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Math 8702: Complex Analysis
Homework # 1

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① Let $f: \mathbb{D} \to \mathbb{D}$ be a holomorphic function, where $\mathbb{D}$ is the open unit disk. Suppose that f has two fixed points α and β. We'll show that f is the identity map.

One of α and β is nonzero because they are distinct. We can assume that $\alpha \neq 0$. Put $\psi_\alpha : \mathbb{D} \to \mathbb{D}$, $\psi_\alpha(z) = \dfrac{\alpha - z}{1-\bar{\alpha} z}$.

Then ψ_α is bijective and holomorphic with $\psi_\alpha^{-1} = \psi_\alpha$. Then $\psi_\alpha^{-1} \circ f \circ \psi_\alpha$ is a holomorphic function from $\mathbb{D}$ to $\mathbb{D}$. We have $\psi_\alpha(0) = \alpha$ and $\psi_\alpha(\alpha) = 0$.

Thus $\psi_\alpha^{-1} \circ f \circ \psi_\alpha(0) = \psi_\alpha^{-1} f(\alpha) = \psi_\alpha^{-1}(\alpha) = 0$. Thus we can apply Schwarz's lemma to $g := \psi_\alpha^{-1} \circ f \circ \psi_\alpha$. ~~There exists $\theta \in \mathbb{R}$ such that $\psi_\alpha^{-1} \circ f \circ \psi_\alpha(z) = e^{i\theta} z$ for all $z \in \mathbb{D}$. We have~~ Because $\beta \neq \alpha = \psi_\alpha(0)$, there is $z_0 = \psi_\alpha^{-1}(\beta)$ with $z_0 \neq 0$.

$$\cancel{f(z) = \psi_\alpha \circ (e^{i\theta} z) \circ \psi_\alpha^{-1} = \psi_\alpha(e^{i\theta} \psi_\alpha^{-1}(z)) \quad (*)}$$

~~Because $f(\beta) = \beta$, we have $\psi_\alpha(e^{i\theta} \psi_\alpha^{-1}(\beta)) = \beta$. Thus $e^{i\theta} \psi_\alpha^{-1}(\beta) = \psi_\alpha^{-1}(\beta)$.~~

We have $g(z_0) = \psi_\alpha^{-1} \circ f \circ \psi_\alpha(z_0) = \psi_\alpha^{-1}(f(\beta)) = \psi_\alpha^{-1}(\beta) = z_0$

~~Because $\beta \neq \alpha$, $\psi_\alpha^{-1}(\beta) \neq \psi_\alpha^{-1}(\alpha) = 0$. Thus $e^{i\theta} = 1$.~~

~~Then from $(*)$ we get $f(z) = \psi_\alpha(\psi_\alpha^{-1}(z)) = z$. That means f is the identity map on $\mathbb{D}$.~~

Thus we can apply Schwarz's lemma for g: there is $c \in \mathbb{C}$ with $|c| = 1$ such that $g(z) = cz$ $\forall z \in \mathbb{D}$. Because $g(z_0) = z_0 \neq 0$, $c = 1$. Thus $g(z) \equiv z$, and hence $f(z) \equiv z$.

② Suppose that U and V are conformally equivalent and U is simply connected. Then U and V are homeomorphic because the bijective holomorphic function from U to V is also an homeomorphism, called f.

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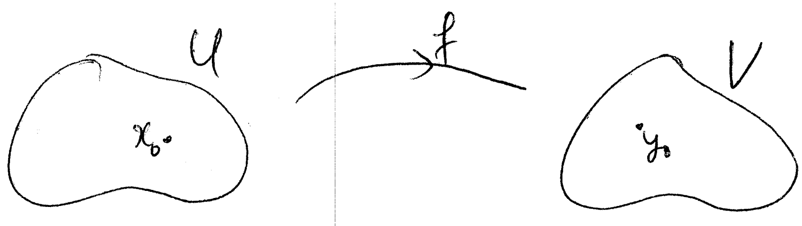

Put $y_0 = f(x_0)$ for some $x_0 \in U$. Assume that U was given to be connected. Then U is path-connected because it is an open connected subset of $\mathbb{C}$. Then so is V. The homeomorphism f induces a group isomorphism on the fundamental groups. Thus $\pi_1(U, x_0) \simeq \pi_1(V, y_0)$.

