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Math 8702: Complex Analysis
Homework 3

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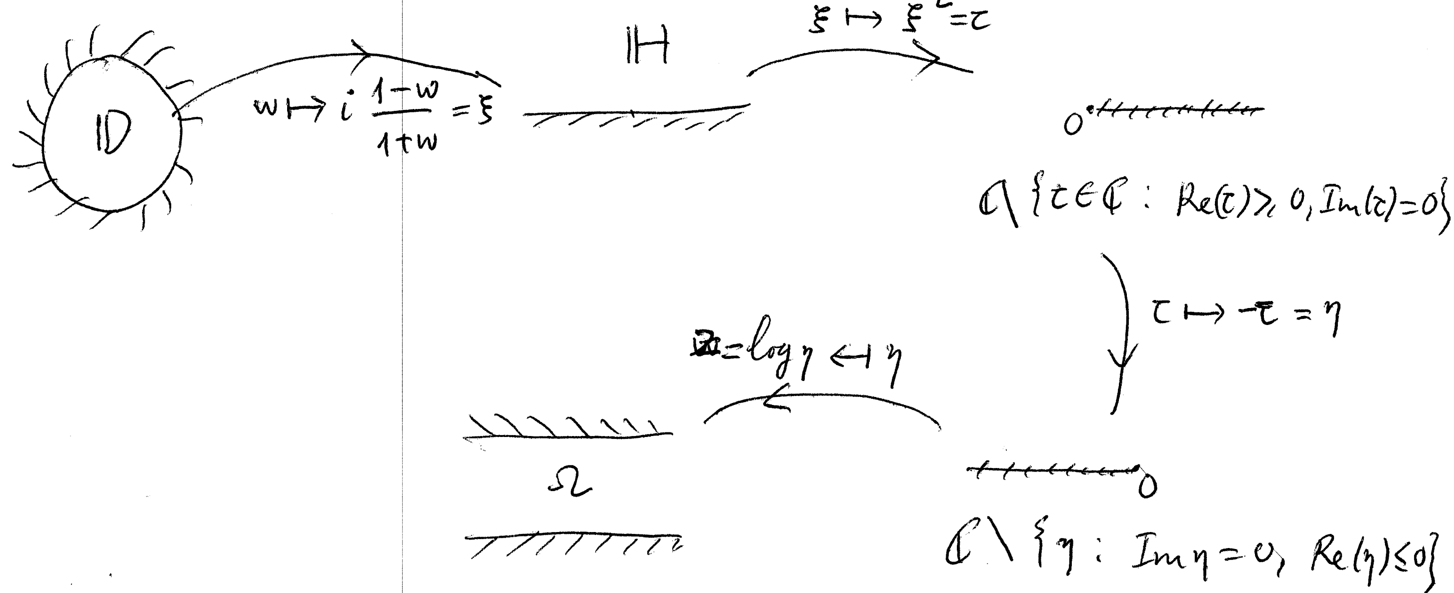
(1) Problem 3, Ahlfors, p. 238

First, we look for a conformal map from the unit disk to a familiar strip, namely $\Omega = \{z \in \mathbb{C} : -\pi < \mathrm{Im}(z) < \pi\}$. We have the following chain of conformal maps



Here we are taking the principal branch of logarithm, namely

$$\log z := \log|z| + i\arg(z), \text{ where } -\pi < \arg z < \pi.$$

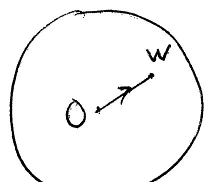
Thus a conformal map from $\mathbb{D}$ to Ω is

$$z = \log \eta = \log(-\tau) = \log(-\xi^2) = \log\left(-i^2\left(\frac{1-w}{1+w}\right)^2\right) = \log\left(\frac{1-w}{1+w}\right)^2.$$

We put $F : \mathbb{D} \to \Omega$, $F(w) = \log\left(\frac{1-w}{1+w}\right)^2$.

Now we try to present $F(w)$ in form of Schwarz-Christoffel formula. By the chain rule, we have $F'(w) = \dfrac{4}{w^2-1} = \dfrac{4}{(w-1)(w+1)}$.

