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Math 8701: Complex Analysis
Problem Set 9

19/19

Bonus +5

① Problem 4, Ahlfors, p. 186.

5/5

Put $f(z) = \frac{1}{e^z - 1}$.

Then $f(z)$ has poles at $z = 0, \pm i2\pi, \pm i4\pi, \pm i6\pi, \ldots$ Thus $f(z)$ is analytic in the annulus $0 < |z| < 2\pi$. Hence, $f(z)$ admits a Laurent expansion in this annulus. We can write

$$f(z) = \sum_{n=\infty}^{\infty} A_n z^n \quad \text{where} \quad A_n = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^{n+1}} dz$$

We have
$$A_n = \frac{1}{2\pi i} \int_{|z|=1} \frac{z^{-n-1}}{e^z - 1} dz$$

- For $n \leq -2$, the integrand has a removable singularity at $z = 0$, and no other singularity enclosed in the unit circle. Thus $A_n = 0$ by Cauchy's Integral formula.

- For $n = -1$, $A_{-1} = \frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{e^z - 1} = \operatorname{Res}_{z=0} \left(\frac{1}{e^z - 1} \right)$

$$= \frac{z}{e^z - 1} \Big|_{z=0} = 1$$

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- For $n \geq 0$, then $z = 0$ is a pole of order $n+2$ of the function $\frac{z^{-n-1}}{e^z - 1}$.

Therefore,

$$A_n = \operatorname{Res}_{z=0}\left(\frac{z^{-n-1}}{e^z-1}\right) = \frac{1}{(n+1)!}\frac{d^{n+1}}{dz^{n+1}}\left(z^{n+2}\frac{z^{-n-1}}{e^z-1}\right)\Big|_{z=0}$$

$$= \frac{1}{(n+1)!}\frac{d^{n+1}}{dz^{n+1}}\left(\frac{z}{e^z-1}\right)\Big|_{z=0}$$

Put
$$h(z) = \begin{cases} \dfrac{z}{e^z-1} & \text{for } z \in \mathbb{C}\setminus\{0\} \\ 0 & \text{for } z=0 \end{cases}$$

Then $h(z)$ is an entire function. We have $A_n = \dfrac{h^{(n+1)}(0)}{(n+1)!}$ for $n \geqslant 0$.

$h(z)$ admits a Taylor expansion $h(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ with $a_n = \dfrac{h^{(n)}(0)}{n!}$.

Therefore $A_n = a_{n+1}$ for $n \geqslant 0$. Thus we need to find a_1, a_2, a_3, a_4, which correspond to A_0, A_1, A_2, A_3. We have

$$a_0 + a_1 z + a_2 z^2 + \cdots = h(z) = \frac{z}{e^z-1} = \frac{z}{\frac{z}{1!} + \frac{z^2}{2!} + \cdots} = \frac{1}{\frac{1}{1!} + \frac{z}{2!} + \frac{z^2}{3!} + \cdots}$$

Therefore
$$\left(\frac{1}{1!} + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \frac{z^4}{5!} + [z^5]\right)\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + [z^5]\right) = 1 \quad (*)$$

Equating the constant term gives $a_0 = 1$.

Equating the coefficient of z^1 gives $\frac{a_0}{2!} + \frac{a_1}{1!} = 0$. Thus $a_1 = -\frac{a_0}{2} = -\frac{1}{2}$.

Equating the coefficient of z^2 gives $\frac{a_0}{3!} + \frac{a_1}{2!} + \frac{a_2}{1!} = 0$. Thus

$$a_2 = -\left(\frac{a_0}{6} + \frac{a_1}{2}\right) = -\left(\frac{1}{6} - \frac{1}{4}\right) = \frac{1}{12}$$

We see that $h(-z) = \dfrac{-z}{e^{-z}-1} = \dfrac{-ze^{z}}{1-e^{z}} = \dfrac{ze^{z}}{e^{z}-1}$. Thus

$$h(-z)-h(z) = \frac{ze^{z}}{e^{z}-1} - \frac{z}{e^{z}-1} = z.$$ Thus we have

$$z = \left(a_0 - a_1 z + a_2 z^2 - a_3 z^3 + \cdots\right) - \left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots\right)$$

$$= -2a_1 z - 2a_3 z^3 - 2a_5 z^5 - \cdots$$

Thus $a_{2k+1} = 0$ for all $k \geq 1$. Thus, $A_{2k} = 0$ for all $k \geq 1$.