Because U is simply connected, $\pi_1(U, x_0) \simeq \{e\}$. Thus $\pi_1(V, y_0) \simeq \{e\}$. Thus V is also simply connected.

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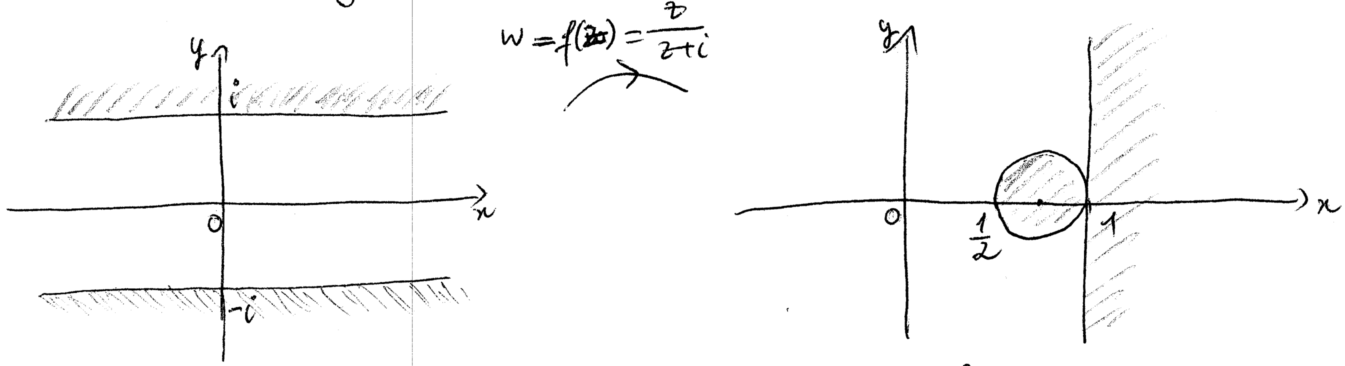

z-plane w-plane

Put $\Omega = \{z \in \mathbb{C} : -1 < \operatorname{Im} z < 1\}$,

$\Gamma = \{ w \in \mathbb{C} : |w - \frac{3}{4}| > \frac{1}{4} \text{ and } \operatorname{Re}(w) < 1\}$

We'll show that $f(\Omega) = \Gamma$.

The function f, if viewed as a map from the extended plane to the extended plane, is a bijective map. Moreover, since $f(z)$ is a linear fractional transformation, it maps a circle in the extended plane to a circle to the extended plane. Also, f is a conformal map on $\mathbb{C} \cup \{\infty\}$, in sense that it preserves angles.

A point on the y-axis in z-plane is of the form iy, with $y \in \mathbb{R}$. Then $f(iy) = \frac{iy}{iy+i} = \frac{y}{y+1} \in \mathbb{R}$. Thus f map the ~~y-axis~~ immaginary axis in z-plane to the real axis in w-plane. Moreover,

$$-i \xmapsto{f} \infty$$

$$\infty \xmapsto{f} 1$$

The the line $\operatorname{Im} z = -1$ in z-plane is mapped to a line passing through 1 in w-plane. Since the line $\operatorname{Im} z = -1$ is perpendicular to the imaginary axis in z-plane, its image is perpendicular to the real axis in w-plane. Thus the line $\operatorname{Im} z = -1$ is mapped to the vertical line $\operatorname{Re}(w) = 1$.

We have $i \xmapsto{f} \frac{1}{2}$, $\infty \xmapsto{f} 1$. Thus the line $\operatorname{Im} z = 1$ in z-plane is mapped to a circle in w-plane passing through $1/2$ and 1. Because the lines $\operatorname{Im} z = 1$ and $\operatorname{Im} z = -1$ are parallel in z-plane, the circle must be tangent to the line $\operatorname{Re}(w) = 1$ in w-plane. Thus the line $\operatorname{Im} z = 1$ is mapped to the circle centered at $3/4$ and with radius $\frac{1}{4}$.