Since F is analytic in the unit disk and $F(0) = 0$, we have

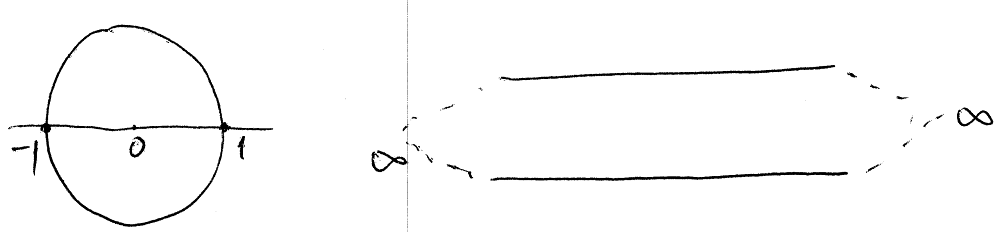


$$F(w) = \int_0^w F'(\xi)\,d\xi = 4\int_0^w \frac{d\xi}{(\xi-1)(\xi+1)} \qquad (*)$$

The formula $(*)$ is now in form of Schwarz-Christoffel formula. The generic conformal map from $\mathbb{D}$ to a polygon whose outer angles are $\beta_1, \cdots \beta_n$ is

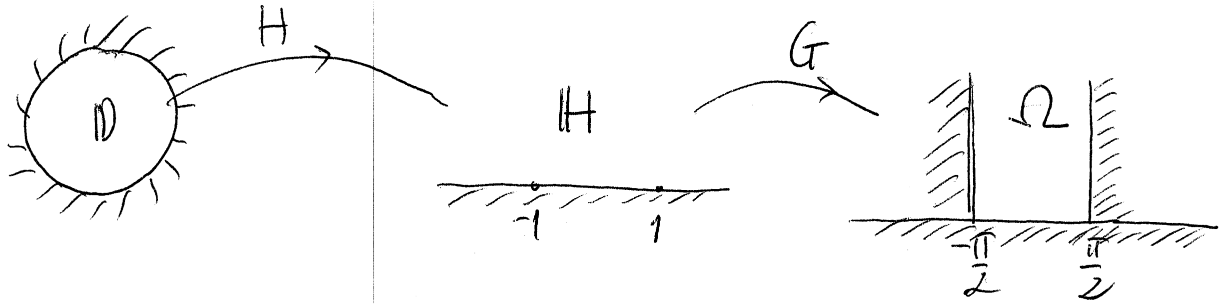
$$G(w) = C\int_0^w \frac{d\xi}{(\xi-w_1)^{\beta_1}\cdots(\xi-w_n)^{\beta_n}} + C'$$

Therefore $(*)$ is a degenerate form with $n=2$, $w_1=-1$, $w_2=1$, $\beta_1=\beta_2=1$.



The image of $\mathbb{D}$ under F is a "polygon" with only two vertices at ∞ (therefore the inner angles are $\alpha_1 = 0$, $\alpha_2 = 0$) Two "line segments" connecting these vertices should be parallel because they have two intercepts, both at infinity. Thus the polygon looks like an infinite strip.

Next, we'll look for a conformal map from the unit disk to a familiar infinite half strip $\Omega = \{z \in \mathbb{C} : \operatorname{Im} z > 0, -\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}\}$.



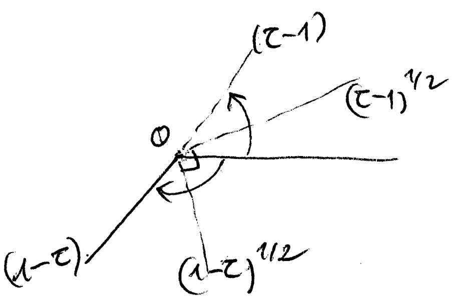
We put $H : \mathbb{D} \longrightarrow \mathbb{H}$, $H(w) = i\,\frac{1-w}{1+w}$,

$G : \mathbb{H} \longrightarrow \Omega$, $G(\xi) = \int_0^{\xi} \frac{d\tau}{(1-\tau^2)^{1/2}}$ (principal branch of logarithm)

Then we know (from the lecture note) that H and G are conformal maps.

Thus $F = G \circ H : \mathbb{D} \longrightarrow \Omega$ is a conformal map. For any $\tau \in \mathbb{H}$, we have $-\pi < \arg(1-\tau) < 0$, and $0 < \arg(1+\tau) < \pi$. Thus $-\pi < \arg(1-\tau) + \arg(1+\tau) < \pi$.

Thus $\log(1-\tau^2) = \log(1-\tau) + \log(1+\tau)$. Thus $(1-\tau^2)^{1/2} = (1-\tau)^{1/2}(1+\tau)^{1/2}$.



Because $-\pi < \arg(1-\tau) < 0$, we have

$$(1-\tau)^{1/2} = -i\,(\tau-1)^{1/2}$$

Then $(1-\tau^2)^{1/2} = -i\,(\tau-1)^{1/2}(\tau+1)^{1/2}$. Thus,

$$G(\xi) = i \int_0^{\xi} \frac{d\tau}{(\tau-1)^{1/2}(\tau+1)^{1/2}}.$$

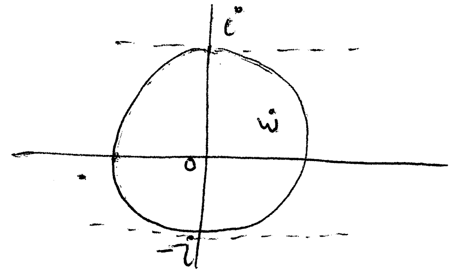
We have $H'(w) = \frac{-2i}{(1+w)^2}$, $H(w) - 1 = (i+1)(i-w)/(1+w)$,

$H(w) + 1 = (i-1)(-i-w)/(1+w)$.

