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Math 8702: Complex Analysis

Homework 4

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① Problem 1, Ahlfors p. 247

We'll show that the function $u: \mathbb{R}^2 \to \mathbb{R}$, $u(x,y) = |x|$ is subharmonic. We see that u is continuous and $|x| = \max\{x, -x\}$ for all $x \in \mathbb{R}$. Put $u_1, u_2 : \mathbb{R}^2 \to \mathbb{R}$, $u_1(x,y) = x$, $u_2(x,y) = -x$ for all $x, y \in \mathbb{R}$. Then $u = \max(u_1, u_2)$. Since u_1 and u_2 are linear, they are harmonic and thus subharmonic. Thus u is also subharmonic (the elementary property 3 at the bottom of page 246, Ahlfors).

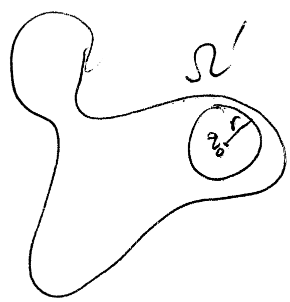
To show that $|z|^\alpha$ $(\alpha \geq 0)$ and $\log(1+|z|^2)$ are subharmonic, we apply the result of problem ② for the domain $\Omega = \mathbb{C}$ and $f(z) = z$.

② Problem 2, Ahlfors p. 248

Let $\Omega \subset \mathbb{C}$ be an open connected subset, $f: \Omega \to \mathbb{C}$ be analytic, α be a nonnegative real number. ~~Define~~

(a) Define $v: \Omega \to \mathbb{R}$, $v(z) = |f(z)|^\alpha$. We'll show that v is subharmonic. For $\alpha = 0$, v is a constant function on Ω $(v \equiv 1)$. Thus v is subharmonic. We consider the case $\alpha > 0$. Take any open connected subset $\Omega' \subset \Omega$ and any harmonic function $u: \Omega' \to \mathbb{R}$ such that $v - u$ attains a maximum at some $z_0 \in \Omega'$. We want to show that $v - u \equiv$ constant throughout Ω'.

Put $A = v(z_0) - u(z_0) = \max\{v(z) - u(z) : z \in \Omega'\}$. By replacing u with $u + A$, we can assume $A = 0$. Put $S = \{z \in \Omega' : v(z) - u(z) = 0\}$. Then $S \neq \emptyset$ because $z_0 \in S$. Because v and u are continuous in Ω', the limit in Ω' of any sequence in S must lie in S. Thus S is closed in Ω'. To show that $S = \Omega'$, we only need to show that S is open because Ω' is connected. Thus we only need to show that $v(z) - u(z) = 0$ on a neighborhood of z_0 (the proof is the same for other elements of S).



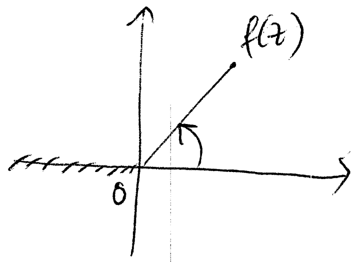
Take a closed disk $\overline{D}(z_0, r) \subset \Omega'$. We'll show that $v(z) - u(z) = 0$ on $D(z_0, r)$. We have

$$v(z) = |f(z)|^\alpha \le u(z) \quad \forall z \in D(z_0, r), \qquad (*)$$
$$v(z_0) = |f(z_0)|^\alpha = u(z_0)$$

• If $f(z_0) = 0$ then $\begin{cases} u(z) \ge 0 & \forall z \in D(z_0, r), \\ u(z_0) = 0. \end{cases}$

Thus u attains a minimum in $D(z_0, r)$. By the minimum principle for harmonic functions, we conclude that $u(z) = 0$ $\forall z \in D(z_0, r)$. Then $(*)$ implies $0 = v(z) = u(z)$ $\forall z \in D(z_0, r)$.

• If $f(z_0) \neq 0$ then there is $\theta \in [0, 2\pi]$ such that $e^{i\theta} f(z_0) = |f(z_0)| > 0$. Replacing $f(z)$ by $\tilde{f}(z) = e^{i\theta} f(z)$ does not change the function v. Thus, we can assume that $f(z_0) > 0$. Shrink r if necessary so that $f(z) \neq 0$ and $|\arg f(z)| < \frac{\pi}{\alpha}$.