Because $f(z)$ is conformal, it preserves orientations. Thus it maps the half plane $\{z : \operatorname{Im} z < 1\}$ to either the inside or the outside of the circle $|w - \frac{3}{4}| = \frac{1}{4}$. Because $f(0) = 0$, which lies in the ~~inside~~ outside of the circle $|w - \frac{3}{4}| = \frac{1}{4}$, we have

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$$f(\{z : \operatorname{Im} z < 1\}) = \{w : |w - \tfrac{3}{4}| > \tfrac{1}{4}\} \qquad (1)$$

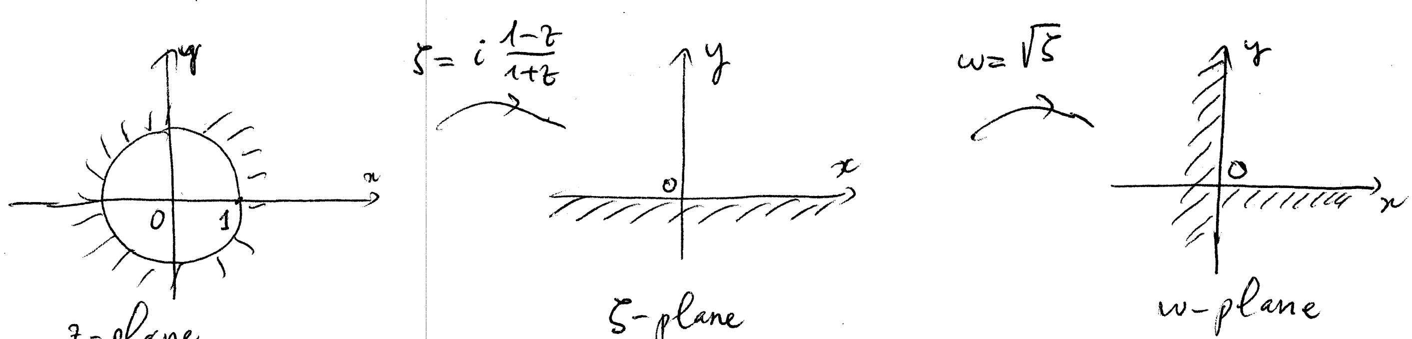
Similarly, f maps the half plane $\operatorname{Im} z > -1$ to either the right half plane or the left half plane of ~~R~~ the line $\operatorname{Re}(w) = 1$. Because $f(0) = 0$, which lies in the left half plane, we have

$$f(\{z : \operatorname{Im} z > -1\}) = \{w : \operatorname{Re}(w) < 1\} \qquad (2)$$

Taking the intersections from (1) and (2), we get

$$f(\{z : -1 < \operatorname{Im} z < 1\}) = \{w : |w - \tfrac{3}{4}| > \tfrac{1}{4} \text{ and } \operatorname{Re}(w) < 1\}.$$

(4) We'll look for a conformal map between the unit disk to the quadrant $\{z : \operatorname{Im} z > 0, \operatorname{Re} z > 0\}$.

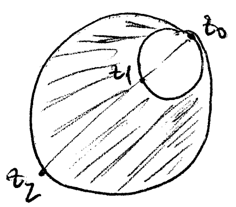


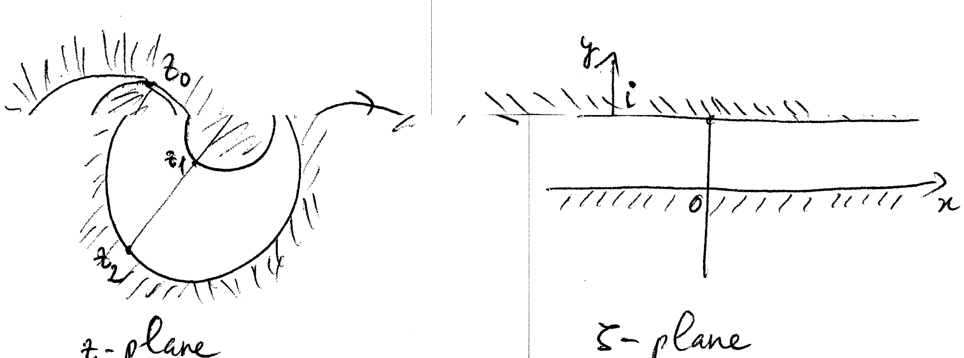
We know that the map $F(z) = \frac{i-z}{i+z}$ maps topologically the ~~uni~~ upper half plane to the unit disk. Thus the inverse $z \mapsto \zeta = i\frac{1-z}{1+z}$ maps topologically the unit disk to the upper half plane.