Equating the coefficient of z^4 at $(*)$ gives $\dfrac{a_0}{5!} + \dfrac{a_1}{4!} + \dfrac{a_2}{3!} + \dfrac{a_3}{2!} + \dfrac{a_4}{1!} = 0$

Thus $a_4 = -\dfrac{a_0}{5!} - \dfrac{a_1}{4!} - \dfrac{a_2}{3!} - \dfrac{a_3}{2!} = -\dfrac{1}{120} + \dfrac{1}{48} - \dfrac{1}{72} = -\dfrac{1}{720}$

Therefore, ~~we have~~ Equating the coefficient of z^6 at $(*)$ gives

$$\frac{a_0}{7!} + \frac{a_1}{6!} + \frac{a_2}{5!} + \underbrace{\frac{a_3}{4!}}_{0} + \frac{a_4}{3!} + \underbrace{\frac{a_5}{2!}}_{0} + \frac{a_6}{1!} = 0$$

Thus $$a_6 = -\frac{a_4}{3!} - \frac{a_2}{5!} - \frac{a_1}{6!} - \frac{a_0}{7!} = \frac{1}{3!}\left(\frac{1}{720} \underbrace{- \frac{1}{12\cdot 4\cdot 5} + \frac{1}{2\cdot 4\cdot 5\cdot 6}}_{0} - \frac{1}{4\cdot 5\cdot 6\cdot 7}\right)$$

$$= \frac{1}{3!}\left(\frac{1}{720} - \frac{1}{4\cdot 5\cdot 6\cdot 7}\right)$$

$$= \frac{1}{3!}\,\frac{1}{6\cdot 4\cdot 5\cdot 6\cdot 7} = \frac{1}{6\cdot 7!}$$

Therefore, $A_0 = -\frac{1}{2}$, $A_1 = \frac{1}{12}$, $A_3 = -\dfrac{1}{720} = -\dfrac{1}{6!}$, $A_5 = \dfrac{1}{6\cdot 7!}$ and $A_{2k} = 0$ for all $k \geq 1$.

We get $$f(z) = \sum_{n=-\infty}^{\infty} A_n z^n = A_{-1} z^{-1} + A_0 z^0 + A_1 z^1 + \cdots$$

$$= \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} A_{2k-1} z^{2k-1}$$

By putting $B_k = (-1)^{k-1}(2k)!\,A_{2k-1}$ for $k \geq 1$, we get

$$f(z) = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}, \text{ where}$$

$$B_1 = 2!\,A_1 = 2 \cdot \frac{1}{12} = \frac{1}{6},$$

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$$B_2 = -4!\,A_3 = -4!\left(-\frac{1}{6!}\right) = \frac{1}{30},$$

$$B_3 = 6!\,A_5 = 6!\,\frac{1}{6 \cdot 7!} = \frac{1}{42}.$$

(2) Problem 1, Ahlfors, p. 190.

We will compare the coefficients in the Laurent expansion of $\pi \cot \pi z$ to its expression as a sum of partial fractions

5\5

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2},$$

to find the values of $\sum_1^{\infty} \frac{1}{n^2}$, $\sum_1^{\infty} \frac{1}{n^4}$ and $\sum_1^{\infty} \frac{1}{n^6}$.