Then the map $z \mapsto \log(f(z))$ is well-defined on $D(z_0,r)$ and analytic. Define $g: D(z_0,r) \to \mathbb{C}$,

$$g(z) = f(z)^\alpha := \cancel{\log}\, \exp(\alpha \log f(z))$$

Then g is analytic and $g(z_0) = f(z_0)^\alpha = u(z_0) \in \mathbb{R}$,

$$\operatorname{Re} g(z) \le |g(z)| = |f(z)|^\alpha \le u(z) \quad (**) \text{ for all } z \in D(z_0,r).$$

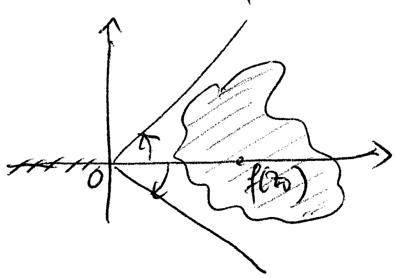
Thus $\operatorname{Re} g(z) - u(z)$ is a harmonic function on $D(z_0,r)$ such that it is ≤ 0 in $D(z_0,r)$ and $=0$ at z_0. Thus by maximum principle, $\operatorname{Re} g(z) - u(z) \equiv 0$ on $D(z_0,r)$. From $(**)$, we have $g(z) = u(z)$ for all $z \in D(z_0,r)$. Thus $v(z) = |f(z)|^\alpha = |g(z)| = u(z)$ on $D(z_0,r)$.

(b) Define $w: \Omega \to \mathbb{R}$, $w(z) = \log(1+|f(z)|^2)$. we want to show that w is subharmonic. The set up of the first part is the same as part (a): we take an arbitrary closed disk $\overline{D}(z_0,r) \subset \Omega$ and a harmonic function $u: D(z_0,r) \to \mathbb{R}$ such that $w-u: D(z_0,r) \to \mathbb{R}$ attains a maximum (assumed to be zero) at z_0. we want to show that $w(z)-u(z) = 0$ for all $z \in D(z_0,r)$. We have

$$w(z) = \log\left(1 + |f(z)|^2\right) \le u(z) \quad \forall z \in D(z_0, r), \qquad (***)$$

$$w(z_0) = \log\left(1 + |f(z_0)|^2\right) = u(z_0).$$

- If $f(z_0) = 0$ then $u(z) \geq 0$ $\forall z \in D(z_0, r)$ and $u(z_0) = 0$. By the minimum principle, $u(z) \equiv 0$ on $D(z_0, r)$. By $(***)$, we get $0 = w(z) = u(z)$ for all $z \in D(z_0, r)$.
- If $f(z_0) \neq 0$ then there is $t \in [0, 2\pi]$ such that $e^{it} f(z_0) = |f(z_0)| > 0$. Replacing $f(z)$ by $\tilde{f}(z) = e^{it} f(z)$ doesn't change the function w. Thus we can assume $f(z_0) > 0$. Shrink r if necessary so that $f(z) \neq 0$ and $|\arg(f(z))| < \frac{\pi}{4}$ for all $z \in D(z_0, r)$.

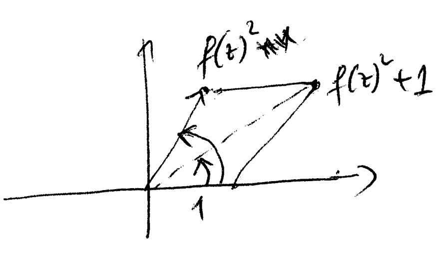


Then $|\arg(f(z)^2)| < \frac{\pi}{2}$. Thus,

$$\arg\left(1 + f(z)^2\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Thus we can define an analytic function

$$h : D(z_0, r) \to \mathbb{R}, \quad h(z) = \log\left(1 + f(z)^2\right).$$



We have

$$h(z_0) = \log\left(1 + f(z_0)^2\right) = \log\left(1 + |f(z_0)|^2\right) = u(z_0) \in \mathbb{R}.$$

Then $\operatorname{Re} h(z_0) = u(z_0)$. Also,

$$\begin{aligned} \operatorname{Re} h(z) = \operatorname{Re}\left[\log\left(1 + f(z)^2\right)\right] &= \log\left|1 + f(z)^2\right| \\ &\le \log\left(1 + |f(z)|^2\right) \\ &\le u(z), \qquad (****) \\ &\forall z \in D(z_0, r) \end{aligned}$$

Thus $\mathrm{Re}\, h(z) - u(z)$ is a harmonic function such that it is ≤ 0 on $D(z_0, r)$ and $= 0$ at z_0. By the maximum principle, $\mathrm{Re}\, h(z) = u(z)$ for all $z \in D(z_0, r)$. Thus all equalities in (****) happens at once. Thus,

$$w(z) = \log\left(1 + |f(z)|^2\right) = u(z) \quad \forall z \in D(z_0, r).$$

③ Problem 3, Alhfors page 248.

Let Ω be an open connected subset of $\mathbb{C}$ and $v : \Omega \to \mathbb{R}$ be a function in $C^2(\Omega)$. We'll show that v is ~~har~~ subharmonic iff $\Delta v \geq 0$ on Ω.