On the upper half plane, we can define an analytic square root function by choosing the square root with positive imaginary part. On this half plane, the square root function is also bijective. Thus we obtain a composite map

$f(z) = \sqrt{i\frac{1-z}{1+z}}$ which maps topologically ~~the~~ the unit disk ~~the~~ to the first quadrant $\{z : Re(z) > 0, Im(z) > 0\}$. ✓

(5) Suppose that we have two circles which are tangent at z_0. Let z_1 be the intersection of the radial line and the smaller circle, and z_2 be the intersection of that line and the larger one. Denote by U the shaded region in the figure. We'll find a conformal map between U and the unit disk.





Consider the linear fractional transformation that map $z_0 \mapsto \infty$, $z_1 \mapsto 0$, $z_2 \mapsto i$. In terms of cross-ratio, this map is $(\zeta, \infty, 0, i) = (z, z_0, z_1, z_2)$. Thus

$$\frac{\zeta}{\zeta - i} = k\frac{z - z_1}{z - z_2}, \text{ where } k = \frac{z_0 - z_2}{z_0 - z_1} > 1. \qquad (1)$$

Applying the property "$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{a-b} = \frac{c}{c-d}$", we have

$$\frac{\zeta}{\zeta - (\zeta - i)} = \frac{kz - kz_1}{kz - kz_1 - (z - z_2)}$$

Thus $$\zeta = \frac{ik(z - z_1)}{(k-1)z - (kz_1 - z_2)} \qquad (2)$$

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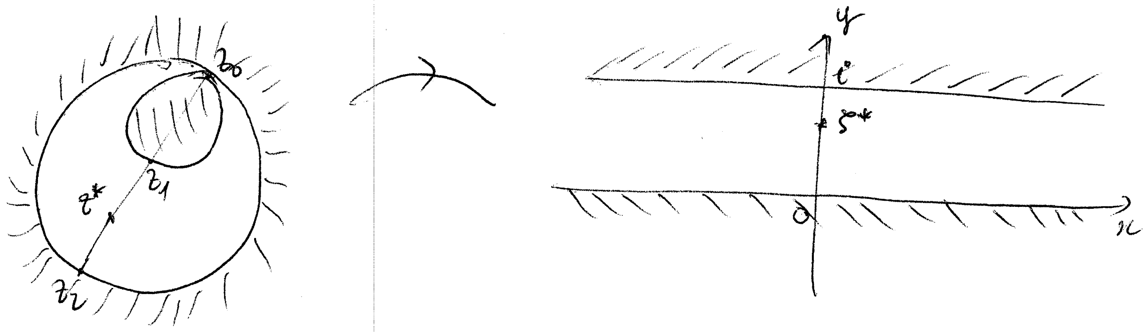


z-plane ζ-plane

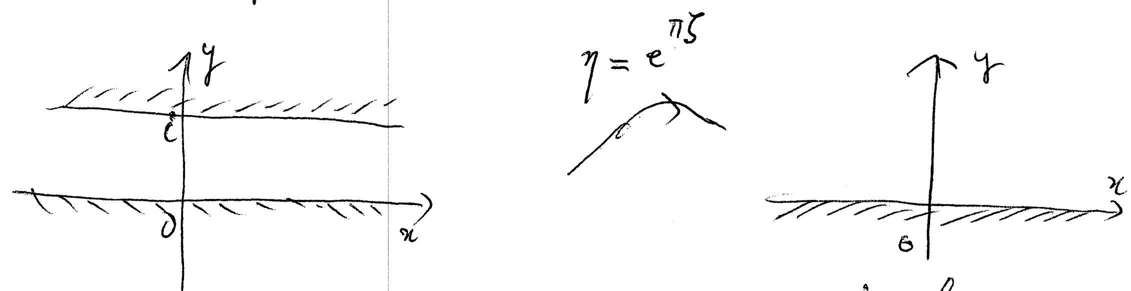
Because $z_0 \mapsto \infty$, $z_1 \mapsto 0$, $z_2 \mapsto i$, the ~~line~~ radial line connecting z_0, z_1, z_2 in z-plane is mapped to the imaginary axis in ζ-plane.

