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Math 8702: Complex Analysis
Homework 5

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(1) Problem 1, Ahlfors, p. 274

Let $f(z)$ be an even elliptic function whose the set of periods is

$$\Lambda = \{ n\omega_1 + m\omega_2 ; \ m, n \in \mathbb{Z} \}.$$

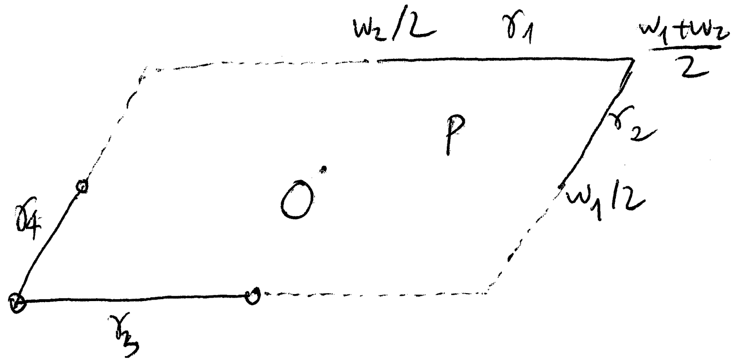
Put
$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right), \quad \forall z \in \mathbb{C} \setminus \Lambda.$$

Then $\wp(z)$ is the Weierstrass $\wp$-function corresponding to the lattice Λ.

(a) Suppose that 0 is not a zero nor a pole of $f(z)$. We'll show that there are $a_1, \dots, a_n, b_1, \dots, b_n, C \in \mathbb{C}$ such that

$$f(z) = C \frac{(\wp(z) - \wp(a_1)) \cdots (\wp(z) - \wp(a_n))}{(\wp(z) - \wp(b_1)) \cdots (\wp(z) - \wp(b_n))} \quad \forall z \in \mathbb{C} \setminus \Lambda.$$

We will consider a special fundamental parallelogram. Put



$$\gamma_1(t) = \frac{\omega_2}{2} + t\frac{\omega_1}{2}, \quad 0 < t < 1$$

$$\gamma_2(t) = \frac{\omega_1}{2} + t\frac{\omega_2}{2}, \quad 0 < t < 1$$

$$\gamma_3(t) = -\frac{\omega_2}{2} - t\frac{\omega_1}{2}, \quad 0 < t < 1$$

$$\gamma_4(t) = -\frac{\omega_1}{2} - t\frac{\omega_2}{2}, \quad 0 < t < 1$$

$$P_{int} = \left\{ z = \frac{\omega_1}{2}t + \frac{\omega_2}{2}s : \ -1 < t, s < 1 \right\}$$

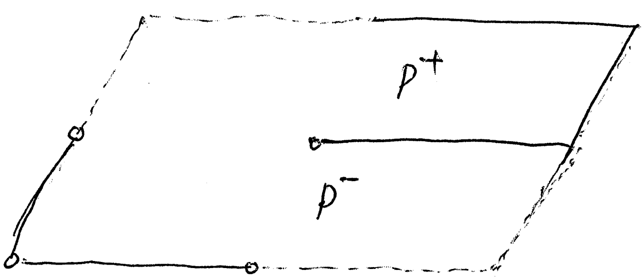
Then we get a fundamental parallelogram of the lattice

$$P = P_{int} \cup \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \cup \left\{ \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2} \right\}.$$

Put $P^+ = P \cap \{ z = \frac{\omega_1}{2} t + \frac{\omega_2}{2} s : (-1 \le t \le 1, s > 0) \text{ or } (0 < t \le 1, s = 0) \}$

$P^- = P \setminus (P^+ \cup \{0\})$

Thus P is a disjoint union $P = P^+ \cup P^- \cup \{0\}$.



Put $A = \left\{ \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2} \right\}$.