Then $F'(w) = G'(H(w))\,H'(w) = \frac{i}{(H(w)-1)^{1/2}(H(w)+1)^{1/2}} \; \frac{-2i}{(1+w)^2}$

$$= \frac{2}{\left[(i+1)(i-w)\right]^{1/2}\left[(i-1)(-i-w)\right]^{1/2}(1+w)}$$

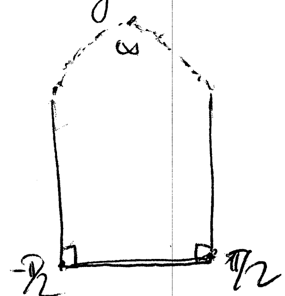
$$= \frac{2}{-i\sqrt{2}\,(w-i)^{1/2}(w+i)^{1/2}(w+1)}$$



Thus,

$$F(w) = \int_0^w F'(\xi)\,d\xi + F(0) = i\sqrt{2}\int_0^w \frac{d\xi}{(\xi-i)^{1/2}(\xi+i)^{1/2}(\xi+1)} + F(0)$$

This expression is now in form of Schwarz Christoffel's formula with $n=3$, $w_1 = i$, $w_2 = -i$, $w_3 = -1$; $\beta_1 = \frac{1}{2}$, $\beta_2 = \frac{1}{2}$, $\beta_3 = 1$. We can think of the image as a triangle with three vertices at $\frac{\pi}{2}$, $\frac{-\pi}{2}$ and ∞. Because $\alpha_1 = \alpha_2 = \frac{1}{2}$, this triangle has two right angles.



② Problem 5, Ahlfors p. 238.

We'll show that the map $F(w) = \int_0^w (1-\xi^n)^{-2/n}\,d\xi$ maps $|w|<1$ onto the interior of a regular polygon of n sides.

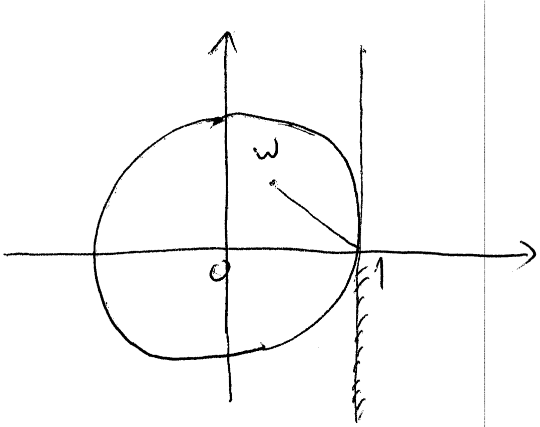
First, we notice that for two different branches of the logarithm, ~~function~~ the results for $(1-\xi^n)^{-2/n}$ differ by a constant factor c with $|c| = 1$ for all $\xi \in \mathbb{D}$. Thus F should be defined up to a constant factor c with

$|c| = 1$. Then the images of F corresponding to different choices of branches of logarithm differ by rotations about the origin. Such rotations, however, do not change the shape of $F(\mathbb{D})$. If $F(\mathbb{D})$ is the interior of a regular polygon of n sides, then $F(\mathbb{D})$ will still be the interior of another regular polygon of n sides when a different branch of logarithm is chosen. Thus we can assume

that's what we want. It cannot be assumed!

$$(1-w)^{-2/n} = \exp\left(-\frac{2}{n}\log(1-w)\right) \quad \forall w \in \mathbb{D}$$

with the choice of $\log z$ such that $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$.



For each $w \in \mathbb{D}$, we have

$$\log(1-w) = i\pi + \log(w-1).$$

Thus
$$(1-w)^{-2/n} = \exp\left(-\frac{2\pi i}{n} + -\frac{2}{n}\log(w-1)\right) = c_1 (w-1)^{-2/n}$$

where $c_1 = \exp\left(-\frac{2\pi i}{n}\right)$.

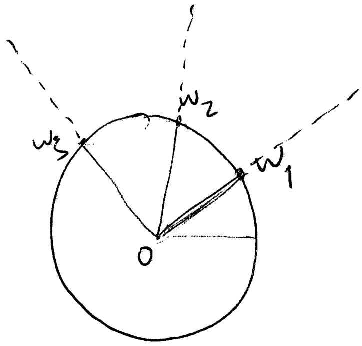
Thus,
$$F(w) = \int_0^w (1-\xi^n)^{-2/n}\, d\xi = c_1 \int_0^w (\xi^n - 1)^{-2/n}\, d\xi.$$

Since the constant factor c_1 just plays a role as a rotation, it doesn't matter by the reason we have mentioned. Thus we can ignore c_1 (by consider c_1 as if 1).

you are using the assumption!

