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Math 8702: Complex Analysis
Homework #6

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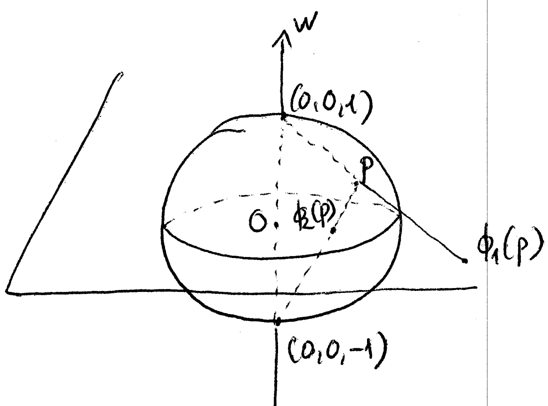
(1) Problem I.1.G, Miranda p.7

$$S^2 = \{(x,y,w) \in \mathbb{R}^3 : x^2+y^2+w^2 = 1\}$$



We define two maps ϕ_1 and ϕ_2 as follow.

$$\phi_1 : S^2 \setminus \{(0,0,1)\} \to \mathbb{C}$$

$$\phi_1(x,y,w) = \frac{x}{1-w} + i\frac{y}{1-w},$$

$$\phi_2 : S^2 \setminus \{(0,0,-1)\} \to \mathbb{C}$$

$$\phi_2(x,y,w) = \frac{x}{1+w} - i\frac{y}{1+w}.$$

We want to show that $\phi_2 \circ \phi_1^{-1}(z) = 1/z$. First, we will determine the inverse function of ϕ_1. Define $\Psi_1 : \mathbb{C} \to S^2 \setminus \{(0,0,1)\}$

$$\Psi_1(z) = \left(\frac{2\mathrm{Re}(z)}{|z|^2+1}, \frac{2\mathrm{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right).$$

Note that Ψ_1 is well-defined because

$$\left(\frac{2\mathrm{Re}(z)}{|z|^2+1}\right)^2 + \left(\frac{2\mathrm{Im}(z)}{|z|^2+1}\right)^2 + \left(\frac{|z|^2-1}{|z|^2+1}\right)^2 = \frac{4|z|^2}{(|z|^2+1)^2} + \frac{(|z|^2-1)^2}{(|z|^2+1)^2} = \frac{(|z|^2+1)^2}{(|z|^2+1)^2} = 1,$$

and that $\Psi_1(z)$ is never equal to $(0,0,1)$. We have

$$\phi_1 \circ \Psi_1(z) = \phi_1\left(\underbrace{\frac{2\mathrm{Re}(z)}{|z|^2+1}}_{x}, \underbrace{\frac{2\mathrm{Im}(z)}{|z|^2+1}}_{y}, \underbrace{\frac{|z|^2-1}{|z|^2+1}}_{w} \right) = \frac{x}{1-w} + i\frac{y}{1-w} \quad (*)$$

Because $1-w = 1-\dfrac{|z|^2-1}{|z|^2+1} = \dfrac{2}{|z|^2+1}$, we have

$$(*) = \frac{2\,\mathrm{Re}(z)}{2} + i\,\frac{2\,\mathrm{Im}(z)}{2} = z.$$

Thus $\phi_1 \circ \Psi_1 = id$. We have

$$\Psi_1 \circ \phi_1(\overbrace{*}^{x,y,w}) = \Psi_1\left(\underbrace{\frac{x}{1-w}}_{\mathrm{Re}(z)} + i\underbrace{\frac{y}{1-w}}_{\mathrm{Im}(z)}\right) = \left(\frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right) \quad (**)$$

Because $|z|^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 = \dfrac{x^2+y^2}{(1-w)^2} = \dfrac{1-w^2}{(1-w)^2} = \dfrac{1+w}{1-w}$, we get

$$|z|^2+1 = \frac{2}{1-w} \quad \text{and} \quad |z|^2-1 = \frac{2w}{1-w}.$$

Thus
$$(**) = \left(\frac{2x}{2}, \frac{2y}{2}, \frac{2w}{2}\right) = (x, y, w).$$