These are the half periods in P. We have an important property:

$$a \in P^+ \setminus A \iff -a \in P^-$$

For each $c \in P \setminus \{0\}$, $\wp(z) - \wp(c)$ is an elliptic function with periods ω_1 and ω_2. Because $z = 0$ is the only pole, which is a double pole, of $\wp(z) - \wp(c)$ in P, it is of order 2. If $c \notin A$ then $\pm c$ are zeros of $\wp(z) - \wp(c)$ in P. These are the only zeros of $\wp(z) - \wp(c)$ in P because $\#(\text{zeros}) = \#(\text{poles})$. If $c \in A$, we know that $\wp'(c) = 0$ and c is a single zero of $\wp'$. Thus c is a double zero of $\wp(z) - \wp(c)$. It is the only zero of $\wp(z) - \wp(c)$ in P because $\#(\text{zeros}) = \#(\text{poles})$.

Because 0 is not a zero nor a pole of $f(z)$, all zeros and poles of $f(z)$ lie in $P^+ \cup P^-$. We see that $a \in P^+ \setminus A$ is a zero of $f(z)$ iff $-a \in P^-$

is also a zero. By the previous paragraph, $\pm a$ are the only zeros of $\wp(z)-\wp(a)$. Thus $\dfrac{f(z)}{\wp(z)-\wp(a)}$ is an elliptic function whose orders of zeros at $\pm a$ decrease by 1 compared to that of $f(z)$. This function gives rise to a double zero at $z=0$. Other than 0 and $\pm a$, $\dfrac{f(z)}{\wp(z)-\wp(a)}$ and $f(z)$ have the same zeros with the same orders.

Let $a_1,\dots,a_r$ be all zeros of $f(z)$ in $P^+\setminus A$ (enumerated with multiplicity). Then $-a_1,\dots,-a_r$ are all zeros of $f(z)$ in P^- (enumerated with multiplicity). The function

$$f_1(z)=\frac{f(z)}{(\wp(z)-\wp(a_1))\cdots(\wp(z)-\wp(a_r))}$$

thus has no zero in $P\setminus(A\cup\{0\})$. Note that $f_1(z)$ is also elliptic.

We do similarly to poles: $b\in P^+\setminus A$ is a pole of $f(z)$ iff $-b\in P^-$ is also a pole. Also, $\pm b$ are the only zeros of $\wp(z)-\wp(b)$. Thus $f(z)(\wp(z)-\wp(b))$ is elliptic function whose orders of poles at $\pm b$ decrease by 1 compared to that of $f(z)$. This function gives rise to a double pole at $z=0$. Other than 0 and $\pm b$, $f(z)(\wp(z)-\wp(b))$ and $f(z)$ have the same poles with the same orders.

Let $b_1,\dots,b_s$ be all poles of $f(z)$ in $P^+\setminus A$ (counted with multiplicity)

Then $-b_1, \ldots, -b_s$ are all poles of $f(z)$ in P^- (counted with multiplicity).

Then the function

$$f_2(z) = f(z) \frac{(\wp(z)-\wp(b_1)) \cdots (\wp(z)-\wp(b_s))}{(\wp(z)-\wp(a_1)) \cdots (\wp(z)-\wp(a_r))}$$

is elliptic and has no pole nor zero in $P \setminus (A \cup \{0\})$.

Now we will show that the order of poles and zeros at the half period must be even (which can be 0). We write the Laurent series of $f(z)$ about $\omega_1/2$, $\omega_2/2$, $(\omega_1+\omega_2)/2$ as follow.

$$f(z) = \alpha_1 \left(z - \frac{\omega_1}{2}\right)^{h_1} + \cdots$$

$$f(z) = \alpha_2 \left(z - \frac{\omega_2}{2}\right)^{h_2} + \cdots$$

$$f(z) = \alpha_3 \left(z - \frac{\omega_1+\omega_2}{2}\right)^{h_3} + \cdots$$

If $h_i > 0$ then we have a zero; $h_i < 0$ a pole; $h_i = 0$ neither a zero nor a pole. Since $f(z)$ is elliptic, $\#(\text{zeros}) = \#(\text{poles})$. in P. Thus,

$$\begin{aligned} 0 = \#(\text{zeros}) - \#(\text{poles}) &= \#(\text{zeros in } P \setminus (A \cup \{0\})) - \#(\text{poles in } P \setminus (A \cup \{0\})) \\ &\quad + \#(\text{zeros in } A) - \#(\text{poles in } A) \\ &= 2r - 2s + (h_1 + h_2 + h_3) \end{aligned}$$