Now let $w_k = \exp\left(\frac{2\pi i k}{n}\right)$, $k = 1, 2, \dots, n$ be the n'th roots of unity. Then $\xi^n - 1 = (\xi - w_1)\cdots(\xi - w_n)$ and

$$F(w) = \int_0^w \frac{d\xi}{[(\xi - w_1)\cdots(\xi - w_n)]^{2/n}}$$



For each $k = 1, \ldots, n$, we define $\log_k$ to be a single valued function $\log_k z = \log|z| + i \arg z$ with

$$\frac{2\pi(k-1)}{n} < \arg z < 2\pi + \frac{2\pi(k-1)}{n}.$$

On this branch of logarithm, we can speak of $(\xi - w_k)^{2/n}$. Consider the following map $\eta : \mathbb{D} \to \mathbb{C}$, $\eta(\xi) = \log[(\xi - w_1)\cdots(\xi - w_n)] - \sum_{k=1}^{n} \log_k(\xi - w_k)$.

For each value $\xi \in \mathbb{D}$, $\eta(\xi)$ is an integer multiple of $2\pi i$. Since η is analytic, it must be a constant function. Thus there is c_2 such that $\eta(\xi) = c_2$ for all $\xi \in \mathbb{D}$. Then $-\frac{2}{n}\eta(\xi) = -\frac{2c_2}{n} = -\frac{2}{n}\log[(\xi - w_1)\cdots(\xi - w_n)] + \frac{2}{n}\sum_{k=1}^{n} \log_k(\xi - w_k)$.

Now take the exponential both sides:

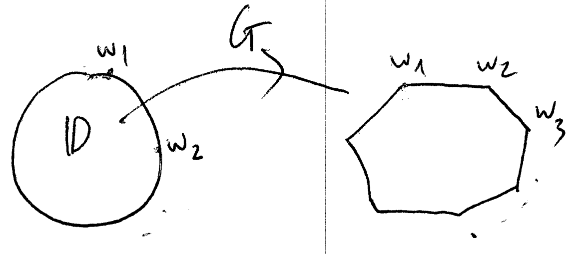
$$[(\xi - w_1)\cdots(\xi - w_n)]^{-2/n} = (\xi - w_1)^{-2/n}\cdots(\xi - w_n)^{-2/n} \exp\left(-\frac{2c_2}{n}\right)$$

Thus

$$F(w) = \underbrace{\exp\left(-\frac{2c_2}{n}\right)}_{c_3} \int_0^w \frac{d\xi}{(\xi - w_1)^{+2/n}\cdots(\xi - w_n)^{+2/n}}$$

Since c_2 is an integer multiple of $2\pi i$, c_3 has modulus 1 and accounts for a rotation about the origin. As we mentioned, this factor doesn't matter, Thus, we ~~should~~ again! ignore it (by considering c_3 as if 1). Then we have $F(w) = \int_0^w \frac{d\xi}{(\xi - w_1)^{2/n} \cdots (\xi - w_n)^{2/n}}$.

Let Ω be the interior of the regular polygon whose vertices are $w_1, \dots, w_n$. By Riemann mapping theorem, there is a conformal map G from $\mathbb{D}$ to Ω. Because Ω has inner angles equal to $\frac{(n-2)\pi}{n}$, then $\beta_1 = \beta_2 = \dots = \beta_n = \frac{2}{n}$.



By Schwartz-Christoffel's theorem, G has the form

$$G(w) = C \int_0^w \frac{d\xi}{(\xi - w_1)^{2/n} \cdots (\xi - w_n)^{2/n}} + C'$$

$$= C\,F(w) + C'.$$

Thus the image of G is obtained by a rotation, a dialation followed by a translation of the image of F. Since $G(\mathbb{D}) = \Omega$, $F(\mathbb{D})$ is also a ~~p~~ regular polygon of n-sides.

③ Problem 6, Ahlfors, p. 238.