Thus, $\Psi_1 \circ \phi_1 = id$. Therefore, $\Psi_1 = \phi_1^{-1}$. Next, we show that $\phi_2 \circ \phi_1^{-1}(z) = 1/z$.

$$\phi_2 \circ \phi_1^{-1}(z) = \phi_2 \circ \Psi_1(z) = \phi_2\left(\underbrace{\frac{2\,\mathrm{Re}(z)}{|z|^2+1}}_{x}, \underbrace{\frac{2\,\mathrm{Im}(z)}{|z|^2+1}}_{y}, \underbrace{\frac{|z|^2-1}{|z|^2+1}}_{w}\right)$$

$$= \frac{x}{1+w} - i\,\frac{y}{1+w} \qquad (***)$$

We have $1+w = \dfrac{2|z|^2}{|z|^2+1} = \dfrac{2z\bar{z}}{|z|^2+1}$. Thus

$$(***) = \frac{2\,\mathrm{Re}(z)}{2z\bar{z}} - i\,\frac{2\,\mathrm{Im}(z)}{2z\bar{z}} = \frac{2\bar{z}}{2z\bar{z}} = \frac{1}{z}.$$

Therefore, $\phi_2 \circ \phi_1^{-1}(z) = 1/z$.

(2) Problem I.2.C, Miranda p.12

Define a function $f: \mathbb{P}^1 \to S^2$,

$$f([z:w]) = \frac{(2\operatorname{Re}(w\bar{z}), 2\operatorname{Im}(w\bar{z}), |w|^2 - |z|^2)}{|w|^2 + |z|^2}.$$

We want to show that f is a homeomorphism. However, we will show that f is an isomorphism of Riemann surfaces (a stronger statement). First, let us check that f is well-defined. For each $\lambda \in \mathbb{C} \setminus \{0\}$ we have

$$f([\lambda z : \lambda w]) = \frac{(2\operatorname{Re}(\lambda\bar{\lambda} w\bar{z}), 2\operatorname{Im}(\lambda\bar{\lambda} w\bar{z}), |\lambda|^2|w|^2 - |\lambda|^2|z|^2)}{|\lambda|^2|w|^2 + |\lambda|^2|z|^2}$$

$= f([z:w])$ (after cancelling $|\lambda|^2$ in the numerator and denominator).

Thus the value of f at $[z:w]$ doesn't depend on the choice of representatives of $[z:w]$. Moreover,

$$|f([z:w])|^2 = \frac{(2\operatorname{Re}(w\bar{z}))^2 + (2\operatorname{Im}(w\bar{z}))^2 + (|w|^2 - |z|^2)}{(|w|^2 + |z|^2)^2}$$

$$= \frac{4|w\bar{z}|^2 + (|w|^2 - |z|^2)^2}{(|w|^2 + |z|^2)^2} = \frac{(|w|^2 + |z|^2)^2}{(|w|^2 + |z|^2)^2} = 1$$

Thus $f([z:w]) \in S^2$. Therefore, f is well-defined.

By definition, the projective line $\mathbb{P}^1 = \{[z:w] \mid z, w \in \mathbb{C}, |z|^2 + |w|^2 \neq 0\}$ has a complex atlas consisting of two charts (U_+, ϕ_+) and (U_-, ϕ_-) defined by

$$U_+ = \{[z:w] \mid w \neq 0\},$$

$$U_- = \{[z:w] \mid z \neq 0\},$$

$$\phi_+ : U_+ \longrightarrow \mathbb{C}, \quad \phi_+([z:w]) = \frac{z}{w},$$

$$\phi_- : U_- \longrightarrow \mathbb{C}, \quad \phi_-([z:w]) = \frac{w}{z}.$$

The inverse maps are $\phi_+^{-1}(z) = [z:1]$ and $\phi_-^{-1}(w) = [1:w]$.