First, we'll find the Laurent expansion of $\cot z$ around $z=0$. We see that $f(z) = \cot z = \frac{\cos z}{\sin z}$ is a meromorphic function with (simple) poles at $z = k\pi$ for $k \in \mathbb{Z}$. Therefore $f(z)$ is analytic in the annulus $0 < |z| < \pi$ and admits a Laurent expansion around 0 in this annulus. We write

$$f(z) = \cot z = \sum_{n=-\infty}^{\infty} A_n z^n \quad \text{for all } 0 < |z| < \pi,$$

where $$A_n = \frac{1}{2\pi i}\int_{|z|=1} \frac{\cot z}{z^{n+1}}\,dz = \frac{1}{2\pi i}\int_{|z|=1} \frac{\cos z}{z^{n+1}\sin z}\,dz.$$

For $n \leq -2$, $z=0$ is a removable singularity of the above integrand (because $\frac{\sin z}{z}$ has a removable singularity at 0). Therefore $A_n = 0$ by Cauchy's Integral formula. For $n=-1$, we have

$$A_{-1} = \frac{1}{2\pi i}\int_{|z|=1} \frac{\cos z}{\sin z}\,dz = \operatorname{Res}_{z=0}\left(\frac{\cos z}{\sin z}\right) = \lim_{z\to 0} z\frac{\cos z}{\sin z} = 1.$$

Thus,
$$\cot z = \frac{1}{z} + A_0 + A_1 z + A_2 z^2 + \cdots.$$

Moreover, we see that $\cot z$ is an odd function, $A_0 = A_2 = A_4 = \cdots = 0$.

Thus, $\cot z = \frac{1}{z} + A_1 z + A_3 z^3 + A_5 z^5 + \cdots$ (1)

We'll try to find the first three coefficients A_1, A_3, A_5 by the following observation $\cos z = \sin z \cot z$, which is equivalent to

$$1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + [z^8] = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + [z^7]\right)\left(\frac{1}{z} + A_1 z + A_3 z^3 + A_5 z^5 + [z^7]\right)$$

for all $0 < |z| < \pi$. Equating the coefficients of z^2, we get

$$-\frac{1}{2!} = -\frac{1}{3!} + A_1 \qquad (2)$$

Equating the coefficients of z^4, we get $\frac{1}{4!} = A_3 - \frac{A_1}{3!} + \frac{1}{5!}$ (3)

Equating the coefficients of z^6, we get

$$-\frac{1}{6!} = A_5 - \frac{A_3}{3!} + \frac{A_1}{5!} - \frac{1}{7!} \qquad (4)$$

From (2), we obtain $A_1 = \frac{1}{6} - \frac{1}{2} = \frac{1-3}{6} = \frac{-1}{3}$.

From (3), we obtain $A_3 = \frac{1}{4!} + \frac{A_1}{3!} - \frac{1}{5!} = \frac{1}{24} - \frac{1}{18} - \frac{1}{120} = -\frac{1}{45}$,

From (4), we obtain $A_5 = \frac{A_3}{3!} - \frac{A_1}{5!} - \frac{1}{6!} + \frac{1}{7!} = -\frac{2}{945}$.

Now we replace z in (1) by πz to get

$$\cot \pi z = \frac{1}{\pi z} + A_1 \pi z + A_3 \pi^3 z^3 + A_5 \pi^5 z^5 + \cdots \quad \text{for } 0 < |z| < 1$$

Thus,
$$\pi \cot \pi z = \frac{1}{z} + A_1 \pi^2 z + A_3 \pi^4 z^3 + A_5 \pi^6 z^5 + \cdots \quad \text{for } 0 < |z| < 1$$

On the other hand, we know that $\pi \cot \pi z$ has an expression as a sum of partial fractions

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

Identifying the right hand members of the above two equations, we get

$$A_1 \pi^2 z + A_3 \pi^4 z^3 + A_5 \pi^6 z^5 + \cdots = \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \quad \text{for } 0 < |z| < 1.$$

Thus
$$A_1 \pi^2 + A_3 \pi^4 z^2 + A_5 \pi^6 z^4 + \cdots = \sum_{n=1}^{\infty} \frac{2}{z^2 - n^2} \quad \text{for } 0 < |z| < 1. \qquad (*)$$

Because both sides of (*) are analytic functions on the disk $|z|<1$, the identity (*) is also valid at $z=0$. We even know that the convergence of these two series is uniform on any disk $|z|<r$ with $r<1$.