Thus $h_1 + h_2 + h_3$ ~~are~~ is even. (1)

Also, we know that $\sum \text{zeros} - \sum \text{poles} \in \Lambda$. Thus we must have

$$\Lambda \ni \sum \text{zeros} - \sum \text{poles} = \sum\left(\text{zeros in } P\setminus(A\cup\{0\})\right) - \sum\left(\text{poles in } P\setminus(A\cup\{0\})\right)$$
$$+ \sum\left(\text{zeros in } A\right) - \sum\left(\text{poles in } A\right)$$
$$= h_1\frac{\omega_1}{2} + h_2\frac{\omega_2}{2} + h_3\frac{\omega_1+\omega_2}{2}.$$

Here we use the fact that $\pm a_1, \dots, \pm a_r$ are all zeros of $f(z)$ in $P\setminus(A\cup\{0\})$ in $P\setminus(A\cup\{0\})$. That is why there sume is zero. The same is for the poles $\pm b_1, \dots, \pm b_s$. Thus, $\frac{\omega_1}{2}(h_1+h_3) + \frac{\omega_2}{2}(h_2+h_3) \in \Lambda$.

Thus h_1+h_3 and h_2+h_3 are even. (2)

From (1) and (2), we conclude that h_1, h_2, h_3 are even. Thus the poles and zeros of $f(z)$ in A are of even orders. Let $a_{r+1}, \dots, a_{r+k}$ be all zeros of $f(z)$ in A which are enumerated with half of the multiplicity. E.g. if a is a zero of order 4 then a will be listed only twice. ~~Then~~ ~~the~~ Since $\wp(z) - \wp(a_{r+1})$ has a double zero at a_{r+1}, the function

$$\frac{f(z)}{(\wp(z)-\wp(a_{r+1}))\cdots(\wp(z)-\wp(a_{r+k}))}$$

has no zeros in A.

Let $b_{s+1}, \dots, b_{s+l}$ be all poles of $f(z)$ in A which are enumerated with half of the multiplicity. Then

$$f(z)\,\frac{(\wp(z)-\wp(b_{s+1}))\cdots(\wp(z)-\wp(b_{s+l}))}{(\wp(z)-\wp(a_{r+1}))\cdots(\wp(z)-\wp(a_{r+k}))}$$

has no zero nor pole in A. Then the function

$$g(z) = f(z)\ \frac{(\wp(z)-\wp(b_1))\cdots(\wp(z)-\wp(b_{s+\ell}))}{(\wp(z)-\wp(a_1))\cdots(\wp(z)-\wp(a_{r+k}))}$$

is elliptic and has no pole nor zero in $P\setminus\{0\}$. We have

$$h_1+h_2+h_3 = \#(\text{zeros in } A) - \#(\text{poles in } A) = 2k-2\ell$$

By (1), we get $0 = 2r-2s+(2k-2\ell)$. Thus $r+k = s+\ell$ ($=$ some n).

Thus,
$$g(z) = f(z)\ \frac{(\wp(z)-\wp(b_1))\cdots(\wp(z)-\wp(b_n))}{(\wp(z)-\wp(a_1))\cdots(\wp(z)-\wp(a_n))}.$$

Because the numerator and denominator have pole at 0 of order $2n$, the fraction has no pole nor zero at 0. Thus $g(z)$ has no pole nor zero at every point in P. Thus $g(z)$ is holomorphic and doubly periodic in $\mathbb{C}$. Thus $g(z)\equiv C$ - which is a nonzero constant. Thus,

$$f(z) = C\ \frac{(\wp(z)-\wp(a_1))\cdots(\wp(z)-\wp(a_n))}{(\wp(z)-\wp(b_1))\cdots(\wp(z)-\wp(b_n))} \qquad \cancel{\forall z\in\mathbb{C}\setminus\Lambda}.$$

(b) Now suppose that $f(z)$ can have zeros or poles at $z=0$. We note the $f(z)$ is even. Thus the Laurent series of $f(z)$ about $z=0$ contains only the even powers of z. We are interested in the first term only.

$$f(z) = \alpha z^{2h} + \cdots$$

$\wp(z)$ also has Laurent series about $z=0$, namely $\wp(z) = z^{-2}+\cdots$

Thus $f(z)\,\wp(z)^h = (\alpha z^{2h}+\cdots)(z^{-2}+\cdots)^h = (\alpha z^{2h}+\cdots)(z^{-2h}+\cdots) = \alpha+\cdots$

This is an elliptic function with no poles nor zeros at $z=0$.