By definition, the Riemann sphere $S^2 = \{(a,b,c) \mid a^2+b^2+c^2 = 1\}$ has a complex atlas consisting of two charts (V_+, Ψ_+) and (V_-, Ψ_-) defined as follow.

$$V_+ = \{(a,b,c) \in S^2 \mid c \neq -1\},$$

$$V_- = \{(a,b,c) \in S^2 \mid c \neq +1\},$$

$$\Psi_+ : V_+ \to \mathbb{C}, \quad \Psi_+(a,b,c) = \cancel{\frac{a+ib}{1-c}}, \ \frac{a-ib}{1+c},$$

$$\Psi_- : V_- \to \mathbb{C}, \quad \Psi_-(a,b,c) = \cancel{\frac{a-ib}{1+c}} \ \frac{a+ib}{1-c}.$$

By computation, the inverse functions are obtained.

$$\Psi_+^{-1}(z) = \frac{(2\mathrm{Re}(z), -2\mathrm{Im}(z), -|z|^2+1)}{|z|^2+1},$$

$$\Psi_-^{-1}(z) = \frac{(2\mathrm{Re}(z), +2\mathrm{Im}(z), -1+|z|^2)}{1+|z|^2}.$$

Define a function $g : S^2 \longrightarrow \mathbb{R}^1$,

$$g(a,b,c) = \begin{cases} [(1-c):(a+ib)] & \text{on } V_- \\ [(a-ib):(1+c)] & \text{on } V_+. \end{cases}$$

First, we'll show that g is well-defined. On $V_+ \cap V_-$, $c \neq \pm 1$ and so $a^2+b^2 \neq 0$. We have

$$\frac{1-c}{a-ib} = \frac{(1-c)(a+ib)}{(a-ib)(a+ib)} = \frac{(1-c)(a+ib)}{a^2+b^2} = \frac{(1-c)(a+ib)}{1-c^2} = \frac{a+ib}{1+c}.$$

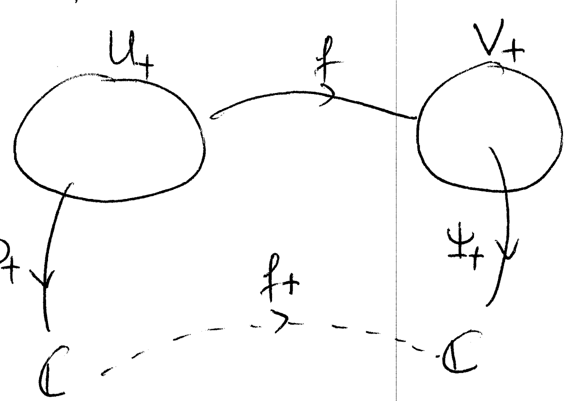
Thus, $[(1-c):(a+ib)] = [(a-ib):(1+c)]$. Thus the values of g by two formulae are the same in the overlap. Thus, g is well-defined.

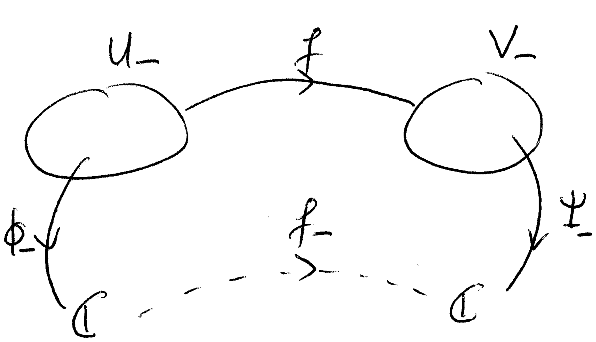
By the definition of f and g, we see that $f(U_+) \subset V_+$, $f(U_-) \subset V_-$, $g(V_+) \subset U_+$ and $g(V_-) \subset U_-$. These observations help us compute $f \circ g$ and $g \circ f$ conveniently by computing them on V_+, V_- or U_+, U_-. Then we get $f \circ g = id_{S^2}$ and $g \circ f = id_{\mathbb{P}^1}$.

Thus f is bijective and g is its inverse. Thus,

$$V_+ = f \circ g(V_+) \subset f(U_+),$$
$$V_- = f \circ g(V_-) \subset f(U_-).$$

Thus $f(U_+) = V_+$, $f(U_-) = V_-$, $g(V_+) = U_+$ and $g(V_-) = U_-$.