Substituting $z=0$ into (*), we get

$$A_1 \pi^2 = \sum_{n=1}^{\infty} \frac{2}{-n^2}$$

Thus $$-\frac{\pi^2}{3} = -\sum_{n=1}^{\infty}\frac{2}{n^2},$$ which gives $$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Taking the derivative of both sides of (*), we get

$$2A_3\pi^4 z + 4A_5\pi^6 z^3 + \cdots = \sum_{n=1}^{\infty}\frac{-4z}{(z^2-n^2)^2} \quad \text{for all } |z|<1.$$

Thus, $$2A_3\pi^4 + 4A_5\pi^6 z^2 + \cdots = \sum_{n=1}^{\infty}\frac{-4}{(z^2-n^2)^2} \quad (**) \quad \text{for all } |z|<1.$$

Substituting $z=0$ into (**), we get

$$2A_3\pi^4 = \sum_{n=1}^{\infty}\frac{-4}{n^4}$$

Thus $$-\frac{2\pi^4}{45} = \sum_{n=1}^{\infty}\frac{-4}{n^4},$$ which gives $$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

Taking the derivative of both sides of (**), we get

$$8A_5\pi^6 z + [z^3] = \sum_{n=1}^{\infty}\frac{16z}{(z^2-n^2)^3} \quad \text{for all } |z|<1.$$

Thus, $$8A_5\pi^6 \cancel{+} [z^2] = \sum_{n=1}^{\infty}\frac{16}{(z^2-n^2)^3} \quad \text{for all } |z|<1.$$

Substituting $z=0$ into the above identity, we get

$$8A_5\pi^6 = \sum_{n=1}^{\infty} \frac{16}{-n^6}$$

Thus, $-\dfrac{16\pi^6}{945} = -\sum_{n=1}^{\infty}\dfrac{16}{n^6}$, which gives $\sum_{n=1}^{\infty}\dfrac{1}{n^6} = \dfrac{\pi^6}{945}$.

③ Problem 2, Ahlfors, p. 190

We will express the series $\sum_{n=-\infty}^{\infty}\dfrac{1}{z^3-n^3}$ in closed form, i.e. a form without sum notations. First, we'll prove that the above series represents a meromorphic function.

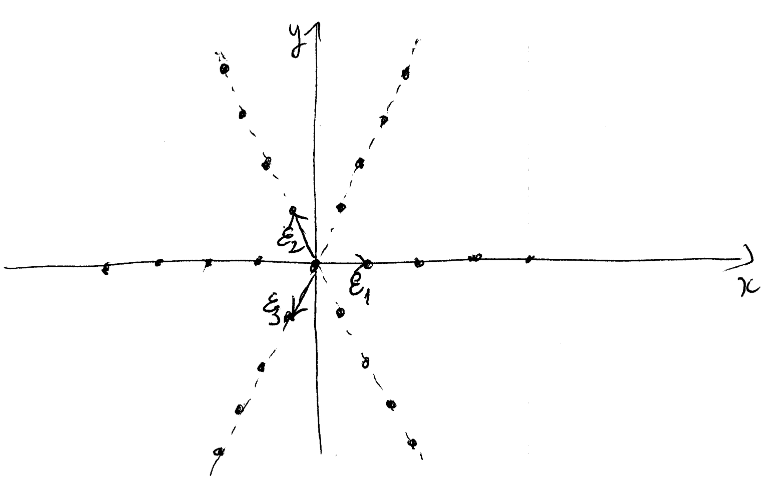
The equation $z^3-n^3=0$ is equivalent to $z=n\varepsilon_i$ where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are the third roots of unit.

$$\varepsilon_1 = \cos 0 + i\sin 0 = 1,$$

5/5

$$\varepsilon_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

$$\varepsilon_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$



Put $\Omega = \mathbb{C}\setminus\{n\varepsilon_i \mid n\in\mathbb{Z},\ i=1,2,3\}$.

and $f(z) = \sum_{n=-\infty}^{\infty}\dfrac{1}{z^3-n^3}$ for all $z\in\Omega$.

