z_1)f(z) = \frac{e^{\pi}z_1}{4(z_1-z_2)} = \frac{e^{\pi}z_1}{4.2z_1} \quad (\text{note that } z_2=-z_1)$$

$$= \frac{e^{\pi}}{8}.$$

Similarly, $$\operatorname{Res}_{z=z_2} f(z) = \lim_{z\to z_2}\frac{e^{\pi}z_2}{4(z_2-z_1)} = \lim\frac{e^{\pi}z_2}{4(z_2-z_1)} = \frac{e^{\pi}z_2}{4.2z_2} = \frac{e^{\pi}}{8}.$$

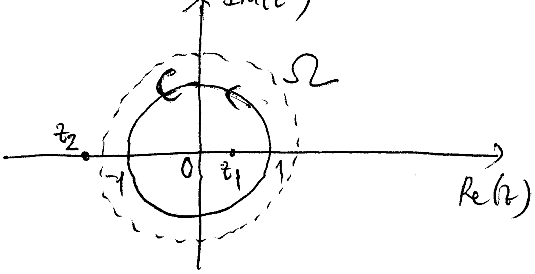
Thus $$(2\pi i)^{-1}\int_C f(z)\,dz = \frac{e^{\pi}}{8}+\frac{e^{\pi}}{8} = \frac{e^{\pi}}{4} \quad\text{and}\quad \int_C f(z)\,dz = \frac{e^{\pi}2\pi i}{4} = \frac{\pi e^{\pi}}{2}i$$

(b) We'll compute $\int_C f(z)\,dz$ where C is the same path as above and

$$f(z)=\frac{e^z}{\left(z^2+z-\frac{3}{4}\right)^2}.$$

The equation $z^2+z-\frac{3}{4}=0$ gives us two distinct roots $z_1=\frac{1}{2}$ and $z_2=-\frac{3}{2}$.

None of them vanishes e^z. Thus z_1 and z_2 are poles of order two of $f(z)$.



Let Ω be the disk centered at 0 of radius $\frac{5}{4}$. Then z_2 lies outside Ω, C is homologous to 0 with respect to Ω, and $n(C,z_1)=1$.

Therefore, $(2\pi i)^{-1}\int_C f\,dz = \operatorname{Res}_{z=z_1} f(z)$.

Since z_1 is a pole of order two, we have $\operatorname{Res}_{z=z_1} f(z) = \frac{d}{dz}\left((z-z_1)^2 f(z)\right)\Big|_{z=z_1}$.

We have $f(z) = \dfrac{e^z}{(z-z_1)^2(z-z_2)^2}$. Thus,

$$\frac{d}{dz}\left((z-z_1)^2 f(z)\right) = \frac{d}{dz}\left(\frac{e^z}{(z-z_2)^2}\right) = \frac{e^z(z-z_2)^2 - e^z 2(z-z_2)}{(z-z_2)^4} = \frac{e^z(z-z_2)-2e^z}{(z-z_2)^3}$$

$$= e^z \frac{z-2-z_2}{(z-z_2)^3}$$

Hence, $\operatorname{Res}_{z=z_1} f(z) = e^{z_1}\dfrac{z_1-2-z_2}{(z_1-z_2)^3} = 0$ because $z_1 - z_2 = 2$.

Thus $\int_C f\,dz = 0$. A consequence of this nice accidental result is that f has an antiderivative in Ω.

(3') We will find the following integral $I = \int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$.

First, we rename θ by t because we wish to consider I as the definition form of the ~~integral~~ line-integral of some complex function:

$$I = \int_0^{2\pi} \frac{dt}{2+\sin t}$$

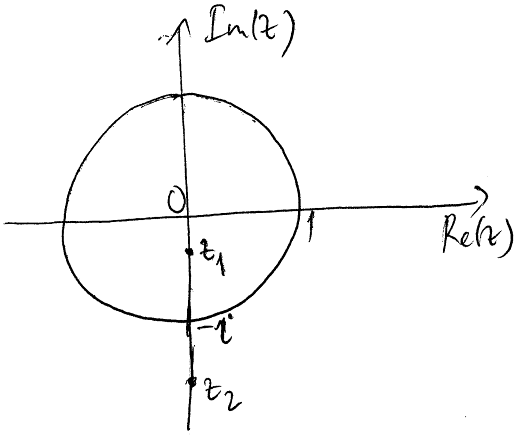
The interval $[0, 2\pi]$ hints us to choose the path γ as the unit circle, i.e. $\gamma(t) = e^{it}$. With this path, we want to find a complex function $f(z)$ such that $f(\gamma(t))\,\gamma'(t) = \dfrac{1}{2+\sin t}$, or equivalently $f(\gamma(t))\,d(\gamma(t)) = \dfrac{dt}{2+\sin t}$

Put $z = \gamma(t) = e^{it}$, we have $\sin t = \dfrac{z^2-1}{2iz}$ and $dt = \dfrac{dz}{iz}$. Thus,

$$\frac{dt}{2+\sin t} = \frac{\frac{dz}{iz}}{2+\frac{z^2-1}{2iz}} = \frac{dz}{2iz+\frac{z^2-1}{2}} = \frac{2}{z^2+4iz-1}\,dz.$$

Thus we choose $f(z) = \dfrac{2}{z^2+4iz-1}$. Then

$$I = \int_0^{2\pi} f(\gamma(t))\,\gamma'(t)\,dt = \int_\gamma f(z)\,dz.$$



The equation $z^2+4iz-1=0$ has discriminant $\Delta' = (2i)^2+1 = -3$. Thus it has two distinct complex solutions $z_1 = -2i+\sqrt{3}i = (-2+\sqrt{3})i$ and $z_2 = -2i-\sqrt{3}i = (-2-\sqrt{3})i$. Then we can write $z^2+4iz-1 = (z-z_1)(z-z_2)$.