Put $f_\pm : \mathbb{C} \to \mathbb{C}$, $f_\pm(z) = \Psi_\pm \circ f \circ \phi_\pm^{-1}(z)$. We have

$$f_+(z) = \Psi_+ \circ f([z:1]) = \Psi_+\left(\frac{(2\mathrm{Re}(\bar{z}),\ 2\mathrm{Im}(\bar{z}),\ 1-|z|^2)}{1+|z|^2} \right)$$

$$= \Psi_+\left(\frac{(2\mathrm{Re}(z),\ -2\mathrm{Im}(z),\ 1-|z|^2)}{1+|z|^2} \right)$$

$$= \Psi_+(\Psi_+^{-1}(z)) = z.$$

$$f_-(w) = \Psi_- \circ f([1:w]) = \Psi_-\left(\frac{(2\mathrm{Re}(w),\ 2\mathrm{Im}(w),\ |w|^2-1)}{|w|^2+1} \right)$$

$$= \Psi_-(\Psi_-^{-1}(w)) = w.$$

Thus, $f_+ = f_- = \mathrm{id}_{\mathbb{C}}$, which are holomorphic.

Similarly, put $g_\pm : \mathbb{C} \to \mathbb{C}$, $g_\pm(z) = \phi_\pm \circ g \circ \Psi_\pm^{-1}(z)$. We have

$$g_+(z) = \phi_+ \circ g\left(\frac{(2\mathrm{Re}(z),\ -2\mathrm{Im}(z),\ 1-|z|^2)}{1+|z|^2} \right) = \phi_+\left(\frac{2|z|^2}{1+|z|^2} : \frac{2\mathrm{Re}(z) - 2i\mathrm{Im}(z)}{1+|z|^2} \right)$$

$$\begin{tikzcd} V_+ \arrow[r, "g"] \arrow[d, "\Psi_+"'] & U_+ \arrow[d, "\phi_+"] \\ \mathbb{C} \arrow[r, dashed, "g_+"] & \mathbb{C} \end{tikzcd}$$

$$= \phi_+([|z|^2 : \bar{z}])$$

$$= \phi_+([z\bar{z} : \bar{z}])$$

$$= \phi_+([z:1])$$

$$= z.$$

$$\begin{tikzcd} V_- \arrow[r, "g"] \arrow[d, "\Psi_-"'] & U_- \arrow[d, "\phi_-"] \\ \mathbb{C} \arrow[r, dashed, "g_-"] & \mathbb{C} \end{tikzcd}$$

$$g_-(z) = \phi_- \circ g\left(\frac{(2\mathrm{Re}(z),\ 2\mathrm{Im}(z),\ |z|^2-1)}{|z|^2-1} \right)$$

$$= \phi_-\left(\left[\frac{2\mathrm{Re}(z) - 2i\mathrm{Im}(z)}{|z|^2-1} : \frac{2|z|^2}{|z|^2-1} \right] \right)$$

$$= \phi_-([\bar{z} : z\bar{z}]) = \phi_-([1:z]) = z.$$

Thus, $g_+ = g_- = id_{\mathbb{C}}$, which are holomorphic. Therefore f and g are holomorphic. Thus, f is an isomorphism.

(3) Problem I.2.J, Miranda p.13

Let $\Omega \subset \mathbb{C}$ be an open subset and $p \in \Omega$. Suppose $\phi: \Omega \to \mathbb{C}$ is an holomorphic function such that $\phi'(p) \neq 0$. We will use the Implicit Function Theorem to show that there is an open neighborhood of p in Ω, namely U, such that $\phi|_U$ is a chart on $\mathbb{C}$.

Write $\phi(z) = u(x,y) + iv(x,y)$ where $u, v : \Omega \to \mathbb{R}$ are the real and imaginary part of $\phi(z)$. Since ϕ is holomorphic, u and v satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \forall (x,y) \in \Omega.$$

Write $p = x_0 + iy_0$ and $\phi(p) = t_0 + is_0$. Then, because $\phi'(p) \neq 0$,

$$|\phi'(p)|^2 = \left(\frac{\partial u}{\partial x}(x_0, y_0)\right)^2 + \left(\frac{\partial v}{\partial x}(x_0, y_0)\right)^2 \neq 0 \qquad (*)$$

We define two functions $f, g: \Omega \times \mathbb{R}^2 \to \mathbb{R}$ which are given by

$$f(x,y,t,s) = u(x,y) - t,$$
$$g(x,y,t,s) = v(x,y) - s, \quad \forall (x,y) \in \Omega, (t,s) \in \mathbb{R}^2.$$

Since u and v are smooth functions, so are f and g. We have

$$\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2,$$

the value of which at $(x,y)=(x_0,y_0)$ is nonzero by $(*)$. Thus,

$$\begin{pmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{pmatrix}$$ is an invertible matrix.