Moreover, $f(z)\,\wp(z)^h$ is even because $f(z)$ and $\wp(z)^h$ are even.

By part (a), there are complex numbers $a_1,\dots,a_n,\ b_1,\dots,b_n,\ C\in\mathbb{C}$ such that

$$f(z)\,\wp(z)^h = C\,\frac{(\wp(z)-\wp(a_1))\cdots(\wp(z)-\wp(a_n))}{(\wp(z)-\wp(b_1))\cdots(\wp(z)-\wp(b_n))}.$$

Therefore,

$$f(z) = C\,\wp(z)^{-h}\,\frac{(\wp(z)-\wp(a_1))\cdots(\wp(z)-\wp(a_n))}{(\wp(z)-\wp(b_1))\cdots(\wp(z)-\wp(b_n))}.$$

(2) Problem 2, Ahlfors p. 275.

Let $f(z)$ be an elliptic function whose periods are

$$\Lambda = \{ n\omega_1 + m\omega_2 \mid m,n\in\mathbb{Z}\}$$

For any $z_1, z_2\in\mathbb{C}$, we shall write $z_1\equiv z_2 \pmod{\Lambda}$ if $z_1-z_2\in\Lambda$.

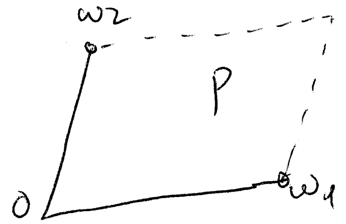
The function $\sigma:\mathbb{C}\to\mathbb{C}$ defined in Ahlfors, p. 274 is

$$\sigma(z) = z\prod_{\omega\in\Lambda^*}\left(1-\frac{z}{\omega}\right)\exp\left(\frac{z}{\omega}+\frac{1}{2}\left(\frac{z}{\omega}\right)^2\right).$$

Accordingly, σ is an entire function whose zeros are exactly at the lattice points. These zeros are of order 1. Moreover, there are complex constant η_1 and η_2 such that

$$\sigma(z+\omega_1) = -\sigma(z)\exp\left(\eta_1 z + \frac{\eta_1\omega_1}{2}\right) \qquad \forall z\in\mathbb{C}, \qquad (1)$$

$$\sigma(z+\omega_2) = -\sigma(z)\exp\left(\eta_2 z + \frac{\eta_2\omega_2}{2}\right) \qquad \forall z\in\mathbb{C}. \qquad (2)$$



Let P be a fundamental parallelogram of the lattice Λ. Because f is elliptic,

$$\#(\text{zeros in } P) = \#(\text{poles in } P).$$

Let $\alpha_1, \dots, \alpha_n$ be the zeros of f in P and $\beta_1, \dots, \beta_n$ be the poles of f in P. These were enumerated with multiplicity. Also, we know that $\alpha_1 + \dots + \alpha_n \equiv \beta_1 + \dots + \beta_n \pmod{\Lambda}$. Thus there are integers k and l such that $\beta_1 + \dots + \beta_n - \alpha_1 - \dots - \alpha_n = k\omega_1 + l\omega_2$. Put

$$a_j = \alpha_j, \quad b_j = \beta_j \quad \forall\, 1 \le j < n,$$

$$a_n = \alpha_n + l\omega_2, \quad b_n = \beta_n - k\omega_1.$$

We have

$$\sum_{j=1}^{n} b_j - \sum_{j=1}^{n} a_j = \sum_{j=1}^{n-1} \beta_j + \underbrace{b_n}_{\beta_n - k\omega_1} - \sum_{j=1}^{n-1} \alpha_j - \underbrace{a_n}_{\alpha_n + l\omega_2}$$