Thus f has only one pole enclosed by γ. This pole is z_1, which is a simple pole and $n(\gamma, z_1) = 1$. Thus,

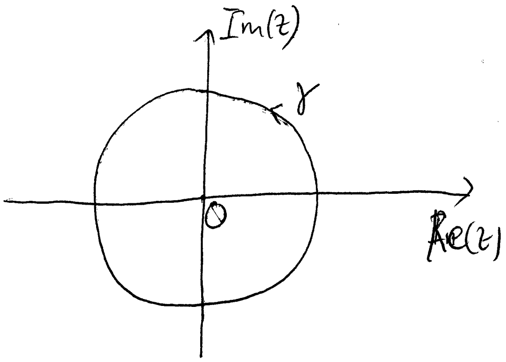
$$I = \int_\gamma f(z)\,dz = 2\pi i \operatorname{Res}_{z=z_1} f(z) = 2\pi i \operatorname{Res}_{z=z_1} \frac{2}{(z-z_1)(z-z_2)}$$

$$= 2\pi i \left.\frac{2}{z-z_2}\right|_{z=z_1}$$

$$= \frac{4\pi i}{z_1 - z_2} = \frac{4\pi i}{2\sqrt{3}\, i} = \frac{2\sqrt{3}}{3}\pi$$

④ Problem 1, Ahlfors p. 154

We will find the number of roots of the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ in the disk $|z| < 1$.



Put $f(z) = z^7 - 2z^5 + 6z^3 - z + 1$. Then f is analytic everywhere in the plane. Take γ to be the unit circle $\gamma(t) = e^{i2\pi t}$, for $0 \le t \le 2\pi$. We want to find the number of zeros of f enclosed by γ. To do so, we put $P(z) = 6z^3$. On γ, we have

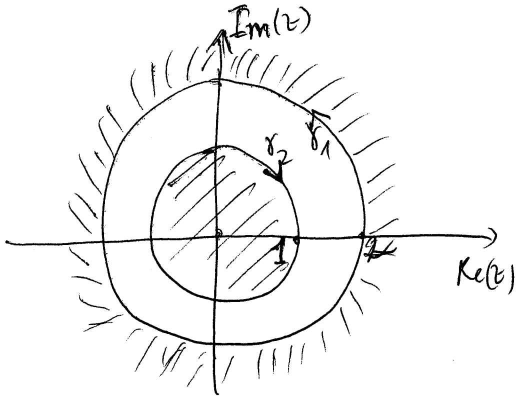
$$|P(z) - f(z)| = |z^7 - 2z^5 - z + 1| \le |z|^7 + 2|z|^5 + |z| + 1 = 5 < \underset{|P(z)|.}{\underset{\|}{6}}$$

Thus, by Rouché's theorem, f and P have the same number of zeros in the disk $|z| < 1$. Since the equation $P(z) = 0$ gives a triple solution $z = 0$, this number is 3. Therefore f has exactly 3 roots in $|z| < 1$.

5/3 ⑤ Problem 2, Ahlfors p. 154

We'll find the number of roots of the equation $z^4 - 6z + 3 = 0$ in the annulus $\mathrm{Ann}(0, 1, 2) = \{z : 1 < |z| < 2\}$.

We'll analyse the problem first, then solve it.



Put $f(z) = z^4 - 6z + 3$. We want to find the number of zeros of f in $Ann(0,1,2)$. To apply Rouché's theorem, we need a cycle γ on which we have some inequality.

The only cycle that we can choose is exactly to boundary of $Ann(0,1,2)$, which is composed of two circles. Let γ_1 be the circle of radius 2 directed counterclockwise $\gamma_1(t) = 2e^{i2\pi t}$ and γ_2 be the circle of radius 1 directed clockwise $\gamma_2(t) = e^{-i2\pi t}$. Then

$$n(\gamma_2, a) = \begin{cases} -1, & a \in \text{disk } |z|<1 \\ 0, & a \text{ in } Ann(0,1,2) \\ 0, & a \text{ in } |z|>2 \end{cases}$$

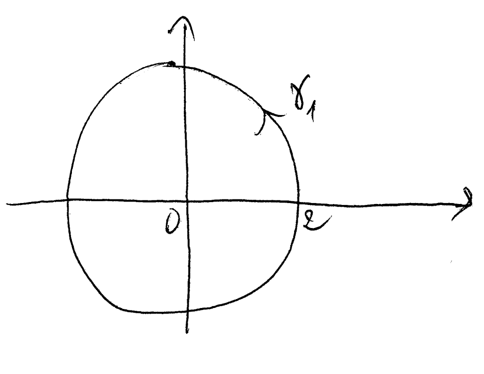
$$n(\gamma_1, a) = \begin{cases} 1, & a \text{ in the disk } |z|<1 \\ 1, & a \text{ in } Ann(0,1,2) \\ 0, & a \text{ in } |z|>2 \end{cases}$$