By the Implicit Function Theorem applied to (f,g) for complex functions, there exist an open neighborhood of (x_0,y_0) in Ω, namely U, an open neighborhood of (t_0,s_0) in $\mathbb{R}^2$, namely V, and a holomorphic function $\zeta : V \to U$ such that $\zeta(t_0,s_0) = (x_0,y_0)$ and $\phi(\zeta(t,s)) = (t,s)$ for all $(t,s) \in V$. Because of the last identity, ζ is injective. Thus $\zeta : V \to \zeta(V)$ is a conformal map. Since $\zeta(V)$ is an open neighborhood of (x_0,y_0), we could have taken U to be $\zeta(V)$ from the beginning. If so, $\zeta : V \to U$ is now a conformal map.

Next, we'll show that $\phi|_U : U \to \phi(U)$ is injective. Suppose that we have $(x_1,y_1), (x_2,y_2) \in U$ such that $\phi(x_1,y_1) = \phi(x_2,y_2)$.

Because $\varsigma : V \to U$ is surjective, there are $(t_1, s_1), (t_2, s_2) \in V$ such that $(x_1, y_1) = \varsigma(t_1, s_1)$ and $(x_2, y_2) = \varsigma(t_2, s_2)$. Thus,

$$\phi(x_1, y_1) = \phi(\varsigma(t_1, s_1)) = (t_1, s_1),$$

$$\phi(x_2, y_2) = \phi(\varsigma(t_2, s_2)) = (t_2, s_2).$$

Thus, $(t_1, s_1) = (t_2, s_2)$. Applying ς to both sides, we get $(x_1, y_1) = (x_2, y_2)$. Hence $\phi|_U$ is injective. Because it is also surjective and holomorphic, it is a conformal map between U and $\phi(U)$. Thus, $\phi|_U$ is a chart on C.

(4) Problem I.3.A, Miranda p. 18.

On $\mathbb{C}^3$, we define a relation $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ if and only if there is $\lambda \in \mathbb{C}^*$ such that $(x_1, y_1, z_1) = \lambda(x_2, y_2, z_2)$. This turns out to be an equivalent relation. The complex projective plane $\mathbb{P}^2$ was defined to be the set $\mathbb{C}^3/\sim$ with quotient topology.

Denote by $[x : y : z]$ the equivalende class of (x, y, z). We define

$$U_0 = \{[x : y : z] \in \mathbb{P}^2 \mid x \neq 0\},$$

$$U_1 = \{[x : y : z] \in \mathbb{P}^2 \mid y \neq 0\},$$

$$U_2 = \{[x : y : z] \in \mathbb{P}^2 \mid z \neq 0\},$$

and the maps $\phi_0 : U_0 \to \mathbb{C}^2$, $\phi_0([x : y : z]) = \left(\frac{y}{x}, \frac{z}{x}\right)$,

$$\phi_1 : U_1 \to \mathbb{C}, \quad \phi_1([x:y:z]) = \left(\frac{x}{y}, \frac{z}{y}\right),$$

$$\phi_2 : U_2 \to \mathbb{C}, \quad \phi_2([x:y:z]) = \left(\frac{x}{z}, \frac{y}{z}\right).$$

These maps are well-defined because their values at $[x:y:z]$ do not depend on the choice of repsentatives. We'll show that ϕ_0 is a homeomorphism, and similarly for ϕ_1 and ϕ_2. First, we show that ϕ_0 is bijective. Put $\Psi_0 : \mathbb{C}^2 \to U_0$, $\Psi_0(u,v) = [1:u:v]$.

$$\phi_0 \circ \Psi_0(u,v) = \phi_0([1:u:v]) = (u,v) \quad \forall (u,v) \in \mathbb{C}^2,$$

$$\Psi_0 \circ \phi_0([x:y:z]) = \Psi_0\left(\frac{y}{x}, \frac{z}{x}\right) = \left[1 : \frac{y}{x} : \frac{z}{x}\right] = [x:y:z], \quad \forall [x:y:z] \in U_0.$$

Thus ϕ_0 is bijective and its inverse map is Ψ_0.