$$= \sum_{j=1}^{n} \beta_j - k\omega_1 - \sum_{j=1}^{n} \alpha_j - l\omega_2$$

$$= 0.$$

By the definition of a_j and b_j, we have $a_j \equiv \alpha_j \pmod{\Lambda}$ and $b_j \equiv \beta_j \pmod{\Lambda}$. Then, the function

$$g(z) = \frac{\sigma(z-a_1) \cdots \sigma(z-a_n)}{\sigma(z-b_1) \cdots \sigma(z-b_n)} \quad \forall\, z \notin \{\beta_1, \dots, \beta_n\} \pmod{\Lambda}$$

has zeros $\alpha_1, \dots, \alpha_n$ and poles $\beta_1, \dots, \beta_n$ in P. Thus the quotient $\frac{f(z)}{g(z)}$ has no pole nor zero in P. We'll show that g is doubly periodic.

For $j = 1, 2, \ldots, n$ and $i = 1, 2$, we have

$$\sigma(z - a_j + \omega_i) = -\sigma(z - a_j)\exp\left(\eta_i(z - a_j) + \frac{\eta_i \omega_i}{2}\right),$$

$$\sigma(z - b_j + \omega_i) = -\sigma(z - b_j)\exp\left(\eta_i(z - b_j) + \frac{\eta_i \omega_i}{2}\right).$$

Thus, $$\frac{\sigma(z - a_j + \omega_i)}{\sigma(z - b_j + \omega_i)} = \frac{\sigma(z - a_j)}{\sigma(z - b_j)}\exp\left(\eta_i(b_j - a_j)\right)$$

Thus, $$g(z + \omega_i) = \prod_{j=1}^{n}\frac{\sigma(z - a_j + \omega_i)}{\sigma(z - b_j + \omega_i)} = g(z)\exp\Big(\eta_i \underbrace{\Big(\sum_{j=1}^{n} b_j - \sum_{j=1}^{n} a_j\Big)}_{0}\Big)$$

$$= g(z).$$

Therefore g is doubly periodic. Thus the quotient map $f(z)/g(z)$ is doubly periodic and has no pole nor zero in P. Thus $f(z)/g(z)$ is holomorphic. Thus it must be a constant C. Therefore,

$$f(z) = C g(z) = C\prod_{k=1}^{n}\frac{\sigma(z - a_k)}{\sigma(z - b_k)}. \checkmark$$

(3) Problem 1, Ahlfors p. 276

Let Λ be the lattice generated by ω_1 and ω_2:

$$\Lambda = \{n\omega_1 + m\omega_2 : m, n \in \mathbb{Z}\}.$$

With $\wp$ and σ defined as in the previous problems, we'll show that

$$\wp(z) - \wp(u) = -\frac{\sigma(z - u)\sigma(z + u)}{\sigma(z)^2 \sigma(u)^2} \quad \forall z, u \not\equiv 0 \pmod{\Lambda}.$$

Regarding u as a constant, we put $f(z) = \wp(z) - \wp(u)$ $\forall z \not\equiv 0 \pmod{\Lambda}$

and $$g(z) = \frac{\sigma(z-u)\,\sigma(z+u)}{\sigma(z)^2\sigma(u)^2} \quad \forall z \not\equiv 0 \pmod{\Lambda}.$$

We'll use the identities (1) and (2) in Problem 2 to show that g is doubly periodic. For $i=1,2$, we have

$$\sigma(z-u+\omega_i) = -\sigma(z-u)\exp\left(\eta_i(z-u)+\frac{\eta_i\omega_i}{2}\right),$$

$$\sigma(z+u+\omega_i) = -\sigma(z+u)\exp\left(\eta_i(z+u)+\frac{\eta_i\omega_i}{2}\right),$$

$$\sigma(z+\omega_i) = -\sigma(z)\exp\left(\eta_i z+\frac{\eta_i\omega_i}{2}\right),$$

$$\sigma(z+\omega_i)^2 = \sigma(z)^2\exp\left(2\eta_i z+\eta_i\omega_i\right).$$

Thus $$\frac{\sigma(z-u+\omega_i)\,\sigma(z+u+\omega_i)}{\sigma(z+\omega_i)^2} = \frac{\sigma(z-u)\,\sigma(z+u)}{\sigma(z)^2}\underbrace{\exp(0)}_{1}$$

thus $g(z+\omega_i) = g(z)$. This means g is doubly periodic with ~~perod~~ periods in Λ.