Put $\Omega = \{ z = x + iy : x, y > 0,\ \min(x, y) < 1 \}$. We'll look for a conformal map $f : \mathbb{H} \to \Omega$. On the real line, we consider four consecutive segments $A = (-\infty, -1)$, $B = (-1, 0)$, $C = (0, 1)$ and $D = (1, \infty)$. On the boundary of Ω, we also name four segments

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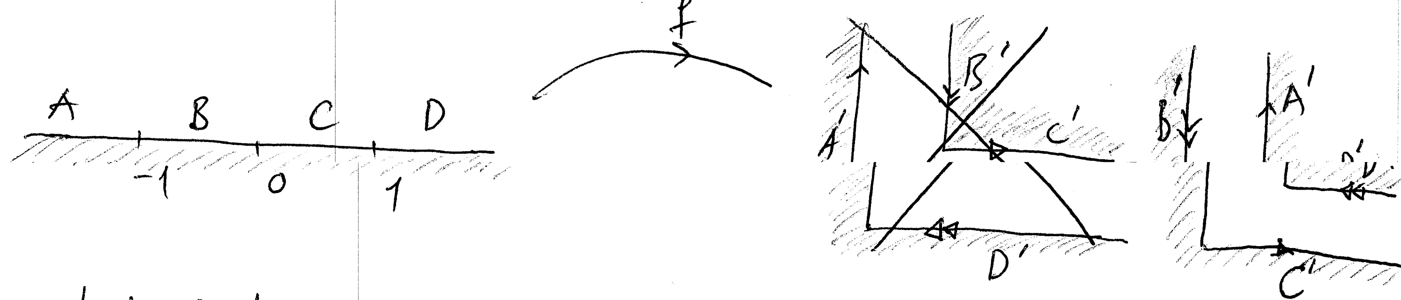
$A' = \{z = 0 + iy : y > 0\}$,

$B' = \{z = 1 + iy : y > 1\}$,

$C' = \{z = x + i : x > 1\}$,

$D' = \{z = x + 0 \cdot i : x > 0\}$.



We want to find a conformal map $f : \mathbb{H} \to \Omega$ that maps A to A', B to B', C to C' and D to D' with the direction sepecifred in the picture. The argument function $\arg : \mathbb{C} \setminus \{0\} \to \mathbb{R}/2\pi\mathbb{Z}$ is additive. We have

$$\arg f'(z) = \arg \frac{df}{dz} = \arg df - \arg dz.$$

Thus we have

$$\arg f'(z) = \begin{cases} \frac{\pi}{2} \pmod{2\pi} & \text{on } A \\ -\frac{\pi}{2} \pmod{2\pi} & \text{on } B \\ 0 \pmod{2\pi} & \text{on } C \\ \pi \pmod{2\pi} & \text{on } D \end{cases}$$

This hints us to put $f'(z) = \alpha e^{\pi i \theta} (z+1)^{\eta_1} z^{\eta_2} (z-1)^{\eta_3}$ where $\alpha > 0$, $\theta \in \mathbb{R}$, $-1 \leq \eta_1, \eta_2, \eta_3 \leq 1$ and the power functions are defined on the principal branch of the logarithm, i.e. $-\pi < \arg(\cdot) < \pi$.

We see that $\forall z \in \mathbb{H}$, $0 < \arg z, \arg(z-1), \arg(z+1) < \pi$. Moreover,

$$\arg(z+1)^{\eta_1} = \eta_1 \arg(z+1) \pmod{2\pi},$$

$$\arg z^{\eta_2} = \eta_2 \arg z \pmod{2\pi},$$

$$\arg(z-1)^{\eta_3} = \eta_3 \arg(z-1) \pmod{2\pi}.$$

Thus we get $\arg f'(z) = \pi\theta + \eta_1 \arg(z+1) + \eta_2 \arg z + \eta_3 \arg(z-1) \pmod{2\pi}$.

On A, we have $\frac{\pi}{2} = \pi\theta + \eta_1\pi + \eta_2\pi + \eta_3\pi \pmod{2\pi}$,

on B, we have $-\frac{\pi}{2} = \pi\theta + 0 + \eta_2\pi + \eta_2\pi \pmod{2\pi}$,

On C, we have $0 = \pi\theta + \pi\eta_3 \pmod{2\pi}$,

On D, we have $\pi = \pi\theta \pmod{2\pi}$.

Thus

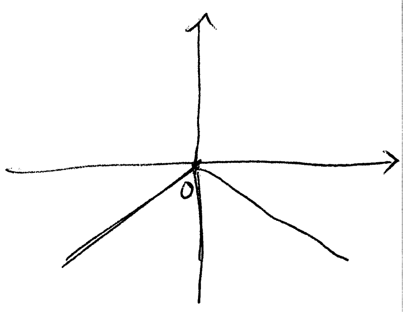
$$\begin{cases} \frac{1}{2} = \theta + \eta_1 + \eta_2 + \eta_3 \pmod 2 \\ -\frac{1}{2} = \theta + \eta_2 + \eta_3 \pmod 2 \\ 0 = \theta + \eta_3 \pmod 2 \\ 1 = \theta \pmod 2 \end{cases}$$

Since $f(z)$ blows up to infinity when z approaches ± 1 on the real line, we want $f'(z)$ does to. Thus we want $\eta_1, \eta_3 < 0$. Then we can choose $\theta = 1$, $\eta_1 = \eta_3 = -1$, $\eta_2 = -\frac{1}{2}$. Thus

$$f'(z) = \frac{-\alpha}{\sqrt{z}(z+1)(z-1)} \quad \text{for } z \in \mathbb{H}.$$

The function $z \mapsto \sqrt{z}$ can extend analytically to a domain covering $\mathbb{R} \setminus \{0\}$

by defining $\sqrt{z} := \exp\left(\frac{1}{2}\log|z| + \frac{1}{2} i \,\widetilde{\arg}\, z\right)$, where $-\frac{\pi}{4} < \widetilde{\arg}\, z < \frac{5\pi}{4}$.