<u>Show that ϕ_0 is continuous</u>

Put $W_0 = \{(x,y,z) \in \mathbb{C}^3 \mid x \neq 0\}$. Then $U_0 = W_0/\sim$.

Let $q : W_0 \to W_0/\sim$ be the quotient map and put $f = \phi_0 \circ q$. By the characterization of quotient topology, ϕ_0 is continuous iff f is continuous. Because $f : W_0 \subset \mathbb{C}^3 \to \mathbb{C}^2$ and $f(x,y,z) = \left(\frac{y}{x}, \frac{z}{x}\right)$, it is obviously continuous.

$$\begin{tikzcd} W_0 \arrow[d, "q"'] \arrow[dr, "f"] & \\ W_0/\sim \arrow[r, "\phi_0"'] & \mathbb{C}^2 \end{tikzcd}$$

<u>Show that Ψ_0 is continuous</u>

Put $g : \mathbb{C}^2 \to W_0 \subset \mathbb{C}^3$, $g(u,v) = (1,u,v)$.

Then g is continuous and $\Psi_0 = q \circ g$. Thus Ψ_0 is also continuous.

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "g"] \arrow[dr, "\Psi_0"'] & W_0 \arrow[d, "q"] \\ & W_0/\sim \end{tikzcd}$$

Next, we show that $\mathbb{P}^2$ is Hausdorff. We have

$$\mathbb{P}^2 = \{[x:y:z] \mid x, y, z \text{ are not zero at once}\}$$

Let $q: \mathbb{C}^3 \to \mathbb{C}^3/\sim$ be the quotient map and W_0, W_1, W_2 be the sets described above. Since $q^{-1}(U_i) = W_i$ which is open in $\mathbb{C}^3$, U_i is open in $\mathbb{P}^2$. Because $\mathbb{C}^2$ is Hausdorff and $U_i \overset{\phi_i}{\simeq} \mathbb{C}^2$, each U_i is also Hausdorff. Let $a, b \in \mathbb{P}^2$ such that $a \neq b$. We are looking for two neighborhoods of a and b in $\mathbb{P}^2$ that are disjoint. Since $\mathbb{P}^2 = U_0 \cup U_1 \cup U_2$, there are two cases.

<u>Case 1</u> a and b belong to the same chart, say U_0 for example.

Since U_0 is Hausdorff, there are an open neighborhood of a in U_0, namely U, and an open neighborhood of b in U_0, namely V, such that $U \cap V = \phi$.

Since U_0 is open $\mathbb{P}^2$, U and V are open in $\mathbb{P}^2$.

<u>Case 2</u> a and b do not belong to the same chart.

WLOG, we assume $a \in U_0$ and $b \in U_1$. Then $a \in U_0 \setminus U_1$ and $b \in U_1 \setminus U_0$.

Thus, $a = [1:0:z_1]$ and $b = [0:1:z_2]$. It is true that for $\alpha, \beta \in \mathbb{C}$,

$$|\alpha| \leq |\beta| \Leftrightarrow |\lambda\alpha| \leq |\lambda\beta| \quad \forall \lambda \in \mathbb{C}^*,$$

and $|\alpha| < |\beta| \Leftrightarrow |\lambda\alpha| < |\lambda\beta| \quad \forall \lambda \in \mathbb{C}^*$.

Thus, the ordering of $|x|, |y|, |z|$ on $[x:y:z]$ is well-defined. Thus we can put $U = \{[x:y:z] : |x| > |y|\}$ and $V = \{[x:y:z] : |x| < |y|\}$.

Then
$$q^{-1}(U) = \{(x,y,z) \in \mathbb{C}^3 : |x| > |y|\},$$
$$q^{-1}(V) = \{(x,y,z) \in \mathbb{C}^3 : |x| < |y|\},$$
which are open and disjoint subsets of $\mathbb{C}^3$. Thus, U and V are open and disjoint on $\mathbb{P}^2$. Since $a = [1:0:z_1] \in U$ and $b = [0:1:z_2] \in V$, U and V are the sets we were looking for.