To show that $f(z)$ is analytic in Ω, we'll show that $\sum_{n=0}^{\infty} \frac{1}{z^3-n^3}$ and $\sum_{n=-\infty}^{0} \frac{1}{z^3-n^3}$ uniformly converge on compact subsets of Ω. Let A be a compact subset of Ω. Then there exists $n_0 \in \mathbb{N}$ such that $n_0 > 2 \max_{z\in A} |z|$.

For $n > n_0$, we have $|z| < \frac{n}{2}$ for all $z \in A$ and

$$\left| \frac{1}{z^3-n^3} \right| \leq \frac{1}{n^3-|z|^3} \leq \frac{1}{n^3-\left(\frac{n}{2}\right)^3} = \frac{8}{7n^3}$$

Thus $\sum_{n=n_0}^{\infty} \frac{1}{z^3-n^3}$ converges uniformly on A ~~to an analytic function on Ω~~. Then so is the series $\sum_{n=0}^{\infty} \frac{1}{z^3-n^3}$.

Similarly,
$$\sum_{n=-\infty}^{0} \frac{1}{z^3-n^3} = \sum_{n=0}^{\infty} \frac{1}{z^3-(-n)^3} = \underbrace{\sum_{n=0}^{n_0} \frac{1}{z^3+n^3}}_{\text{analytic on } \Omega} + \sum_{n=n_0}^{\infty} \frac{1}{z^3+n^3}$$

Because
$$\left| \frac{1}{z^3+n^3} \right| \leq \frac{1}{n^3-|z|^3} \leq \frac{1}{n^3-\left(\frac{n}{2}\right)^3} = \frac{8}{7n^3}$$
for all $n > n_0$, $z \in A$,

the second series also converges uniformly on A.

Therefore, the series $\sum_{n=-\infty}^{\infty} \frac{1}{z^3-n^3}$ represents an analytic function on Ω.

Moreover, the singular part at $z=0$ is $\frac{1}{z^3}$ and at $z = n\xi_i$ for $n \neq 0$ is $\frac{1}{z-n\xi_i}$.

Thus the series $\sum_{n=-\infty}^{\infty} \frac{1}{z^3-n^3}$ represents a meromorphic function on Ω with the singular parts at 0 and $n\xi_i$ $(n \neq 0)$ mentioned above.

Next, we'll express $f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{z^3 - n^3}$ in a closed form. To do so, we'll find constants A, B, C independent of n and z satisfying

$$\frac{z^2}{z^3 - n^3} = \frac{A}{z - n\varepsilon_1} + \frac{B}{z - n\varepsilon_2} + \frac{C}{z - n\varepsilon_3}$$

This is equivalent to

$$z^2 = A(z - n\varepsilon_2)(z - n\varepsilon_3) + B(z - n\varepsilon_3)(z - n\varepsilon_1) + C(z - n\varepsilon_1)(z - n\varepsilon_2)$$

Equating the coefficients of $z^2, z, 1$ from bothe sides, we get

$$\begin{cases} A + B + C = 1 \\ \varepsilon_1 A + \varepsilon_2 B + \varepsilon_3 C = 0 \\ \varepsilon_2\varepsilon_3 A + \varepsilon_3\varepsilon_1 B + \varepsilon_1\varepsilon_2 C = 0 \end{cases}$$

Knowing that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ and $\varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 + \varepsilon_1\varepsilon_2 = 0$, we can choose $A = B = C = \frac{1}{3}$. Therefore, we get the identity

$$\frac{3z^2}{z^3 - n^3} = \frac{1}{z - n\varepsilon_1} + \frac{1}{z - n\varepsilon_2} + \frac{1}{z - n\varepsilon_3} \quad \text{for all } z \in \Omega. \qquad (*)$$