This definition agrees with the function $\sqrt{z}$ from the beginning on $\mathbb{H}$.

~~The~~ A function f we are looking for should look like

$$f(z) = \alpha \int_0^z \frac{d\xi}{\sqrt{\xi}(\xi-1)(\xi+1)} + \beta \qquad (*)$$

Note that with this definition, $f(z)$ is continuous at 0. We'll adjust α, β such that $\mathbb{H}$ is maps exactly to Ω. We have

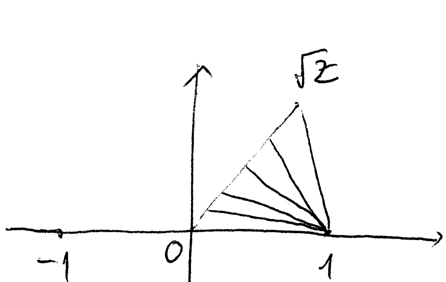
$$\int_0^z \frac{d\xi}{\sqrt{\xi}(\xi-1)(\xi+1)} = \frac{1}{2}\int_0^z \frac{d\xi}{\sqrt{\xi}(\xi-1)} - \frac{1}{2}\int_0^z \frac{d\xi}{\sqrt{\xi}(\xi+1)}$$

Put $u = \sqrt{\xi}$ for each $\xi \in \mathbb{H}$. Then u belongs to ~~the first quadrant~~ $\mathbb{H}$.

Thus $0 < \arg(u-1), \arg(u+1) < \pi$. Then

$$\int_0^z \frac{d\xi}{\sqrt{\xi}(\xi-1)(\xi+1)} = \frac{1}{2}\int_0^{\sqrt{z}} \frac{2u\,du}{u(u^2-1)} - \frac{1}{2}\int_0^{\sqrt{z}} \frac{2u\,du}{u(u^2+1)}$$

$$= \frac{1}{2}\int_0^{\sqrt{z}} \frac{du}{u-1} - \frac{1}{2}\int_0^{\sqrt{z}} \frac{du}{u+1} - \int_0^{\sqrt{z}} \frac{du}{u^2+1}$$



$$= \frac{1}{2}\left(\log(\sqrt{z}-1) - \pi i\right) - \frac{1}{2}\left(\log(\sqrt{z}+1)\right) - \int_0^{\sqrt{z}} \frac{du}{u^2+1}$$

Therefore,

$$f(z) = \frac{\alpha}{2}\left[\log(\sqrt{z}-1) - \log(\sqrt{z}+1) - \pi i - 2\int_0^{\sqrt{z}} \frac{du}{u^2+1}\right] + \beta$$

$(**)$

<u>On B</u> Put $f_1(x) = f(-x) = -\alpha \int_0^{-x} \frac{d\xi}{\sqrt{\xi}(\xi^2-1)} + \beta$ for $0<x<1$.

By the substitution $t = -\xi$, we get

$$f_1(x) = -\alpha \int_0^x \frac{-dt}{\sqrt{-t}(t^2-1)} + \beta = i\alpha \int_0^x \frac{dt}{\sqrt{t}(1-t^2)} + \beta.$$

Thus $f_1(0) = \beta$. Because we want $f(0)=0$, we choose $\beta = 0$. Because the map $x \in (0,1) \mapsto \int_0^x \frac{dt}{\sqrt{t}(1-t^2)}$ is an increasing function which goes to infinity as $x \to 1^-$, the image of f_1 is exactly B'. Thus f maps B to B' and respects the desired direction on B'.

<u>On C</u> $f(x) = -\alpha \underbrace{\int_0^x \frac{dt}{\sqrt{t}(t^2-1)}}$ $\forall x \in (0,1)$

$$\begin{cases} = 0 \text{ at } x = 0 \\ \text{increasing as } x \to 1^- \\ \to \infty \text{ as } x \to 1^- \end{cases}$$

Thus f maps C to C' with the desired direction on C'.