Put $D = \{(z,w) : |z|, |w| \leq 1\} \subset \mathbb{C}^2$. We show that
$$\mathbb{P}^2 = \phi_0^{-1}(D) \cup \phi_1^{-1}(D) \cup \phi_2^{-1}(D).$$
All we need is to show that $\mathbb{P}^2$ is contained in the set on the right.

~~Put $\alpha =$~~ Take $[a:b:c] \in \mathbb{P}^2$ and put $\alpha = \max\{|a|, |b|, |c|\} > 0$ for a specific choice of representative. There are 3 cases.

$\alpha = |a|$ Then $[a:b:c] \in U_0$ and $\left|\frac{b}{a}\right|, \left|\frac{c}{a}\right| \leq 1$. Thus
$$\phi_0([a:b:c]) = \left(\frac{b}{a}, \frac{c}{a}\right) \in D$$
Thus, $[a:b:c] \in \phi_0^{-1}(D)$.

$\alpha = |b|$ Then $[a:b:c] \in U_1$ and $\left|\frac{a}{b}\right|, \left|\frac{c}{b}\right| \leq 1$. Thus
$$\phi_1([a:b:c]) = \left(\frac{a}{b}, \frac{c}{b}\right) \in D.$$
Thus, $[a:b:c] \in \phi_1^{-1}(D)$.

$\alpha = |c|$ Then $[a:b:c] \in U_2$ and $\left|\frac{a}{c}\right|, \left|\frac{b}{c}\right| \le 1$. Thus,

$$\phi_2([a:b:c]) = \left(\frac{a}{c}, \frac{b}{c}\right) \in D.$$

Thus, $[a:b:c] \in \phi_2^{-1}(D)$. Therefore, $[a:b:c] \in \phi_0^{-1}(D) \cup \phi_1^{-1}(D) \cup \phi_2^{-1}(D)$.

Because D is a compact subset of $\mathbb{C}^2$, $\phi_0^{-1}(D)$, $\phi_1^{-1}(D)$, $\phi_2^{-1}(D)$ are compact sets. Since $\mathbb{P}^2$ is the union of 3 compact subspaces, it is compact as well.

(5) Problem I.3.C, Miranda p. 18

Let $F: \mathbb{C}^{n+1} \to \mathbb{C}$ be a homogeneous polynomial of degree d. Put $\underline{x} = (x_0, x_1, \dots, x_n) \in \mathbb{C}^{n+1}$. We'll show that $F(\underline{x}) = \frac{1}{d}\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i}$. $\quad (*)$

We know that F is a sum of monomials of degree d. Since $(*)$ is additive in F, it suffices to prove it for monomials. Moreover, since a scalar factor can be cancelled out from both sides of $(*)$, it suffices to consider monic monomials only.

Let $F(\underline{x}) = x_0^{r_0} x_1^{r_1} \cdots x_n^{r_n}$ be a monic monomial with $r_0, \dots, r_n \geq 0$ and $r_0 + r_1 + \dots + r_n = d$. Then $\dfrac{\partial F}{\partial x_0} = r_0 x_0^{r_0 - 1} x_1^{r_1} \cdots x_n^{r_n}$.

Thus $x_0 \dfrac{\partial F}{\partial x_0} = r_0 \, x_0^{r_0} x_1^{r_1} \cdots x_n^{r_n} = r_0 F(\underline{x})$.

Since x_0 and any x_i have the same role, $x_i \dfrac{\partial F}{\partial x_i} = r_i F(\underline{x})$ for all $i = 1, \dots, n$.

Thus, $\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = \sum_{i=0}^{n} r_i F(\underline{x}) = \left(\sum_{i=0}^{n} r_i\right) F(\underline{x}) = d.F(\underline{x})$.