We know that σ has simple zeros at exactly the lattice points. Thus $g(z)$ has zeros at $\pm u \pmod{\Lambda}$, each of which is of order one. When $u=-u \pmod{\Lambda}$, i.e. u is a half period then two simple zeros at u and $-u$ become a double zero at u. By saying $g(z)$ has simple zeros at $\pm u \pmod{\Lambda}$, we have already included this special case. Also, $g(z)$ has double poles at the lattice points. As we discussed in Problem 1, $f(z) = \wp(z)-\wp(u)$ has double poles at the lattice points. Also, $f(z)$ has simple zeros at $\pm u \pmod{\Lambda}$. Therefore, the quotient $f(z)/g(z)$ is an entire and doubly periodic function. Thus, it is a constant C_u.

$$\wp(z) - \wp(u) = C_u \frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2} \qquad \forall z \not\equiv 0 \pmod{\Lambda} \quad (*)$$

We'll show that C_u is independent of u. Recall the definition of σ is as follow $\sigma(z) = z \prod_{w\in\Lambda^*} \left(1 - \frac{z}{w}\right) \exp\left(\frac{z}{w} + \frac{1}{2}\left(\frac{z}{w}\right)^2\right)$.

Thus $$\sigma(-z) = -z \prod_{w\in\Lambda^*} \left(1 + \frac{z}{w}\right) \exp\left(\frac{-z}{w} + \frac{1}{2}\left(\frac{z}{w}\right)^2\right).$$

By replacing w by $-w$, we get $\sigma(-z) = -\sigma(z)$. Thus σ is an odd function. Now we swap z and u in $(*)$ to get

$$\wp(u) - \wp(z) = C_u \frac{\sigma(u-z)\sigma(u+z)}{\sigma(u)^2\sigma(z)^2} \qquad \forall u \not\equiv 0 \pmod{\Lambda}$$

With this form, we proved earlier that the factor C_u is independent of u (because it depends only in z). Thus $C_u = C$, which is independent of both z and u.

$$\wp(z) - \wp(u) = C \frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2} \qquad \forall u, z \not\equiv 0 \pmod{\Lambda} \quad (**)$$

~~Consider only u such that $2u \not\equiv 0 \pmod{\Lambda}$.~~ By the definition of σ, we have $\sigma(z) = z\tilde{\sigma}(z)$ where $\tilde{\sigma}$ is entire and $\tilde{\sigma}(0) = 1$. Consequently,

$$\sigma'(0) = \lim_{z\to 0} \frac{\sigma(z)}{z} = \tilde{\sigma}(0) = 1.$$

Dividing both sides of (**) by $z-u$, we get

$$\frac{\wp(z)-\wp(u)}{z-u} = C\,\frac{\sigma(z-u)}{z-u}\,\frac{\sigma(z+u)}{\sigma(z)^2\sigma(u)^2}$$

Let z approach u, we get

$$\wp'(u) = C\sigma'(0)\frac{\sigma(2u)}{\sigma(u)^4} = C\frac{\sigma(2u)}{\sigma(u)^4} \qquad (***)$$

We know that

$$\wp'(u) = -\frac{2}{u^3} + O(u),$$

$$\frac{\sigma(2u)}{\sigma(u)^4} = \frac{2u\,\tilde\sigma(2u)}{u^4\tilde\sigma(u)^4} = \frac{2}{u^3}\,\frac{\tilde\sigma(2u)}{\tilde\sigma(u)^4} = \frac{2}{u^3}\Bigg(\underbrace{\frac{\tilde\sigma(0)}{\tilde\sigma(0)^4}}_{1} + O(u)\Bigg)$$