<u>On A</u> Put $f_2(x) = f(-x)$ for all $x>0$. By $(**)$ we have

$$f_2(x) = -\frac{\alpha}{2}\left[\log(\sqrt{-x}-1) - \log(\sqrt{-x}+1) - \pi i - 2\int_0^{\sqrt{-x}} \frac{du}{u^2+1}\right]$$

$$= -\frac{\alpha}{2}\left[\log(i\sqrt{x}-1) - \log(i\sqrt{x}+1) - \pi i - 2\int_0^{i\sqrt{x}} \frac{du}{u^2+1}\right] \quad (1)$$

Put $\theta = \arg(i\sqrt{x}-1)$. Then $\frac{\pi}{2} < \theta < \pi$ as show in the picture.

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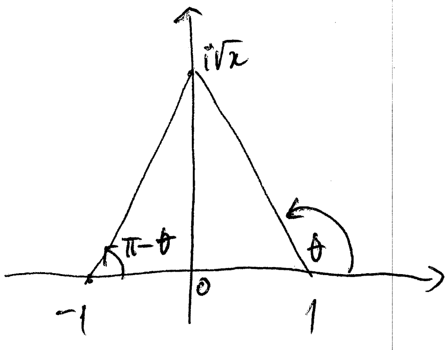


Then $\arg(i\sqrt{x}+1) = -\theta+\pi \in (0,\pi)$. Thus,

$$\arg \frac{i\sqrt{x}-1}{i\sqrt{x}+1} = \arg(i\sqrt{x}-1) - \arg(i\sqrt{x}+1) = 2\theta - \pi$$

Since $\left|\frac{i\sqrt{x}-1}{i\sqrt{x}+1}\right| = 1$, we have

$$\log \frac{i\sqrt{x}-1}{i\sqrt{x}+1} = i \arg \frac{i\sqrt{x}-1}{i\sqrt{x}+1} = i(2\theta-\pi) \qquad (2)$$

We have $\displaystyle\int_0^{i\sqrt{x}} \frac{du}{u^2+1} = \int_0^{i\sqrt{x}} \frac{du}{(1-iu)(1+iu)}$

$$= -\frac{1}{2}\int_0^{i\sqrt{x}} \frac{du}{iu-1} + \frac{1}{2}\int_0^{i\sqrt{x}} \frac{du}{iu+1}$$

$$= \frac{i}{2}\int_0^{i\sqrt{x}} \frac{du}{u+i} - \frac{i}{2}\int_0^{i\sqrt{x}} \frac{du}{u-i}$$

$$= \frac{i}{2}\log(u+i)\Big|_0^{i\sqrt{x}} - \frac{i}{2}\log(u-i)\Big|_0^{i\sqrt{x}}$$

$$= \frac{i}{2}\left[\log(i\sqrt{x}+i) - \log i\right] - \frac{i}{2}\left[\log(i\sqrt{x}-i) - \log(-i)\right]$$

$$= \frac{i}{2}\log(\sqrt{x}+1) - \frac{i}{2}\left[\log(\sqrt{x}-1) + i\pi\right]$$

$$= \frac{i}{2}\log\frac{\sqrt{x}+1}{\sqrt{x}-1} + \frac{\pi}{2} \qquad (3)$$

Substituting (2) and (3) into (1), we get

$$f_2(x) = -\frac{\alpha}{2}\left(i(2\theta-\pi) - \pi i - i\log\frac{\sqrt{x}+1}{\sqrt{x}-1} - \pi\right)$$

Thus $$f_2(x) = \frac{\alpha\pi}{2} + \frac{i\alpha}{2}\Big(\underbrace{\log\frac{\sqrt{x}+1}{\sqrt{x}-1} + 2\pi - 2\theta}\Big)$$

$$\begin{cases} \text{strictly increasing in } x \\ \to \infty \text{ as } x \to 1^+ \\ \to 2\pi - 2\cdot\frac{\pi}{2} = \pi \text{ as } x \to \infty \end{cases}$$

Because we want the image of f_2 to be exactly A', we must have

$$\frac{\alpha\pi}{2} = 1 \iff \alpha = \frac{2}{\pi}.$$

Thus we obtain two forms of $f(z)$, one from $(*)$ and one from $(**)$:

$$f(z) = -\frac{2}{\pi}\int_0^z \frac{d\xi}{\sqrt{\xi}(\xi-1)(\xi+1)}, \qquad (4)$$

$$f(z) = -\frac{1}{\pi}\Big[\log(\sqrt{z}-1) - \log(\sqrt{z}+1) - \pi i - 2\int_0^{\sqrt{z}} \frac{du}{u^2+1}\Big]. \qquad (5)$$

<u>On D</u> we have $\forall x > 1$,

$$f(x) = -\frac{1}{\pi}\Big(\log\frac{\sqrt{x}-1}{\sqrt{x}+1} - \pi i - 2\int_0^{\sqrt{x}} \frac{du}{u^2+1}\Big)$$

$$= \frac{1}{\pi}\Big(\underbrace{2\arctan(\sqrt{x}) - \log\frac{\sqrt{x}-1}{\sqrt{x}+1}}\Big) + i$$

$$\begin{cases} \text{derivative} = \dfrac{-2}{\sqrt{x}(x^2-1)} < 0, \\ \to \infty \text{ as } x \to 1^-, \\ \to \pi \text{ as } x \to \infty \end{cases}$$

Thus f maps D to D' and respect the direction on D' as desired.