Therefore, if we put $\gamma = \gamma_1 + \gamma_2$ then

$$n(\gamma, a) = \begin{cases} 0 & \text{if } a \text{ in } |z|<1 \\ 1 & \text{if } a \text{ in } Ann(0,1,2) \\ 0 & \text{if } a \text{ in } |z|>2 \end{cases}$$

Therefore, γ is eligible for Rouché's theorem (in other words, the theorem can apply to γ). The domain (region) enclosed by γ is $\mathrm{Ann}(0,1,2)$ because $n(\gamma, a) = 1$ holds only if $a \in \mathrm{Ann}(0,1,2)$. To use Rouché's theorem, we need to find an analytic function $g(z)$ such that $|f(z) - g(z)| < |g(z)|$ on γ. Thus, we have 2 inequalities "$|f(z) - g(z)| < |g(z)|$ for $|z| = 1$" and "$|f(z) - g(z)| < |g(z)|$ for $|z| = 2$". The problem is that it's hard to find a function $g(z)$ satisfying both inequalities. It seems to be not a good idea to compose γ_1 and γ_2 into one cycle γ and then try to apply Rouché's theorem for γ.

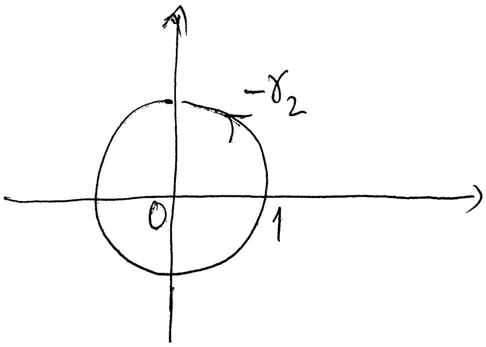
Instead, we'll treat γ_1 and γ_2 separately. By applying the theorem to $-\gamma_2$, we'll find the number of zeros of $f(z)$ in $|z| < 1$. By applying it to γ_1, we'll find the number of zeros in $|z| < 2$. Then by subtraction, with a caution on γ_2, we get the number of zeros in $\mathrm{Ann}(0,1,2)$.



Put $P_1(z) = z^4$. We have

$$|f(z) - P_1(z)| = |-6z+3| \le 6|z| + 3 = 15 < \underset{|P_1(z)|^{*}}{\underset{\parallel}{16}}$$

Thus # of zeros of f in $|z|<2$ is equal to # of zeros of P_1 in $|z|<2$, which is 4.



Put $P_2(z) = -6z$. We have

$$|f(z) - P_2(z)| = |z^4+3| \leq |z|^4 + 3 = 4 < 6 = |P_2(z)|$$

Thus we have 2 conclusions:

- f has no zeros on γ_2,
- # of zeros of f in $|z|<1$ is equal to # of zeros of P_2 in $|z|<1$, which is 1.

Thus, # of zeros of $f(z)$ in $|z| \leq 1$ is 1. Therefore, $f(z)$ has exactly $4-1=3$ zeros in the annulus $\mathrm{Ann}(0,1,2)$.

⑥ Problem 1, Ahlfors p. 161.

(b) We'll find poles and residues of the following function

$$f(z) = \frac{1}{(z^2-1)^2}$$

We have $f(z) = \dfrac{1}{(z-1)^2(z+1)^2}$. Thus f has two poles -1 and 1, each of which has order two. We have

$$\mathrm{Res}_{z=1} f(z) = \frac{d}{dz}\left((z-1)^2 f(z)\right)\Big|_{z=1} = \frac{d}{dz}\frac{1}{(z+1)^2}\Big|_{z=1} = \frac{-2}{(z+1)^3}\Big|_{z=1} = \frac{-2}{2^3} = -\frac{1}{4}$$

$$\mathrm{Res}_{z=-1} f(z) = \frac{d}{dz}\left((z+1)^2 f(z)\right)\Big|_{z=-1} = \frac{d}{dz}\frac{1}{(z-1)^2}\Big|_{z=-1} = \frac{-2}{(z-1)^3}\Big|_{z=-1} = \frac{-2}{-8} = \frac{1}{4}$$

(e) We'll find all poles and residues of the following function

$$f(z) = \frac{1}{\sin^2 z}$$

Write $z = x+iy$. We'll solve the equation $\sin(x+iy) = 0$ for x and y.

$$0 = \sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

where $\sinh(y) = \frac{e^y - e^{-y}}{2}$ and $\cosh(y) = \frac{e^y + e^{-y}}{2}$. The equation is equivalent to

$$\begin{cases} \sin x \cosh y = 0 \\ \cos x \sinh y = 0 \end{cases} \iff \begin{cases} \sin x = 0 \\ \sinh y = 0 \end{cases} \iff \begin{cases} x = n\pi \text{ for } n \in \mathbb{Z} \\ y = 0 \end{cases}$$

