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Math 8702: Complex Analysis

Homework #2

1

50

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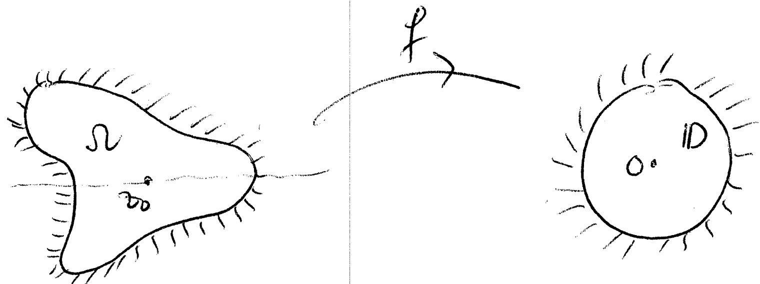
(1) Problem 1, Ahlfors, p. 232

Let Ω be a symmetric domain with respect to the real line and $z_0 \in \Omega \cap \mathbb{R}$.

Let $f: \Omega \to \mathbb{D}$ be the conformal map such that $f(z_0) = 0$ and $f'(z_0) > 0$.



Put $g: \Omega \to \mathbb{D}$, $g(z) = \overline{f(\bar{z})}$. Then g is well-defined because both Ω and $\mathbb{D}$ are symmetric with respect to the real axis. For any $a \in \Omega$, we have

$$\lim_{z \to a} \frac{g(z) - g(a)}{z - a} = \lim_{z \to a} \frac{\overline{f(\bar{z})} - \overline{f(\bar{a})}}{z - a} = \overline{\lim_{z \to a} \left(\frac{f(\bar{z}) - f(\bar{a})}{\bar{z} - \bar{a}} \right)} = \overline{f'(\bar{a})}.$$

Thus g is analytic and $g'(z) = \overline{f'(\bar{z})}$ for all $z \in \Omega$.

Moreover, g is bijective because

$$\begin{aligned} & g(z) = w \\ \iff & \overline{f(\bar{z})} = w \\ \iff & f(\bar{z}) = \bar{w} \\ \iff & \bar{z} = f^{-1}(\bar{w}) \\ \iff & z = \overline{f^{-1}(\bar{w})} \end{aligned}$$

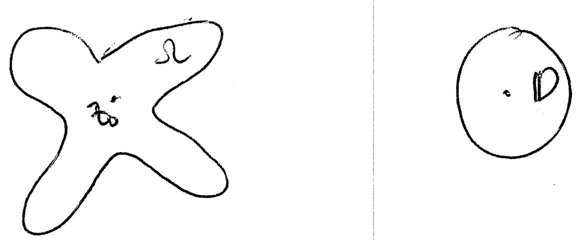
Thus g is a conformal map. Moreover,

$$g(z_0) = \overline{f(\bar{z}_0)} = \overline{f(z_0)} = \bar{0} = 0,$$

$$g'(z_0) = \overline{f'(\bar{z}_0)} = \overline{f'(z_0)} = f'(z_0) > 0.$$

By the uniqueness, $g=f$. Therefore, $\overline{f(\bar{z})} = f(z)$ for all $z \in \Omega$.

(2) Suppose that Ω is a domain symmetric with respect to the point z_0. Let $f : \Omega \to \mathbb{D}$ be the conformal map such that $f(z_0)=0$, $f'(z_0)>0$. We define $g : \Omega \to \mathbb{D}$, $g(z) = -f(2z_0 - z)$. Then g is well-defined because Ω is symmetric with respect to z_0 and $\mathbb{D}$ is symmetric with respect to 0. We have $g'(z) = f'(2z_0 - z)$ by the chain rule. Thus g is analytic.



In addition, g is bijective because f is bijective. Thus g is a conformal map. Moreover,

$$g(z_0) = -f(2z_0 - z_0) = -f(z_0) = 0,$$
$$g'(z_0) = f'(2z_0 - z_0) = f'(z_0) > 0.$$

By the uniqueness, $g=f$. Therefore, $-f(2z_0 - z) = f(z)$ for all $z \in \Omega$.

As a comment, if Ω is symmetric with ~~repect~~ respect to the real line then f maps symmetric points to symmetric points (wrt to the real line); if Ω is symmetric with respect to z_0 then f maps symmetric points wrt z_0 to symmetric points wrt the origin.