Since $z_0 \mapsto \infty$ and $z_1 \mapsto 0$, the smaller circle C_1 is mapped to a line through 0 in ζ-plane. Since C_1 and the radial line are perpendicular, the image of C_1 is also perpendicular to the imaginary axis. Thus C_1 is mapped to the real axis in ζ-plane. Similarly, since $z_0 \mapsto \infty$ and $z_2 \mapsto i$, the larger circle C_2 is mapped to a line through i and perpendicular to the imaginary axis in ζ-plane. Then C_1 is mapped to the line $\operatorname{Im} \zeta = 1$.

Put $z^* = \dfrac{z_1 + z_2}{2} \in U$, which lies outside the ~~small~~ circle C_1 and inside the circle C_2. We have the image of z^*

$$\frac{\zeta^*}{\zeta^* - i} = k\,\frac{z^* - z_1}{z^* - z_2} = -k$$

Thus $\zeta^* = -k\zeta^* + ik$. Thus $\zeta^* = i\dfrac{k}{k+1}$ which lies inside the strip determined by $\operatorname{Im} \zeta = 0$ and $\operatorname{Im} \zeta = 1$. Thus ζ^* lies in the domain $\operatorname{Im} \zeta < 1$ and $\operatorname{Im} \zeta > 0$. Thus the outside of C_1 is mapped to the domain $\operatorname{Im} \zeta > 0$, and the inside of C_2 is mapped to the domain $\operatorname{Im}\zeta < 1$. Taking the intersections, we conclude that U is mapped to the strip $\{\zeta : 0 < \operatorname{Im}\zeta < 1\}$. Moreover, this map is topological.

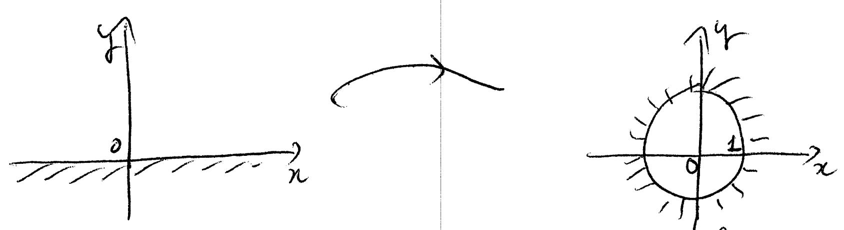


ζ-plane $\qquad\qquad$ η-plane

Consider the map $\eta = e^{\pi\zeta}$. The strip is now mapped to

$$\{\eta = e^{\pi\zeta} \mid 0 < \operatorname{Im}\zeta < 1\} = \{\eta = e^{\pi x} e^{i\pi y} \mid x \in \mathbb{R},\ 0 < y < 1\}$$
$$= \{\eta \in \mathbb{C} : 0 < \arg\eta < \pi\}$$
$$= \mathbb{H}.$$

The map $\eta = e^{\pi\zeta}$ is topological from the strip to $\mathbb{H}$. The final step is to map the upper half plane to the unit disk by $w = \dfrac{i-\eta}{i+\eta}$.