Thus, the ~~zeros~~ poles of $f(z)$ are $z_n = n\pi$ for any $n \in \mathbb{Z}$. Since $\sin(z-n\pi) = (-1)^n \sin z$, we get $\sin^2(z-n\pi) = \sin^2 z$ and thus

$$\lim_{z\to z_n}(z-z_n)^2 f(z) = \lim_{z\to n\pi}(z-n\pi)^2\frac{1}{\sin^2 z} = \lim_{z\to n\pi}\frac{(z-n\pi)^2}{\sin^2(z-n\pi)} = \lim_{w\to 0}\frac{w^2}{\sin^2 w} = 1$$

Hence $n\pi$ is a pole of order 2. Then

$$\mathrm{Res}_{z=n\pi} f(z) = \frac{d}{dz}\left((z-n\pi)^2 f(z)\right)\Big|_{z=n\pi} = \frac{d}{dz}\left(\frac{(z-n\pi)^2}{\sin^2 z}\right)\Big|_{z=n\pi}$$

$$= \left((z-n\pi)^2(-2)\cos z \sin^{-3} z + 2(z-n\pi)\sin^{-2} z\right)\Big|_{z=n\pi}$$

$$= \frac{-2\cos z\,(z-n\pi)^2 + 2(z-n\pi)\sin z}{\sin^3 z}\Big|_{z\to n\pi}$$

$$= \frac{-2\cos z\, 2(z-n\pi) + 2\sin z\,(z-n\pi)^2 + 2\sin z + 2(z-n\pi)\cos z}{3\cos z \sin^2 z}\Bigg|_{z\to n\pi} \quad \text{(L'Hospital)}$$

$$= \frac{-2(\cos z)(z-n\pi) + 2\sin z\left[(z-n\pi)^2+1\right]}{3\cos z\sin^2 z}\Bigg|_{z\to n\pi}$$

$$= \frac{2(\sin z)(z-n\pi) - 2\cos z + 2\cos z\left[(z-n\pi)^2+1\right] + 2\sin z\cdot 2(z-n\pi)}{-3\sin^3 z + 6\cos^2 z\sin z}\Bigg|_{z\to n\pi} \quad \text{(L'Hospital)}$$

$$= \frac{6(\sin z)(z-n\pi) + 2(\cos z)(z-n\pi)^2}{(-3\sin^2 z + 6\cos^2 z)\sin z}\Bigg|_{z\to n\pi}$$

$$= \left(\lim_{z\to n\pi}\frac{z-n\pi}{\sin z}\right)\left(\lim_{z\to n\pi}\frac{6\sin z + 2(z-n\pi)\cos z}{-3\sin^2 z + 6\cos^2 z}\right)$$

$$= 1.0$$

$$= 0$$

(f) We'll find the poles and residues of the following function

$$f(z) = \frac{1}{z^m(1-z)^n} \quad (m, n \text{ positive integers})$$

We see that f has two poles, $z=0$ of order m and $z=1$ of order n. We have

$$\operatorname{Res}_{z=0} f(z) = \cancel{\lim}\ \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\frac{1}{(1-z)^n}\Bigg|_{z=0}$$, which is equal to 1 if $m=1$.

We'll show by induction on $k \geq 1$ that

$$\frac{d^k}{dz^k}\left(\frac{1}{(1-z)^n}\right) = n(n+1)\cdots(n+k-1)\frac{1}{(1-z)^{n+k}} \quad (*)$$

For $k=1$, we have $\frac{d}{dz}\left(\frac{1}{(1-z)^n}\right) = \frac{d}{dz}\left((1-z)^{-n}\right) = (-n)(-1)(1-z)^{-n-1} = n\frac{1}{(1-z)^{n+1}}$.

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Suppose that (*) is true for k. We'll show that it's true for $k+1$.

$$\frac{d^{k+1}}{dz^{k+1}}\left(\frac{1}{(1-z)^n}\right) = \frac{d}{dz}\left(\frac{d^k}{dz^k}\frac{1}{(1-z)^n}\right) = \frac{d}{dz}\left(n(n+1)\cdots(n+k-1)\frac{1}{(1-z)^{n+k}}\right)$$

$$= n(n+1)\cdots(n+k-1)[-(n+k)](-1)\,(1-z)^{-n-k-1}$$

$$= n(n+1)\cdots(n+k)\frac{1}{(1-z)^{n+k+1}}$$

Thus (*) is also true for $k+1$, and hence it's true for every $k \geq 1$.

