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Math 8702: Complex Analysis
Homework #7

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① Problem II.1.C, Miranda, p. 30

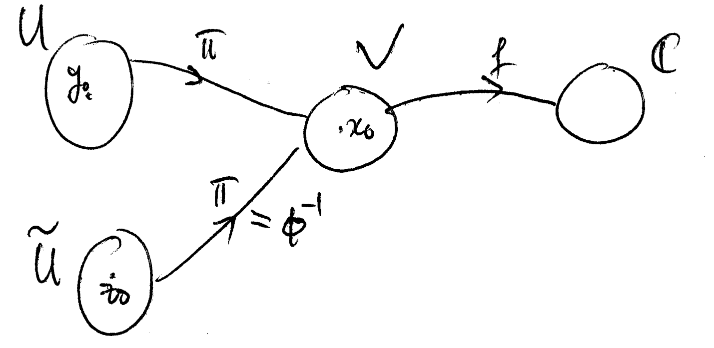
Let L be a lattice in $\mathbb{C}$ and $X = \mathbb{C}/L$ be the complex torus, determined by the quotient map $\pi : \mathbb{C} \to X$. Then π is a holomorphic map by the construction of complex structure on X. We will show that a function f on X is meromorphic if and only if $f \circ \pi$ is meromorphic on $\mathbb{C}$.

Now let f be a meromorphic function on X. We'll show that $f \circ \pi$ is a meromorphic function on $\mathbb{C}$. Since X is compact, f has only finitely many poles. Let $I = \{x_1, \dots, x_n\}$ be the set of all poles of f on X. If $I = \phi$ then f is holomorphic; and so $f \circ \pi$ is holomorphic in $\mathbb{C}$. We consider the case $I \neq \phi$. Then $f : X \setminus I \to \mathbb{C}$ is holomorphic. Put $J = \pi^{-1}(I)$. We will show that J is a discrete subset of $\mathbb{C}$. For each $k = 1, 2, \dots, n$, $\pi^{-1}(x_k) = x_k + L$, which is a discrete subset of $\mathbb{C}$. Since $J = \bigcup_{k=1}^{n} \pi^{-1}(x_k)$ is the union of finitely many discrete sets, J is also a discrete subset of $\mathbb{C}$. We have

$\boxed{2}$

$$\mathbb{C}\setminus J \xrightarrow{\pi} X\setminus I \xrightarrow{f} \mathbb{C},$$

so $f\circ\pi$ is holomorphic on $\mathbb{C}\setminus J$. Take $y_0 \in J$. We'll show that y_0 is a pole of $f\circ\pi$. Put $x_0 = \pi(y_0) \in I$. Then x_0 is a pole of f.



By definition of what is a pole, there exists a chart $(V;\ \phi: V \to W)$ where $x_0 \in V$ and $f\circ\phi^{-1}: W \to \mathbb{C}$ has a pole at $\phi(x_0)$.

By the definition of complex structure on X, ϕ is the inverse of a restriction of π on some subset of $\mathbb{C}$. In other words, there is an open set $\tilde{U} \subset \mathbb{C}$ such that $\phi = (\pi|_{\tilde{U}})^{-1}$. Put $z_0 = \phi(x_0)$. Then $\pi(z_0) = \pi\circ\phi(x_0) = x_0 = \pi(y_0)$. Thus $y_0 - z_0 \in L$.