η-plane $\qquad\qquad$ w-plane

We compose all steps together to get a topological map from U to the unit disk

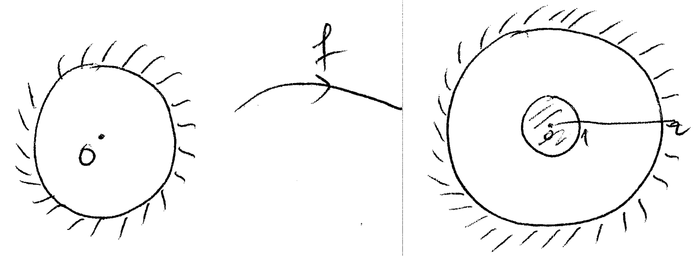
$$w = \frac{i-\eta}{i+\eta} = \frac{i - e^{\pi\zeta}}{i + e^{\pi\zeta}} = \left[i - \exp\left(\frac{\pi i k (z - z_1)}{(k-1)z - (kz_1 - z_2)} \right) \right] \Big/ \left[i + \exp\left(\frac{\pi i k (z - z_1)}{(k-1)z - (kz_1 - z_2)} \right) \right]$$

where $k = \dfrac{z_0 - z_2}{z_0 - z_1} > 1$.

(6) Denote $\Omega_1 = \{z \in \mathbb{C} : 0 < |z| < 1\}$,

$\Omega_2 = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

Suppose by contradiction that there is a conformal map $f : \Omega_1 \to \Omega_2$.



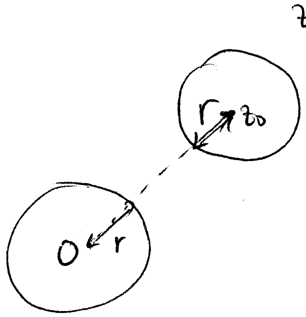
Then 0 is an isolated singularity of $f(z)$. We have

$|(z-0)f(z)| = |z|\,|f(z)| \le 2|z|$.

Thus $\lim_{z\to 0} (z-0)f(z) = 0$. Thus by Cauchy's theorem, 0 is a removable singularity of f. Thus f can be extended to an analytic map ~~$\bar{f} : \mathbb{D} \to \Omega_2 \cup \{a\}$~~ $\bar{f} : \mathbb{D} \to \Omega_2 \cup \{a\}$, where $a = \lim_{z\to 0} f(z) = \bar{f}(0)$.

Since $\mathbb{D}$ is connected and $\bar{f}$ is continuous, $\Omega_2 \cup \{a\} = \bar{f}(\mathbb{D})$ must also be connected. Since $\bar{f}$ is not a constant function, it is an open map. Thus $\Omega_2 \cup \{a\} = \bar{f}(\mathbb{D})$ is an open subset of $\mathbb{C}$. Therefore, a cannot lie on the boundary of Ω_2, nor can it lie outside Ω_2. Thus $a \in \Omega_2$. Because f is surjective, there exists $z_0 \in \Omega_1$ such that $a = f(z_0)$. Thus we have

$$\lim_{z\to 0} f(z) = f(z_0) \qquad (1)$$



Put $r = \frac{1}{3}\min\{|z_0|, 1-|z_0|\}$. Then $B(z_0, r) \subset \Omega_1$ and $B(0,r) \cap B(z_0, r) = \phi$. Because f is injective,

$$f(B(0,r)\setminus\{0\}) \cap f(B(z_0,r)) = \phi. \qquad (2)$$

Because f is analytic and nonconstant, it is an open map.

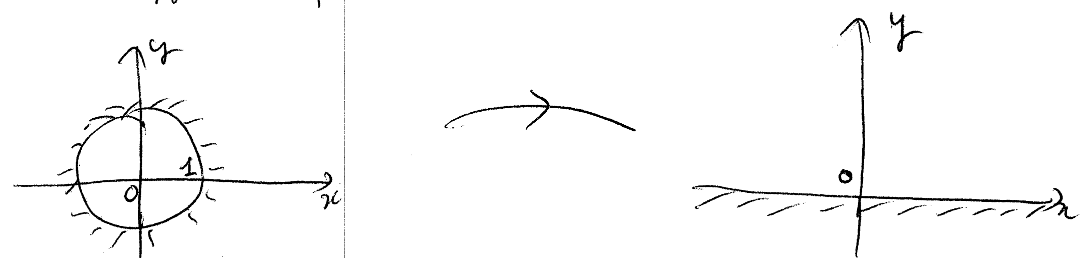
Thus $f(B(z_0, r))$ is an open set in $\mathbb{C}$ containing $f(z_0)$. Thus there exists $\delta > 0$ such that $B(f(z_0), \delta) \subset f(B(z_0, r))$. Together with (2), we have

$$f(B(0,r)\setminus\{0\}) \cap B(f(z_0),\delta) = \phi$$

In other word, for all $z \in \Omega_1$ such that $|z| < r$, $f(z) \notin B(f(z_0), \delta)$.