We know that

$$g(z) := \pi \cot \pi z = \lim_{m \to \infty} \sum_{n=-m}^{m} \frac{1}{z - n}$$

Then, for each $i = 1, 2, 3$ we have

$$\frac{1}{\varepsilon_i} g\left(\frac{z}{\varepsilon_i}\right) = \lim_{m \to \infty} \sum_{n=-m}^{m} \frac{1}{\varepsilon_i} \frac{1}{z/\varepsilon_i - n} = \lim_{m \to \infty} \sum_{n=-m}^{m} \frac{1}{z - n\varepsilon_i}$$

Taking the ~~same~~ sum over $[-m, m]$ of $(*)$, we get

$$\sum_{n=-m}^{m} \frac{3z^2}{z^3-n^3} = \sum_{n=-m}^{m} \frac{1}{z-n\varepsilon_1} + \sum_{n=-m}^{m} \frac{1}{z-n\varepsilon_2} + \sum_{n=-m}^{m} \frac{1}{z-n\varepsilon_3}$$

Taking the limit both side, we have

$$\lim_{m\to\infty} \sum_{n=-m}^{m} \frac{3z^2}{z^3-n^3} = \frac{1}{\varepsilon_1} g\left(\frac{z}{\varepsilon_1}\right) + \frac{1}{\varepsilon_2} g\left(\frac{z}{\varepsilon_2}\right) + \frac{1}{\varepsilon_3} g\left(\frac{z}{\varepsilon_3}\right) \qquad (**)$$

The left hand member is just $3z^2 \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{1}{z^3-n^3} = 3z^2 f(z)$. Thus, $(**)$ gives

$$f(z) = \frac{1}{3z^2}\left[\frac{1}{\varepsilon_1} g\left(\frac{z}{\varepsilon_1}\right) + \frac{1}{\varepsilon_2} g\left(\frac{z}{\varepsilon_2}\right) + \frac{1}{\varepsilon_3} g\left(\frac{z}{\varepsilon_3}\right)\right] \ldots$$

Therefore,

$$\sum_{n=-\infty}^{\infty} \frac{1}{z^3-n^3} = \frac{1}{3z^2}\left[\frac{\pi}{\varepsilon_1}\cot\left(\frac{\pi z}{\varepsilon_1}\right) + \frac{\pi}{\varepsilon_2}\cot\left(\frac{\pi z}{\varepsilon_2}\right) + \frac{\pi}{\varepsilon_3}\cot\left(\frac{\pi z}{\varepsilon_3}\right)\right].$$

4 Problem 1, Ahlfors, p. 193

We'll show that $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right) = \frac{1}{2}$.

By definition, the above infinite product is $P = \lim_{m\to\infty} P_m$, where P_m's partial products $P_m = \prod_{n=2}^{m}\left(1-\frac{1}{n^2}\right)$.

For $m \geq 6$, we have

$$P_m = \prod_{n=2}^{m} \frac{n^2-1}{n^2} = \prod_{n=2}^{m} \frac{(n-1)(n+1)}{n^2} = \underbrace{\left(\prod_{n=2}^{m}(n-1)\right)}_{(m-1)!}\underbrace{\left(\prod_{n=2}^{m}(n+1)\right)}_{\frac{1}{2}(m+1)!}\underbrace{\left(\prod_{n=2}^{m}\frac{1}{n^2}\right)}_{\frac{1}{(m!)^2}}$$

Thus, $P_m = \dfrac{(m-1)!\,(m+1)!}{2m!\,m!} = \dfrac{m+1}{2m}$.

Therefore, $\lim_{m\to\infty} P_m = \frac{1}{2}$.

(5) Problem 3, Ahlfors, p. 193.

We'll show that the infinite product $\prod_1^\infty \left(1+\frac{z}{n}\right) e^{-z/n}$ converges absolutely and uniformly on every compact set of $\mathbb{C}$.