(3) Problem 1, additional problem.

Put $U = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ ~~and~~ consider a holomorphic map $f : U \to \mathbb{D}$ with $f(1)=0$. We'll ~~find~~ look for the maximum possible the value of $|f(2)|$.

$$\begin{tikzcd} \mathbb{D} \arrow[r, "\varphi_1"] \arrow[rrd, dashed, "g"'] & \mathbb{H} \arrow[r, "\varphi_2"] & U \arrow[d, "f"] \\ & & \mathbb{D} \end{tikzcd}$$

We know that the map $\varphi_1 : \mathbb{D} \to \mathbb{H}$,

$$\varphi_1(z) = i\,\frac{1-z}{1+z}$$

is a conformal map. By rotating 90° about the origin, we can map $\mathbb{H}$ onto U. Thus $\varphi_2 : \mathbb{H} \to U$,

$$\varphi_2(z) = -iz$$

is a conformal map. Thus $\varphi = \varphi_2 \circ \varphi_1 : \mathbb{D} \to U$ is a conformap. We have

$$\varphi(z) = \varphi_2(\varphi_1(z)) = -i\varphi_1(z) = \frac{1-z}{1+z}.$$

Put $g = f \circ \varphi : \mathbb{D} \to \mathbb{D}$. We have $g(0) = f(\varphi(0)) = f(1) = 0$. Therefore by Schwarz's lemma, $|g(z)| \le |z|$ for all $z \in \mathbb{D}$. Thus, $|f(\varphi(z))| \le |z|$ for all $z \in \mathbb{D}$. If z is replaced by $\varphi^{-1}(z)$ then $|f(z)| \le |\varphi^{-1}(z)|$ for all $z \in U$. We have $\varphi^{-1}(z) = \varphi(z) = \frac{1-z}{1+z}$. Thus,

$$|f(z)| \le \left|\frac{1-z}{1+z}\right|, \text{ for all } z \in U.$$

Thus $|f(2)| \le \frac{1}{3}$. The equality happens when $g(z) = z$, for which $f = \varphi^{-1}$. ~~Thus~~ In that case,

$$f(z) = \frac{1-z}{1+z} \quad \forall z \in U.$$

(4) Put $\mathcal{F} = \{f : \mathbb{D} \to \mathbb{C} \text{ holomorphic}, f(0) = 0, \operatorname{diam}(f(\mathbb{D})) \le 2\}$, where

$$\operatorname{diam}(f(\mathbb{D})) := \sup_{z_1, z_2 \in \mathbb{D}} |f(z_1) - f(z_2)|.$$

We see that $\mathcal{F}$ is a family of holomorphic functions on $\mathbb{D}$. To show that $\mathcal{F}$ is a normal family, by Montel's theorem, it suffices to show that $\mathcal{F}$ is locally bounded. We have

$$|f(z)| = |f(z) - f(0)| \leq \operatorname{diam}(f(\mathbb{D})) \leq 2 \quad \forall z \in \mathbb{D}, \forall f \in \mathcal{F}.$$

Thus $\mathcal{F}$ is uniformly bounded in $\mathbb{D}$, and hence it's a normal family.

To show that $\mathcal{F}$ is compact, we take a sequence (f_n) in $\mathcal{F}$ and show that it has a convergent subsequence to some element in $\mathcal{F}$. Since $\mathcal{F}$ is normal, there exists a subsequence (f_{n_k}) of (f_n) that converges to some $f: \mathbb{D} \to \mathbb{C}$ uniformly on every compact set of $\mathbb{D}$. By Weierstrass's theorem, f is holomorphic. We have $f(0) = \lim_{k\to\infty} f_{n_k}(0) = \lim_{k\to\infty} 0 = 0$. Moreover,

$$|f(z_1) - f(z_2)| = \lim_{k\to\infty} \underbrace{|f_{n_k}(z_1) - f_{n_k}(z_2)|}_{\leq \operatorname{diam}(f_{n_k}(\mathbb{D})) \leq 2} \leq 2, \quad \forall z_1, z_2 \in \mathbb{D}.$$

Thus $\operatorname{diam}(f(\mathbb{D})) \leq 2$. We have showed that $f \in \mathcal{F}$. Therefore, $\mathcal{F}$ is compact.