For any closed disk $\overline{D}(z_0, R) \subset \Omega$, we define a function $\phi_{z_0, R} : [0, R) \to \mathbb{R}$

$$\phi_{z_0, R}(r) = \frac{1}{2\pi} \int_0^{2\pi} v(z_0 + re^{i\theta})\, d\theta$$

Because $\phi_{z_0, R}(0) = v(z_0)$, we have the following characterization of subharmonicity

$$v \text{ is subharmonic} \iff \phi_{z_0, R}(r) \geq \phi_{z_0, R}(0) \quad \forall\, 0 < r < R, \ \forall \text{ choice of } \overline{D}(z_0, R) \subset \Omega) \qquad (*)$$

For a moment, we shall denote $\phi_{z_0, R}$ by ϕ for simplicity. First we'll show that ϕ is continuous on $[0, R)$. Since $|\nabla v|$ is continuous on $\overline{D}(z_0, R)$, it is ~~un~~bounded on $\overline{D}(z_0, R)$. Thus there is $M \in \mathbb{R}$ such that

$$|\nabla v(z)| \leq M, \quad \forall z \in \overline{D}(z_0, R)$$

For any $0 \leq r$, $r + h < R$, we have

$$\phi(r+h) - \phi(r) = \frac{1}{2\pi} \int_0^{2\pi} \left[v(z_0 + re^{i\theta} + he^{i\theta}) - v(z_0 + re^{i\theta}) \right] d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \nabla v(z_0 + re^{i\theta} + \eta e^{i\theta}) \cdot he^{i\theta}\, d\theta \quad \text{for some } 0 \le |\eta| \le |h|$$

Thus, $$|\phi(r+h) - \phi(r)| \le \frac{1}{2\pi}\int_0^{2\pi} |\nabla v(z_0 + re^{i\theta} + \eta e^{i\theta})|\, |h|\, d\theta$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} M|h|\, d\theta = M|h|.$$

Thus $\lim_{h\to 0}(\phi(r+h) - \phi(r)) = 0$. Therefore ϕ is continuous on $[0, R)$.

Next, we'll show that ϕ is differentiable on $(0, R)$.

For $0 < r, r+h < R$, we have

$$\frac{\phi(r+h) - \phi(r)}{h} = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\frac{v(z_0 + re^{i\theta} + he^{i\theta}) - v(z_0 + re^{i\theta})}{h}}_{\psi(h,\theta)}\, d\theta$$

we have $\lim_{h\to 0} \phi(h,\theta) = \frac{\partial v}{\partial e^{i\theta}}(z_0 + re^{i\theta})$, where $e^{i\theta} = (\cos\theta, \sin\theta)$ represents a unit vector in $\mathbb{R}^2$.

As shown in the continuity part, we have

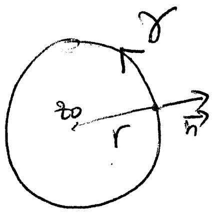
$$|\phi(h,\theta)| \le \frac{M|h|}{|h|} = M \quad \text{for all } h \text{ sufficiently small, } \theta \in [0, 2\pi].$$

Thus, by Lebesgue's Dominated Convergence theorem, we have

$$\lim_{h\to 0} \frac{\phi(r+h) - \phi(r)}{h} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial v}{\partial e^{i\theta}}(z_0 + re^{i\theta})\, d\theta$$

$$= \frac{1}{2\pi r}\int_0^{2\pi} \nabla v(z_0 + re^{i\theta}) \cdot re^{i\theta}\, d\theta$$

dot product in $\mathbb{R}^2$



Put $\gamma(\theta) = z_0 + re^{i\theta}$. Then $\gamma'(\theta) = rie^{i\theta}$. Then

$$\phi'(r) = \frac{1}{2\pi r}\int_0^{2\pi} \nabla v(\gamma(\theta)) \cdot \underbrace{e^{i\theta}}_{\vec{n}} \underbrace{|\gamma'(\theta)|\, d\theta}_{ds}$$

$$= \frac{1}{2\pi r}\int_\gamma \nabla v \cdot \vec{n}\, ds$$

$$= \frac{1}{2\pi r}\int_{D(z_0, r)} (\Delta v)\, dA \quad \text{by Divergence Theorem.}$$

Therefore, ϕ is differentiable on $(0, R)$ and

$$\phi'(r) = \frac{1}{2\pi r}\int_{D(z_0, r)} (\Delta v)\, dA. \qquad (**)$$

Now we return to the problem.

"$\Rightarrow$" Suppose that v is subharmonic in Ω and suppose by contradiction that $\Delta v(z_0) < 0$ for some $z_0 \in \Omega$. Since Δv is continuous, there exists $R > 0$ such that $\bar{D}(z_0, R) \subset \Omega$ and $\Delta v(z) < 0$ for all $z \in \bar{D}(z_0, R)$. Put $\phi = \phi_{z_0, R}$. Then $\Delta v(z) < 0$ for all $z \in D(z_0, r)$ and $0 < r < R$. By $(**)$, we get $\phi'(r) < 0$ for all $0 < r < R$. Thus ϕ is a strictly decreasing function on $[0, R)$. Thus $\phi(r) < \phi(0)$ for all $0 < r < R$. This contradicts the characterization of subharmonicity at $(*)$.