According to Ahlfors, an infinite product $\prod_1^\infty (1+a_n)$ is said to be absolutely convergent if the infinite sum $\sum_1^\infty \log(1+a_n)$ is absolutely convergent, where log is understood as the principal branch. In fact, the infinite sum should be taken from some index n_0 rather than 1 to avoid the case $a_n=-1$ for some first entries a_n's. Returning to the problem: let A be a compact subset of $\mathbb{C}$, we need to show that there exists $n=n_A \in \mathbb{N}$ such that the series
$$\sum_{n=n_A}^\infty \left|\log\left[\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right]\right|$$
uniformly converges on A. We choose $n_A > 2\max_{z\in A}|z|$. Then $\left|\frac{z}{n}\right| < \frac{1}{2}$ for all $n \geq n_A$, $z\in A$.

With the principal branch of logarithm, we have the following property
$$\left[\ \log(y_1 y_2) = \log(y_1) + \log(y_2) \quad \text{if } \operatorname{Re}(y_1), \operatorname{Re}(y_2) > 0\ \right]$$

For $n \geq n_A$ and $z\in A$, we have

$$\mathrm{Re}\left(1+\frac{z}{n}\right) = 1+\mathrm{Re}\left(\frac{z}{n}\right) \geq 1-\left|\frac{z}{n}\right| > 1-\frac{1}{2} > 0,$$

$$\mathrm{Re}\left(e^{-z/n}\right) = \mathrm{Re}\left(e^{w}\right) \quad (\text{where } w = -z/n)$$
$$= e^{\mathrm{Re}(w)} \cos(\mathrm{Im}\, w).$$

We have
$$|\mathrm{Im}(w)| = \left|\mathrm{Im}\left(-\frac{z}{n}\right)\right| \leq \left|\frac{z}{n}\right| < \frac{1}{2} < \frac{\pi}{2}.$$

Thus $\cos(\mathrm{Im}(w)) > 0$, and hence $\mathrm{Re}(e^{-z/n}) > 0$. Therefore,

$$\log\left[\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right] = \log\left(1+\frac{z}{n}\right) + \log e^{-z/n} = \log\left(1+\frac{z}{n}\right) - \frac{z}{n}. \quad (*)$$

We know that the function $f(w) = \log(1+w) - w$ is analytic in the disk $|w| < \frac{1}{2}$. Thus, $f(w)$ admits a Taylor expansion around $w=0$ in this disk. Thus, we can write $f(w) = a_0 + a_1 w + w^2 g(w)$ where $g(w)$ is an analytic function on the disk $|w| < \frac{1}{2}$. By taking $w=0$, we get $a_0 = 0$. Then

$$\log(1+w) - w = a_1 w + w^2 g(w)$$

Taking the derivatives of both sides, we get

$$\frac{1}{1+w} - 1 = a_1 + 2w\, g(w) + w^2 g'(w)$$

Plugging $w=0$, we get $a_1 = 0$. Thus $f(w) = w^2 g(w)$ for $|w| \leq \frac{1}{2}$.

In fact the above identity holds for all $|w| < 1$. Put $M = \max_{|z| \leq \frac{1}{2}} |g(z)|$.

We have ~~from (*)~~ $|f(w)| \leq M|w|^2$ for $|w| \leq \frac{1}{2}$.

Applying this result for $w = \frac{z}{n}$, we have an estimation of $(*)$:

$$\left| \log \left[\left(1 + \frac{z}{n}\right) e^{-z/n} \right] \right| = \left| \log\left(1 + \frac{z}{n}\right) - \frac{z}{n} \right| = \left| f\left(\frac{z}{n}\right) \right| \leq M \frac{|z|^2}{n^2} \leq \frac{M n_A^2}{4n^2}.$$

Since the series $\sum_{n=n_A}^{\infty} \frac{M n_A^2}{4n^2}$ converges, the series $\sum_{n=n_A}^{\infty} \left| \log\left[\left(1+\frac{z}{n}\right) e^{-z/n} \right] \right|$ converges uniformly on A.

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