For $k=m-1$, in particular, we get $\dfrac{d^{m-1}}{dz^{m-1}}\dfrac{1}{(1-z)^n} = (n)(n+1)\cdots(n+m-2)\dfrac{1}{(1-z)^{n+m-1}}$

thus,
$$\operatorname{Res}_{z=0} f(z) = \frac{1}{(m-1)!}\, n(n+1)\cdots(n+m-2)\frac{1}{(1-z)^{n+m-1}}\bigg|_{z=0}$$

$$= \frac{n(n+1)\cdots(n+m-2)}{(m-1)!} \qquad \text{for } m \geq 2$$

We have
$$\operatorname{Res}_{z=1} f(z) = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-1)^n f(z)\right]_{z=1}$$

$$= \frac{(-1)^n}{(n-1)!}\left(\frac{d^{n-1}}{dz^{n-1}}\frac{1}{z^m}\right)\bigg|_{z=1}$$

Put $w = 1-z$, (*) gives us

$$(-1)^k \frac{d^k}{dw^k}\frac{1}{w^n} = n(n+1)\cdots(n+k-1)\frac{1}{w^{n+k}}$$

Replacing n by m, and k by $n-1$, we get

$$(-1)^{n-1}\frac{d^{n-1}}{dw^{n-1}}\frac{1}{w^m} = m(m+1)\cdots(m+n-2)\frac{1}{w^{m+n-1}}$$

Thus, $$\operatorname{Res}_{z=1} f(z) = \frac{(-1)^n}{(n-1)!}(-1)^{n-1} m(m+1)\cdots(m+n-2)\frac{1}{z^{m+n-1}}\Big|_{z=1}$$

$$= -\frac{m(m+1)\cdots(m+n-2)}{(n-1)!} \quad \text{for } n \geq 2$$

For $n=1$, $\operatorname{Res}_{z=1} f(z) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{-1}{z^m} = -1.$

(7) Problem 3, Ahlfors p. 161

5/5 We'll evaluate the following integrals by the method of residues.

(c) $I = \int_{-\infty}^{\infty} f(x)\,dx$ where $f(x) = \dfrac{x^2-x+2}{x^4+10x^2+9}$.

Since $f(z) = \dfrac{z^2-z+2}{z^4+10z^2+9}$ is a rational function ~~on the plan~~ and has no poles on the real line, I can be computed through the residues of $f(z)$. But first we should find the poles of $f(z)$.

$$z^4+10z^2+9 = 0$$
$$\iff (z^2+5)^2-4^2 = 0$$
$$\iff (z^2+1)(z^2+9) = 0$$
$$\iff z=\pm i \text{ or } z=\pm 3i$$

Therefore $f(z)$ has four poles $\pm i, \pm 3i$, each of which are simple. We can write

$$f(z) = \frac{z^2-z+2}{(z-i)(z+i)(z-3i)(z+3i)}$$

Therefore,

$$\operatorname{Res}_{z=i} f(z) = \left.\frac{z^2-z+2}{(z+i)(z-3i)(z+3i)}\right|_{z=i} = \frac{i^2-i+2}{2i(-2i)(4i)} = \frac{1-i}{16i} = \frac{i+1}{-16} = -\frac{1}{16}-\frac{1}{16}i$$

$$\operatorname{Res}_{z=-i} f(z) = \left.\frac{z^2-z+2}{(z-i)(z-3i)(z+3i)}\right|_{z=-i} = \frac{i^2+i+2}{(-2i)(-4i)(2i)} = \frac{1+i}{-16i} = \frac{i-1}{16} = \frac{-1}{16}+\frac{1}{16}i$$

$$\operatorname{Res}_{z=3i} f(z) = \left.\frac{z^2-z+2}{(z-i)(z+i)(z+3i)}\right|_{z=3i} = \frac{9i^2-3i+2}{2i\cdot 4i\cdot 6i} = \frac{-7-3i}{-48i} = \frac{-7i+3}{48} = \frac{3}{48}-\frac{7}{48}i$$

$$\operatorname{Res}_{z=-3i} f(z) = \left.\frac{z^2-z+2}{(z-i)(z+i)(z-3i)}\right|_{z=-3i} = \frac{9i^2+3i+2}{-4i(-2i)(-6i)} = \frac{-7+3i}{48i} = \frac{-7i-3}{-48} = \frac{3}{48}+\frac{7}{48}i$$

Then $I = 2\pi i(\operatorname{Res}_{z=i} + \operatorname{Res}_{z=3i}) = 2\pi i\left(-\frac{10}{48}i\right) = \frac{5\pi}{12}$ ✓

(d) $I = \int_0^\infty f(x)\,dx$, where $f(x) = \dfrac{x^2}{(x^2+a^2)^3}$, a is real.

For $a=0$, we have $f(x) = \dfrac{x^2}{x^6} = \dfrac{1}{x^4}$. Then $I = \int_0^\infty \frac{1}{x^4}\,dx = \infty$.

For $a \neq 0$, we can assume $a>0$ ✓ without loss of generality. Since $f(z) = \dfrac{z^2}{(z^2+a^2)^3}$ is a rational function such that the numerator's degree is two less than the denominator's degree, and $f(z)$ has no zeros on the real line, I can be computed via the residue of $f(z)$. We have

$$f(z) = \frac{z^2}{(z-ia)^3(z+ia)^3}$$

Thus the pole in the upper half domain is ia, which is of order three. Thus

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \operatorname{Res}_{z=ia} f(z) = 2\pi i \frac{1}{2!}\frac{d^2}{dz^2}\frac{z^2}{(z+ia)^3}\Big|_{z=ia}$$

$$= \pi i \frac{d^2}{dz^2}\left(z^2(z+ia)^{-3}\right)\Big|_{z=ia}$$

$$= \pi i \frac{d}{dz}\left(2z(z+ia)^{-3} - 3z^2(z+ia)^{-4}\right)_{z=ia}$$

$$= \pi i\left(2(z+ia)^{-3} - 6z(z+ia)^{-4} - 6z(z+ia)^{-4} + 12z^2(z+ia)^{-5}\right)_{z=ia}$$