"$\Leftarrow$" Suppose that $\Delta v(z) \geq 0$ for all $z \in \Omega$. For each closed disk $\bar{D}(z_0, R) \subset \Omega$, we put $\phi = \phi_{z_0, R} : [0, R) \to \mathbb{R}$. By $(**)$, $\phi'(r) \geq 0$ for all $0 < r < R$. Thus ϕ is an increasing function. Thus $\phi(r) \geq \phi(0)$ for all

$0<r<R$. This means v satisfies $(*)$, and thus is subharmonic.

④ Problem 5, Ahlfors p. 248

Let $\Omega \subset \mathbb{C}$ be an open connected subset. For each $n \in \mathbb{N}$, suppose there is a continuous and subharmonic function $v_n : \Omega \to \mathbb{R}$ such that (v_n) converges to some function $v : \Omega \to \mathbb{R}$ uniformly on every compact subset of Ω. Then v is also continuous and subharmonic.

Proof Take an arbitrary closed disk $\overline{D}(z_0, r) \subset \Omega$. We'll show that v is continuous in $D(z_0, r)$ and $v(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} v(z_0 + re^{i\theta})\,d\theta$.

Because $\overline{D}(z_0, r)$ is compact and $v|_{\overline{D}(z_0,r)}$ is a uniform limit of a sequence of continuous functions, it is also continuous. For each $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that $|v_n(z) - v(z)| < \varepsilon$ for all $n > N(\varepsilon)$ and for all $z \in \overline{D}(z_0, r)$.

Because v_n is subharmonic, $v_n(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} v_n(z_0 + re^{i\theta})\,d\theta$.

Then
$$v(z_0) - \varepsilon < v_n(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} v_n(z_0 + re^{i\theta})\,d\theta \leq \frac{1}{2\pi}\int_0^{2\pi}\left(v(z_0 + re^{i\theta}) + \varepsilon\right)d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} v(z_0 + re^{i\theta})\,d\theta + \varepsilon$$
for all $n > N(\varepsilon)$.

Thus $v(z_0) \leq 2\varepsilon + \frac{1}{2\pi}\int_0^{2\pi} v(z_0 + re^{i\theta})\,d\theta$.

Since ε was chosen arbitrarily positive, $v(z) \leq \frac{1}{2\pi}\int_0^{2\pi} v(z_0 + re^{i\theta})\,d\theta$.

<u>Note</u>: There is a more general definition of subharmonic functions in which the continuity is not necessary. Namely, a function $v : \Omega \to \mathbb{R}$ is called subharmonic if for every closed disk $\overline{D}(z_0, r) \subset \Omega$ and for every continuous function $u : \overline{D}(z_0, r) \to \mathbb{R}$ such that u is harmonic in $D(z_0, r)$ and $v \leq u$ on $\partial D(z_0, r)$, we have $v \leq u$ in $D(z_0, r)$.

With this definition, we can state a more general theorem about a uniform limit of subharmonic functions as follow: Let $\Omega \subset \mathbb{C}$ be an open and connected subset. For each $n \in \mathbb{N}$, suppose there is a subharmonic function $v_n : \Omega \to \mathbb{R}$ such that (v_n) converges to some function $v : \Omega \to \mathbb{R}$ uniformly on every compact subset of Ω. Then v is also subharmonic.

<u>Proof</u> Let $\overline{D}(z_0, r) \subset \Omega$ and $u : \overline{D}(z_0, r) \to \mathbb{R}$ be continuous, harmonic in $D(z_0, r)$ and $v \leq u$ on $\partial D(z_0, r)$. We'll show that $v \leq u$ in $D(z_0, r)$.

For each $\varepsilon > 0$, there is $N(\varepsilon) \in \mathbb{N}$ such that $v_n < u + \varepsilon$ on $\partial D(z_0, r)$ for all $n > N(\varepsilon)$. Then $v_n < v + \varepsilon \leq u + \varepsilon$ on $\partial D(z_0, r)$ for all $n > N(\varepsilon)$.

Since $u + \varepsilon$ is also a harmonic function, and v_n is subharmonic, we have

$$v_n(z) \leq u(z) + \varepsilon \qquad \forall z \in D(z_0, r), \ \forall n > N(\varepsilon)$$

Fix each $z \in D(z_0, r)$ and let n tend to infinity, we get $v(z) \leq u(z) + \varepsilon$.

Since ε was chosen arbitrarily positive, we get $v(z) \leq u(z)$.