5 We consider the family $\mathcal{F} = \{f: \mathbb{D} \to \mathbb{D} \text{ holomorphic}, f(0) = 0\}$. Then $\mathcal{F}$ is a normal family because $|f(z)| < 1$ for all $z \in \mathbb{D}$, $f \in \mathcal{F}$. Consider $f \in \mathcal{F}$ and put $f_n(z) = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}(z)$. Suppose that $\lim_{n\to\infty} f_n(z) = h(z)$ for all $z \in \mathbb{D}$.

We'll show that either $h(z) = z$ for all z, or $h(z) = 0$ for all z.

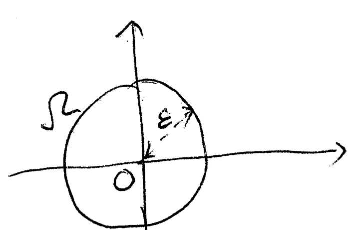
Because f_n is holomorphic and $f_n(0) = f \circ f \circ \cdots \circ f(0) = 0$, $f_n \in \mathcal{F}$. Since $\mathcal{F}$ is normal, there exists a subsequence (f_{n_k}) which converges uniformly to h on every compact set. Thus $h: \mathbb{D} \to \mathbb{C}$ is a holomorphic map by Weierstrass's theorem. Moreover, $h(0) = \lim_{n\to\infty} f_n(0) = 0$ and $|h(z)| = \lim_{n\to\infty} |f_n(z)| \leq 1$. By the Maximum Principle, $|h(z)| < 1$ for all $z \in \mathbb{D}$. Thus $h: \mathbb{D} \to \mathbb{D}$ and $h(0) = 0$. We will consider two cases $|f(0)| = 1$ and $|f'(0)| < 1$.

- $|f'(0)| = 1$.

Applying the Schwarz's lemma for $f(z)$, ~~there~~ we have a constant c with $|c| = 1$ and $f(z) = cz$ for all $z \in \mathbb{D}$. Then $f_n(z) = c^n z$. For $z \neq 0$, the sequence $\{c^n z\}_n$ converges if and only if the sequence $\{c^n\}_n$ converges. In that case, $\lim_{n\to\infty} (c^{n+1} - c^n) = 0$. Consequently, $|c^{n+1} - c^n| \to 0$. We have $|c^{n+1} - c^n| = |c^n|\,|c-1| = |c-1|$. Thus $c = 1$. Then $f(z) = z$, and $f_n(z) = z$, and hence $h(z) = z$ for all $z \in \mathbb{D}$.

- $|f'(0)| < 1$.

Because $|f'(z)|$ is continuous around zero, there exists $\varepsilon > 0$ and $\rho < 1$ such that $|f'(z)| \leq \rho$ for all $z \in \Omega = B(0, \varepsilon)$.



Applying the Schwarz's lemma to f, we have $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. Thus $f(z) \in \Omega$ for all $z \in \Omega$. Thus we can view f as a

6

map from Ω to Ω. Then f_n also map Ω to Ω for any $n \in \mathbb{N}$.

For $z \in \Omega$, we have $f_{n+1}(z) = f(f_n(z))$. Then by the chain rule,

$$f'_{n+1}(z) = f'(f_n(z))\, f'_n(z).$$ Thus $$|f'_{n+1}(z)| \leq |f'(f_n(z))|\,|f'_n(z)| \leq \rho\, |f'_n(z)|.$$

Repeatedly using this result, we get $$|f'_{n+1}(z)| \leq \rho\, |f'_n(z)| \leq \rho^2 |f'_{n-1}(z)| \leq \cdots \leq \rho^n |f'(z)| \leq \rho^{n+1}$$

Thus $|f'_n(z)| \leq \rho^n$ for all $z \in \Omega$. By Weierstrass's theorem, we also know that $h'(z) = \lim_{n\to\infty} f'_n(z)$. Thus $h'(z) = 0$ for all $z \in \Omega$. Since h' is holomorphic, and $\mathbb{D}$ is connected, $h'(z) = 0$ for all $z \in \mathbb{D}$. Thus h is a constant function. Since $h(0) = 0$, h must be the zero function.