$$= \pi i\left(2(2ia)^{-3} - 6(ia)(2ia)^{-4} - 6(ia)(2ia)^{-4} + 12(ia)^2(2ia)^{-5}\right)$$

$$= \pi i (ia)^{-3}\left(2.2^{-3} - 6.2^{-4} - 6.2^{-4} + 12.2^{-5}\right)$$

$$= \frac{\pi i}{i^3 a^3}\left(\frac{1}{4} - \frac{3}{8} - \frac{3}{8} + \frac{3}{8}\right)$$

$$= -\frac{\pi}{a^3}\left(-\frac{1}{8}\right)$$

$$= \frac{\pi}{8a^3}$$

Since f is an even function, $I = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{16a^3}$ ✓

(e) $$I = \int_0^{\infty} \frac{\cos x}{x^2+a^2}\,dx$$

If $a=0$ then $I = \int_0^{\infty} \frac{\cos x}{x^2}\,dx$. On the interval $x \in (0, \frac{\pi}{3})$, we have $\cos x \geq \frac{1}{2}$. Thus $\frac{\cos x}{x^2} \geq \frac{1}{2x^2}$ and hence $\int_0^{\pi/3} \frac{\cos x}{x^2}\,dx \geq \int_0^{\pi/3} \frac{1}{2x^2}\,dx = \infty$

Therefore, I diverges.

For $a \neq 0$, we can assume $a > 0$ without loss of generality. We see that

$$J = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\,dx = \int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx + i\int_{-\infty}^{\infty} \frac{\sin x}{x^2+a^2}\,dx$$

Thus $\quad Re(J) = \int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx = 2\int_{0}^{\infty} \frac{\cos x}{x^2+a^2}\,dx$ (since the integrand is even)

$$= 2I$$

Hence, we'll evaluate J instead of I since J is of the form we know how to deal with. The function $f(z) = \frac{1}{z^2+a^2}$ has only one pole in the upper half plane $z = ia$. Thus,

$$J = 2\pi i \, \mathrm{Res}_{z=ia} f(z).e^{iz} = 2\pi i \left.\frac{e^{iz}}{z+ia}\right|_{z=ia} = 2\pi i \frac{e^{-a}}{2ia} = \frac{\pi}{a} e^{-a}$$

Therefore, $\quad I = \frac{1}{2} Re(J) = \frac{\pi}{2a} e^{-a}$ ✓

(g) $I = \int_0^{\infty} \frac{x^{1/3}}{1+x^2}\,dx$

Put $x = t^3$, then $dx = 3t^2 dt$. We get $I = \int_0^{\infty} \frac{t}{1+t^6} 3t^2 dt = \int_0^{\infty} \frac{3t^3}{1+t^6}\,dt$.

Put $u = t^2$, then $du = 2t\,dt$. We get

$$I = \frac{1}{2}\int_0^{\infty} \frac{3t^2}{1+t^6} 2t\,dt = \frac{3}{2}\int_0^{\infty} \frac{u}{1+u^3}\,du$$

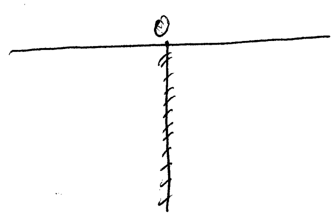
Here we're facing difficulty because what we want is an integral over the whole real line, not the positive half like this. Even the change of variable $v=-u$ doesn't help. We should follow a different approach.

Put $f(z) = \frac{1}{1+z^2}$. Then $f(z)$ is analytic everywhere on the plane except at two poles $z=i$ and $z=-i$. We have $I = \int_0^\infty x^{1/3} f(x)\,dx$.

To relate I to an integral over $(-\infty, \infty)$, we should find an analytic extension of $x^{1/3}$ on the upper half plane and the negative real line. By definition,

$$x^{1/3} := \exp\left(\tfrac{1}{3}\log x\right)$$

Thus, we need an extension of $\log x$ to the upper half plane and the negative real line. $\log z$ is not a good choice because it's not analytic on the negative real line. We choose $g(z) = \log(-iz) + i\frac{\pi}{2}$.



Then $g(z)$ coincides $\log x$ on the positive real ray and $g(z)$ is analytic on the domain we want.

Then $h(z) = \exp\left(\frac{1}{3} g(z)\right)$ is the extension of $x^{1/3}$ that we want. For $y>0$, we have $g(-y) = \log(iy) + i\frac{\pi}{2} = \log y + i\pi = \log(-iy) + i\frac{\pi}{2} + i\pi = g(+y) + i\pi$.