Thus $|f(z) - f(z_0)| \geq \delta$ for all $z \in \Omega_1$, $|z| < r$. This contradicts (1).

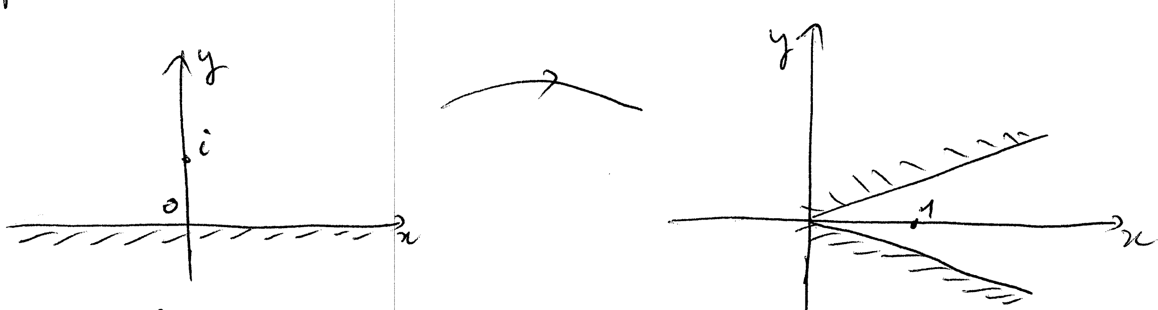
7 We'll look for a conformal map between the unit disk and the wedge $\{z \in \mathbb{C} : |\arg(z)| < 1\}$. As we know, the map $\zeta = i\dfrac{1-z}{1+z}$ maps the unit disk to the upper half plane



z - plane

ζ - plane.

This map carries 0 to i. We need a ~~ma~~ conformal map that carries the upper half plane $\mathbb{H}$ to the wedge $\{w \in \mathbb{C} : |\arg(w)| < 1\}$ such that i is mapped to 1.



ζ - plane

w - plane.

On the upper half plane, we consider the principal branch of the logarithm.

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Define the map $w = e^{-i} \zeta^{\frac{2}{\pi}}$ for all $\zeta \in \mathbb{H}$, where

$$\zeta^{\frac{2}{\pi}} := \exp\left(\frac{2}{\pi} \log \zeta\right).$$

We rewrite $\mathbb{H}$ in terms of polar coordinates $\mathbb{H} = \{\zeta = re^{i\theta} : r > 0, 0 < \theta < \pi\}$.

Then the image of $\mathbb{H}$ under the above map is

$$\{w = e^{-i}(re^{i\theta})^{2/\pi} : r > 0, 0 < \theta < \pi\} = \{w = r^{\frac{2}{\pi}} e^{i\left(\frac{2\theta}{\pi} - 1\right)} : r > 0, 0 < \theta < \pi\}$$

$$= \{w = r' e^{i\theta'} : r' > 0, -1 < \theta' < 1\}$$

$$= \{w \in \mathbb{C} : |\arg(w)| < 1\}.$$

Thus we have a conformal map $\zeta \mapsto w = e^{-i}\zeta^{2/\pi}$ from $\mathbb{H}$ to the domain $\{w \in \mathbb{C} : |\arg(w)| < 1\}$ which carries $\zeta = i$ to

$$w = r^{\frac{2}{\pi}} e^{i\left(\frac{2\theta}{\pi} - 1\right)}$$

$$= 1^{\frac{2}{\pi}} e^{i\left(\frac{2}{\pi}\frac{\pi}{2} - 1\right)}$$

$$= 1.$$

Therefore, we obtain a map $w = e^{-i}\zeta^{2/\pi} = e^{-i}\left(i\frac{1-z}{1+z}\right)^{2/\pi}$ from the unit disk to the wedge $\{w \in \mathbb{C} : |\arg(w)| < 1\}$. This map is conformal and carries $z = 0$ to $w = 1$.