With the computation

$$2\int_0^{\sqrt{z}} \frac{du}{u^2+1} = +i\int_0^{\sqrt{z}} \frac{du}{u+i} - i\int_0^{\sqrt{z}} \frac{du}{u-i}$$

$$= i\log(u+i)\Big|_0^{\sqrt{z}} - i\log(u-i)\Big|_0^{\sqrt{z}}$$

$$= i\left(\log(\sqrt{z}+i) - \log i\right) - i\left(\log(\sqrt{z}-i) - \log(-i)\right)$$

$$= +i\log(\sqrt{z}+i) - i\log(\sqrt{z}-i) + \pi$$

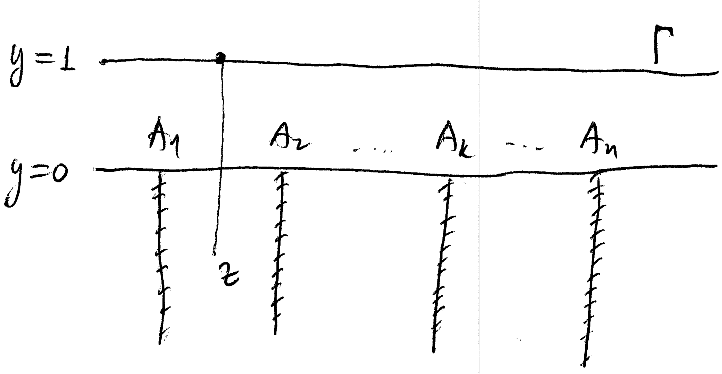
$$= i\log\frac{\sqrt{z}+i}{\sqrt{z}-i} + \pi,$$

the form (5) can be reduced as

$$f(z) = -\frac{1}{\pi}\left(\log\frac{\sqrt{z}-1}{\sqrt{z}+1} - i\log\frac{\sqrt{z}+i}{\sqrt{z}-i}\right) + (1+i).$$

④ Let $A_1, A_2, \ldots, A_n$ be real numbers and $\Omega = \mathbb{C}\setminus\bigcup_{k=1}^{n}\{A_k + iy : y \leq 0\}$.

We'll show that Ω is simply connected. Put $\Gamma = \{z\in\mathbb{C} : \operatorname{Im} z = y = 1\}$.



Then Γ is just a line and thus simply connected. To show that Ω is simply connected, we will show that Γ is a retract of Ω. We define a function H as follow

$$H: \Omega\times[0,1] \to \Omega$$

$$H(z,t) = (x, (1-t)y + t)$$

$$= z + it(1-y)$$

To check if H is well-defined, we take $z = A_k + iy$ $(y>0)$ and check if $H(z,t) \in \Omega$ for all $t \in [0,1]$. We have $H(z,t) = A_k + i[(1-t)y + t]$ and $(1-t)y + t \geq$ ~~y~~ for all $t \in [0,1]$. Thus H is well-defined. We see that H is also continuous and $H(z,0) = z$ $\forall z \in \Omega$, $H(z,1) = x + i$ $\forall z \in \Omega$. Thus H is a deformation retraction of Ω onto Γ. Thus $\pi_1(\Omega,*) \simeq \pi_1(\Gamma,*) \simeq \{0\}$. Therefore Ω is simply connected. While the argument works, it's easier for everyone to stay in the $\mathbb{C}$-analysis realm.

(5) Define a function $u : \mathbb{R}^2 \setminus \{(0,1)\} \to \mathbb{R}$, $u(x,y) = Re\left(\frac{i+z}{i-z}\right)$.

Because u is the real part of an analytic function on $\mathbb{C} \setminus \{i\}$, u is harmonic in $\mathbb{R}^2 \setminus \{0\}$. In particular, u is harmonic in the unit disk. We have

$$u(x,y) = Re\left(\frac{x+(y+1)i}{-x+(1-y)i}\right) = Re\left[\frac{(x+(y+1)i)(-x-(1-y)i)}{x^2+(1-y)^2}\right]$$

$$= Re\left[\frac{-x^2+(1-y^2)}{x^2+(1-y)^2} + i\frac{-2x}{x^2+(1-y)^2}\right]$$

$$= \frac{1-(x^2+y^2)}{x^2+(1-y)^2}$$

For every (x,y) on the unit ~~disk~~ circle except $(0,1)$, we have $u(x,y) = 0$. At the point $(0,1)$, u was defined to be 0 by the problem.