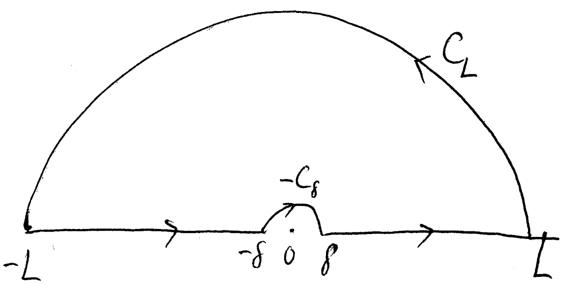
Thus,
$$h(-y) = \exp\left(\tfrac{1}{3} g(-y)\right) = \exp\left(\tfrac{1}{3} g(y) + \tfrac{i\pi}{3}\right) = \exp\left(\tfrac{i\pi}{3}\right)\exp\left(\tfrac{1}{3} g(y)\right)$$
$$= \exp\left(\tfrac{i\pi}{3}\right) h(y).$$

Using the fact that $f(z)$ is even, we get
$$\int_{-\infty}^{0} h(x) f(x)\,dx \overset{y=-x}{=\!=} \int_0^\infty h(-y) f(-y)\,dy = \exp\left(\tfrac{i\pi}{3}\right)\int_0^\infty h(y) f(y)\,dy = \exp\left(\tfrac{i\pi}{3}\right) I$$

Thus, $\text{pr.v.} \int_{-\infty}^{\infty} h(z) f(z)\, dz = \int_{0}^{\infty} h(x) f(x)\, dx + \int_{-\infty}^{0} h(x) f(x)\, dx$

$$= I + \exp\left(\frac{i\pi}{3}\right) I$$

$$= I\left(1 + \exp\left(\frac{i\pi}{3}\right)\right) \qquad (1)$$



Put $\gamma_{L,\delta} = [-L,-\delta] + [\delta, L] - C_\delta + C_L$ as in the figure. We have

$$|h(z)| = \left|\exp\left(\tfrac{1}{3} g(z)\right)\right| = \left|\exp\left(\tfrac{1}{3}\log(-iz)\right)\right| = (\operatorname{Im} z)^{1/3} \le |z|^{1/3}$$

Thus, $\left|\int_{C_L} h(z) f(z)\, dz\right| \le \int_{C_L} |h(z)||f(z)|\,|dz| \le \int_{C_L} |z|^{1/3} \frac{1}{|1+z^2|}\,|dz|$

$$\le \int_{C_L} \frac{|z|^{1/3}}{|z|^2 - 1}\,|dz| = \frac{L^{1/3}}{L^2 - 1}\,\pi L \longrightarrow 0$$

as $L \to \infty$.

On the other hand $\left|\int_{-C_\delta} h(z) f(z)\, dz\right| \le \int_{-C_\delta} |z|^{1/3} \frac{1}{|1+z^2|}\,|dz|$

$$\le \int_{-C_\delta} |z|^{1/3} \frac{1}{1 - |z|^2}\,|dz| = \frac{\delta^{1/3}}{1 - \delta^2}\,\pi\delta \longrightarrow 0 \text{ as } \delta \to 0$$

Thus $\lim_{\substack{\delta \to 0 \\ L \to \infty}} \int_{\gamma_{L,\delta}} h(z) f(z)\, dz = \text{pr.v.} \int_{-\infty}^{\infty} h(z) f(z)\, dz \overset{(1)}{=\!=} I\left(1 + \exp\left(\frac{i\pi}{3}\right)\right). \quad (2)$

Since $h(z)$ has no pole in the ~~the~~ upper half plane and $f(z)$ has only one (simple) pole $z = i$ in the upper half plane, we get

$$\lim_{\substack{\varepsilon,\delta\to 0\\ L\to\infty}} \int_{\gamma_{L,\delta}} h(z)f(z)\,dz = 2\pi i \operatorname{Res}_{z=i} h(z)f(z)$$

$$= 2\pi i\, h(z)\frac{1}{z+i}\Big|_{z=i} = 2\pi i\, h(i)\frac{1}{2i}$$

$$= \pi \exp\left(\frac{1}{3}g(i)\right)$$

$$= \pi \exp\left(\frac{1}{3}\left(\log 1 + i\frac{\pi}{2}\right)\right) = \pi \exp\left(i\frac{\pi}{6}\right)$$

Then together with (2), we get

$$\pi \exp\left(i\frac{\pi}{6}\right) = I\left(1+\exp\left(\frac{i\pi}{3}\right)\right)$$

Thus

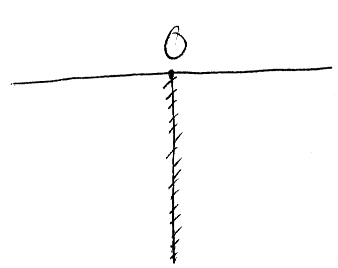
$$I = \pi\frac{\exp(i\pi/6)}{1+\exp(i\pi/3)} = \pi\frac{\sqrt{3}/2 + i\,1/2}{1+1/2+i\sqrt{3}/2} = \pi\frac{\sqrt{3}+i}{3+i\sqrt{3}} = \pi/\sqrt{3}.$$

Therefore,

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\,dx = \frac{\pi\sqrt{3}}{3}.$$

(h) $$I = \int_0^\infty \frac{\log x}{1+x^2}\,dx$$

The idea is similar to the above problem. We want to extend $\log x$ to an analytic function on the upper half plane and the negative real ray. Thus we choose $g(z) = \log(-iz) + i\frac{\pi}{2}$ as above.



For $y>0$, we have $g(-y) = g(y) + i\pi$.