Therefore, $F(\underline{x}) = \frac{1}{d} \sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i}$. ✓

(6) Problem I.3.E, Miranda p.18

Let $F(x,y,z) = ax+by+cz$ and $G(x,y,z) = \alpha x + \beta y + \gamma z$ be two homogeneous polynomials on $\mathbb{C}^3$. The conditions for them to be of degree one are $(a,b,c) \neq \vec{0}$ and $(\alpha,\beta,\gamma) \neq \vec{0}$. Let X_F and X_G be the projective plane curves defined by F and G respectively.

$$X_F = \{[x:y:z] \in \mathbb{P}^2 \mid ax+by+cz=0\},$$

$$X_G = \{[x:y:z] \in \mathbb{P}^2 \mid \alpha x+\beta y+\gamma z=0\}.$$

Suppose $X_F \neq X_G$. We'll show that $X_F \cap X_G$ has only one element.

Put $\vec{u} = (a,b,c)$ and $\vec{v} = (\alpha,\beta,\gamma)$. Then $\vec{u}, \vec{v} \neq \vec{0}$.

If $\vec{u} \times \vec{v} = \vec{0}$ then $\vec{u}$ and $\vec{v}$ are parallel. Thus there is $\lambda \in \mathbb{C}^*$ such that $\vec{u} = \lambda \vec{v}$. Then

$$\begin{aligned} X_F &= \{[x:y:z] \in \mathbb{P}^2 \mid (a,b,c)\cdot(x,y,z) = 0\} \\ &= \{[x:y:z] \in \mathbb{P}^2 \mid \lambda(\alpha,\beta,\gamma)\cdot(x,y,z) = 0\} \\ &= \{[x:y:z] \in \mathbb{P}^2 \mid (\alpha,\beta,\gamma)\cdot(x,y,z) = 0\} \\ &= X_G, \end{aligned}$$

which is a contradiction.

Therefore, $\vec{u}\times\vec{v}\neq\vec{0}$. Define $T:\mathbb{C}^3\to\mathbb{C}^2$ to be a linear map

$$T(x,y,z)=\underbrace{\begin{pmatrix} a & b & c\\ \alpha & \beta & \gamma\end{pmatrix}}_{A}\begin{pmatrix} x\\ y\\ z\end{pmatrix}$$

Then $[x:y:z]\in X_F\cap X_G \Leftrightarrow \begin{cases} T(x,y,z)=0\\ (x,y,z)\neq 0\end{cases}$

$$\Leftrightarrow (x,y,z)\in(\ker T)\setminus\{0\} \qquad (*)$$

We have $\quad \vec{u}\times\vec{v}=\left(\begin{vmatrix} b & c\\ \beta & \gamma\end{vmatrix}, \begin{vmatrix} c & a\\ \gamma & \alpha\end{vmatrix}, \begin{vmatrix} a & b\\ \alpha & \beta\end{vmatrix}\right)\neq 0.$

Thus at least one of the determinants is nonzero. Thus the maximal size of a minor matrix of A that is invertible is 2. Thus $\operatorname{rank}(A)=2$. Thus $\dim(\operatorname{Im}T)=2$. Then $\dim(\ker T)=3-\dim(\operatorname{Im}T)=3-2=1$.

Moreover, $\vec{u}\times\vec{v}=(b\gamma-c\beta, c\alpha-a\gamma, a\beta-b\alpha)$ is in the kernel of T because $\quad a(b\gamma-c\beta)+b(c\alpha-a\gamma)+c(a\beta-b\alpha)=0,$

$$\alpha(b\gamma-c\beta)+\beta(c\alpha-a\gamma)+\gamma(a\beta-b\alpha)=0.$$

Thus, $\ker T=\langle\vec{u}\times\vec{v}\rangle=\{\lambda(\vec{u}\times\vec{v})\mid\lambda\in\mathbb{C}\}$. From $(*)$, we get

$[x:y:z]\in X_F\cap X_G \Leftrightarrow (x,y,z)=\lambda(\vec{u}\times\vec{v})$ for some $\lambda\in\mathbb{C}^*$

$$\Leftrightarrow [x:y:z]=[(b\gamma-c\beta):(c\alpha-a\gamma):(a\beta-b\alpha)]$$

Therefore, X_F intersects X_G at a single point, which is

$$[(b\gamma-c\beta):(c\alpha-a\gamma):(a\beta-b\alpha)].$$