Thus, (***) gives us

$$-\frac{2}{u^3} + O(u) = C\frac{2}{u^3}\big(1+O(u)\big) = \frac{2C}{u^3} + C.\,O(u^{-2}).$$

thus $C=-1$. therefore we have the identity

$$\wp(z)-\wp(u) = -\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2} \qquad \forall\, u,z \not\equiv 0 \pmod{\Lambda}.$$

(4) With the function $\zeta(z)$ defined by $\zeta(z) = \dfrac{\sigma'(z)}{\sigma(z)}$ for all $z \not\equiv 0 \pmod{\Lambda}$, we'll show that $\dfrac{\wp'(z)}{\wp(z)-\wp(u)} = \zeta(z-u)+\zeta(z+u)-2\zeta(z)$ for all $z,u \not\equiv 0$, $z \not\equiv \pm u \pmod{\Lambda}$.

Put $f(z) = \wp(z)-\wp(u)$ and $g(z) = -\dfrac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2}$. For each z_0 such that $z_0 \not\equiv 0, \pm u \pmod{\Lambda}$, there is a neighborhood of z_0 and a constant C such that for all z in that neighborhood,

$$\log g(z) = \log\sigma(z-u) + \log\sigma(z+u) - 2\log\sigma(z) - \log\sigma(u) + C \qquad (1)$$

Here the logarithms do not necessarily refer to the same branch cut. Also, we can take a logarithm of $f(z)$ for z in the neighborhood of z_0:

$$\log f(z) = \log(\wp(z) - \wp(u)). \qquad (2)$$

Also, the logarithms in (1) and (2) may not be the same. However, since they differ by a constant, we can take the derivatives to get an identity. Namely,

$$\frac{d}{dz}\log(\wp(z) - \wp(u)) = \frac{d}{dz}\log\sigma(z-u) + \frac{d}{dz}\log\sigma(z+u) - 2\frac{d}{dz}\log\sigma(z).$$

This is equivalent to

$$\frac{\wp'(z)}{\wp(z) - \wp(u)} = \frac{\sigma'(z-u)}{\sigma(z-u)} + \frac{\sigma'(z+u)}{\sigma(z+u)} - 2\frac{\sigma'(z)}{\sigma(z)}$$

$$= \zeta(z-u) + \zeta(z+u) - 2\zeta(z).$$

⑤ Problem 3, Ahlfors, p. 277.

By the previous problem, when $z, u \not\equiv 0$ and $z \not\equiv \pm u \pmod{\Lambda}$ we have

$$\frac{\wp'(z)}{\wp(z) - \wp(u)} = \zeta(z-u) + \zeta(z+u) - 2\zeta(z) \qquad (3)$$

Because the conditions are symmetric, we can swap z and u to get

$$\frac{\wp'(u)}{\wp(u) - \wp(z)} = \zeta(u-z) + \zeta(u+z) - 2\zeta(u) \qquad (4)$$

As mention in Problem ③, σ is an odd function. Thus σ' is even. Then $\zeta = \sigma'/\sigma$ is odd. Thus $\zeta(z-u) + \zeta(u-z) = 0$.

Summing up (3) and (4) term by term, we get

$$\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)} = 2\zeta(z+u)-2\zeta(z)-2\zeta(u).$$

Therefore,

$$\zeta(z+u) = \zeta(z)+\zeta(u)+\frac{1}{2}\,\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)}. \qquad (5)$$

(6) Problem 4, Ahlfors p. 277

With the relation $\wp=-\zeta'$, we differentiate (5) with respect to z and get

$$-\wp(z+u) = -\wp(z) + \frac{1}{2}\,\frac{\wp''(z)\big(\wp(z)-\wp(u)\big)-\big(\wp'(z)-\wp'(u)\big)\wp'(z)}{\big(\wp(z)-\wp(u)\big)^2} \qquad (6)$$

we swap z and u in (6) to get

$$-\wp(z+u) = -\wp(u) + \frac{1}{2}\,\frac{\wp''(u)\big(\wp(u)-\wp(z)\big)-\big(\wp'(u)-\wp'(z)\big)\wp'(u)}{\big(\wp(z)-\wp(u)\big)^2} \qquad (7)$$