Put $f(z) = \frac{1}{1+z^2}$. Using the fact that f is even, we get

$\boxed{22}$

$$\int_{-\infty}^{0} g(x) f(x)\, dx \underline{\underline{y=-x}} \int_0^{\infty} g(-y) f(y)\, dy$$

$$= \int_0^{\infty} (g(y) + i\pi) f(y)\, dy$$

$$= \int_0^{\infty} g(y) f(y)\, dy + i\pi \int_0^{\infty} \frac{dy}{1+y^2}$$

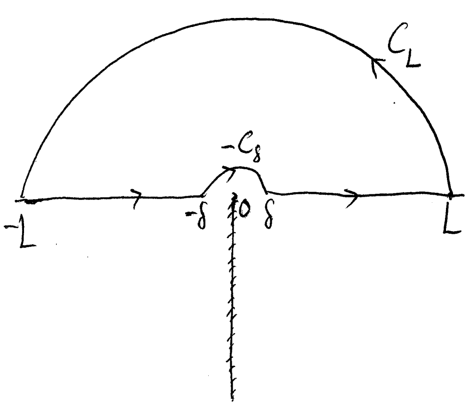
$$= I + i\pi \arctan(y) \Big|_{y=0}^{y\to\infty}$$

$$= I + i\frac{\pi^2}{2}$$

Thus,

$$\text{pr. v.} \int_{-\infty}^{\infty} g(z) f(z)\, dz = \int_0^{\infty} g(z) f(z)\, dz + \int_{-\infty}^{0} g(z) f(z)\, dz$$

$$= I + I + i\frac{\pi^2}{2} = 2I + i\frac{\pi^2}{2} \qquad (1)$$



Put $\gamma_{L\delta} = [-L,-\delta] - C_\delta + [\delta, L] + C_L$ as in the figure. We ~~get~~ have

$$|g(z)| = \left|\log(-iz) + i\frac{\pi}{2}\right|$$

$$= \left|\log(-ix+y) + i\frac{\pi}{2}\right|$$

$$= \left|\log|z| + i\left(\arg(-iz) + \frac{\pi}{2}\right)\right|$$

$$\leq \sqrt{(\log|z|)^2 + \left(\arg(-iz) + \frac{\pi}{2}\right)^2}$$

$$\leq \sqrt{(\log|z|)^2 + \frac{9\pi^2}{4}}$$

We have

$$\left| \int_{C_L} g(z) f(z)\, dz \right| \leq \int_{C_L} |g(z)||f(z)||dz| \leq \int_{C_L} \sqrt{(\log|z|)^2 + \frac{9\pi^2}{4}}\ \frac{1}{|z^2+1|}\, |dz|$$

$$\leq \int_{C_L} \sqrt{\log^2 L + \frac{9\pi^2}{4}}\ \frac{1}{L^2-1}\, |dz|$$

$$= \pi L \sqrt{\log^2 L + \frac{9\pi^2}{4}}\ \frac{1}{L^2-1}$$

$\to 0$ as $L \to \infty$ because the logarithmic function creeps to infinity slower than the identity function. On the other hand,

$$\left| \int_{-C_\delta} g(z) f(z)\, dz \right| \leq \int_{-C_\delta} |g(z)||f(z)||dz| \leq \int_{-C_\delta} \sqrt{(\log|z|)^2 + \frac{9\pi^2}{4}}\ \frac{1}{|1+z^2|}\, |dz|$$

$$\leq \int_{C_L} \sqrt{\log^2 \delta + \frac{9\pi^2}{4}}\ \frac{1}{1-\delta^2}\, |dz|$$

$$= \pi \delta \sqrt{\log^2 \delta + \frac{9\pi^2}{4}}\ \frac{1}{1-\delta^2}$$

$$= \frac{\pi}{1-\delta^2} \sqrt{(\delta \log \delta)^2 + \frac{9\pi^2}{4}\delta^2}$$

$\to 0$ as $\delta \to 0$ because $\delta \log \delta \to 0$ as $\delta \to 0$.

Thus,

$$\lim_{\substack{\delta \to 0 \\ L \to \infty}} \int_{\gamma_{L\delta}} g(z) f(z)\, dz = \text{pr.v.} \int_{-\infty}^{\infty} g(z) f(z)\, dz \overset{(1)}{=} 2I + i\frac{\pi^2}{2} \qquad (2).$$

Since $g(z)$ has no zero in the upper half plane and $f(z)$ has only one pole in this domain, namely $z = i$, which is simple, we have

$$\lim_{\substack{\delta \to 0 \\ L \to \infty}} \int_{\gamma_{L\delta}} g(z) f(z)\, dz = 2\pi i \operatorname{Res}_{z=i} g(z) f(z)$$

$$= 2\pi i \, g(z) \frac{1}{z+i} \Big|_{z=i} = 2\pi i \, g(i) \frac{1}{2i}$$

$$= \pi g(i) = \pi \left(\log 1 + i\frac{\pi}{2} \right) = i\frac{\pi^2}{2}$$

Together with (2), we get $i\frac{\pi^2}{2} = 2I + i\frac{\pi^2}{2}$. Then $I = 0$

Therefore $\displaystyle\int_0^\infty \frac{\log x}{1+x^2}\, dx = 0.$

completion: 8/8