Adding (6) to (7) term by term, we get

$$-2\wp(z+u) = -\wp(z)-\wp(u)+\frac{1}{2}\cdot\frac{\wp''(z)-\wp''(u)}{\wp(z)-\wp(u)} - \frac{1}{2}\left(\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)}\right)^2 \qquad (8)$$

We know that $\wp$ satisfies the following differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

where g_2 and g_3 are constants related to the Eisenstein series. Taking derivatives both sides, we get $2\wp'(z)\wp''(z) = 12\wp'(z)\wp(z)^2 - g_2\wp'(z)$.

For z not equal to half periods, $\wp'(z)\neq 0$. Then

$$2\wp''(z) = 12\wp(z)^2 - g_2$$

Thus $\wp''(z) = 6\wp(z)^2 - \frac{1}{2} g_2$. Replacing z by u, we get

$$\wp''(u) = 6\wp(u)^2 - \frac{1}{2} g_2.$$

Thus $\wp''(z) - \wp''(u) = 6\wp(z)^2 - 6\wp(u)^2 = 6(\wp(z) - \wp(u))(\wp(z) + \wp(u))$.

Thus, $$\frac{\wp''(z) - \wp''(u)}{\wp(z) - \wp(u)} = 6(\wp(z) + \wp(u)).$$

Replacing this identity into (8), we get

$$-2\wp(z+u) = \underbrace{-\wp(z) - \wp(u) + 3(\wp(z) + \wp(u))}_{2(\wp(z) + \wp(u))} - \frac{1}{2}\left(\frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)}\right)^2$$

Therefore, $$\wp(z+u) = -\wp(z) - \wp(u) + \frac{1}{4}\left(\frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)}\right)^2. \quad (9)$$

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By L'Hospital rule, we have $\lim_{z \to u} \frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)} = \frac{\wp''(z)}{\wp'(z)}$.

As z approaches u, (9) becomes

$$\wp(2z) = -2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2.$$

8 In problem 3, we derived Equation (***) and then proved that the constant C therein is -1. Thus, $\wp'(u) = -\frac{\sigma(2u)}{\sigma(u)^4}$ for all $u \not\equiv 0 \pmod{\Lambda}$. Renaming u with z, we write $\wp'(z) = -\frac{\sigma(2z)}{\sigma(z)^4}$.

⑨ The additional problem.

Suppose $\mathrm{Im}\,\tau > 0$. We'll show that the series $\sum_{(m,n)\neq(0,0)} \frac{1}{|n+m\tau|^2}$ diverges.

Suppose by contradiction that it converges. Put $\delta = \max\{1, |\tau|\} > 0$. We have $|n+\tau m| \le |n| + |\tau||m| \le \delta(|n|+|m|)$. Thus,

$$\infty > \sum_{(m,n)\neq(0,0)} \frac{1}{|n+\tau m|^2} \geq \frac{1}{\delta^2}\sum_{(m,n)\neq(0,0)} \frac{1}{(|n|+|m|)^2} \geq \frac{1}{\delta}\sum_{m,n\geq 1} \frac{1}{(n+m)^2}.$$

Thus,

$$A \doteq \sum_{m,n\geq 1} \frac{1}{(n+m)^2} < \infty.$$

For each $k \geq 2$, the term $\frac{1}{k^2}$ occurs exactly $(k-1)$ times in A as

$$\frac{1}{(1+(k-1))^2}, \frac{1}{(2+(k-2))^2}, \dots, \frac{1}{((k-1)+1)^2}.$$

Thus,

$$A = \sum_{k=2}^{\infty} \frac{k-1}{k^2} = \sum_{k=2}^{\infty}\left(\frac{1}{k} - \frac{1}{k^2}\right).$$

Since $A < \infty$,

$$\sum_{k=2}^{\infty} \frac{1}{k} = A + \sum_{k=2}^{\infty} \frac{1}{k^2} < \infty.$$

This is a contradiction.