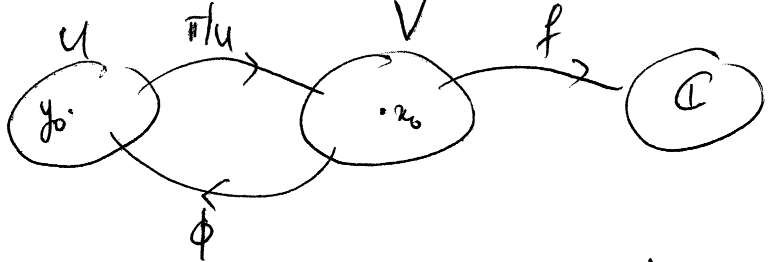
Put $U = \tilde{U} + (y_0 - z_0)$. Then U is an open set in $\mathbb{C}$ that contains y_0. For each $y \in U$, we have

$$f\circ\pi(y) = f\circ\pi(\underbrace{y + z_0 - y_0}_{\in \tilde{U}}) = f\circ\phi^{-1}(y + z_0 - y_0)$$

Since $f\circ\phi^{-1}$ has a pole at z_0, $f\circ\pi$ has a pole at y_0.

Now we will prove the converse. Suppose that $f\circ\pi$ is a meromorphic function on $\mathbb{C}$. We'll show that f is meromorphic on X. Let $X\setminus I$ be the domain of f (included all removable singularity). Put $J = \pi^{-1}(I)$. Then $\mathbb{C}\setminus J$ is the domain of $f\circ\pi$. Since $f\circ\pi$ is meromorphic, J is discrete and is the set of all poles of $f\circ\pi$. Note that J can be empty. We can think of I as the intersection of J with a fundamental parallelogram of $\mathbb{C}$. Then I is also discrete. Then $\mathbb{C}\setminus J \xrightarrow{\pi} X\setminus I \xrightarrow{f} \mathbb{C}$.

With $x_0 \in X\setminus I$, we'll show that f is holomorphic at x_0. Let $y_0 \in \pi^{-1}(x_0) \subset \mathbb{C}\setminus J$. Since $f\circ\pi$ is holomorphic at y_0, there is an open disk $U = \mathbb{D}(y_0, \varepsilon)$ in $\mathbb{C}$ such that $f\circ\pi$ is holomorphic in U. Put $V = \pi(U)$. Then V is open in $X\setminus I$ since it is open in X. A priori, we can shrink ε if necessary so that $\pi|_U : U \to \pi(U)$ is a bijection. Then $\phi = (\pi|_U)^{-1} : V \to U$ is a chart of X around x_0.



We have
$$f\circ\phi^{-1} = f\circ(\pi|_U) = (f\circ\pi)|_U$$
which is holomorphic on U.
Thus f is holomorphic on V.

With $x_0 \in I$, we'll show that f has a pole at x_0. Put $y_0 \in \pi^{-1}(x_0) \subset J$. Then $f\circ\pi$ has a pole at y_0. Then we introduce the set U and V and the coordinate chart ϕ exactly the same as the above case. Then we have $f\circ\phi^{-1}(z) = (f\circ\pi)|_U(z)$ $\forall z \in U\setminus\{$

Since $(f\circ\pi)|_U$ has a pole at y_0, $f\circ\phi^{-1}$ also has a pole at y_0. Thus f has a pole at x_0 by definition.

(2) Problem II.3.I, Miranda, p. 43

Let X be a compact Riemann surface and f a nonconstant meromorphic function on X. We will show that f must have a pole and a zero.

Let $F: X \to \mathbb{C}_\infty$ be the holomorphic function corresponding to f, i.e.

$$F(x) = \begin{cases} f(x) & \text{if } x \text{ is not a pole,} \\ \infty & \text{if } x \text{ is a pole.} \end{cases}$$

Since f is nonconstant, F is nonconstant. Thus F is an open map. Thus, $F(X)$ is open in $\mathbb{C}_\infty$. On the other hand, since X is compact, $F(X)$ is also compact. Since $\mathbb{C}_\infty$ is Hausdorff, $F(X)$ is closed in $\mathbb{C}_\infty$. Since $\mathbb{C}_\infty$ is connected, $F(X) = \mathbb{C}_\infty$, i.e. F is surjective. Thus there is $x_1, x_2 \in X$ such that $F(x_1) = 0$ and $F(x_2) = \infty$. Thus $f(x_1) = 0$ and $f(x_2) = \infty$. This means f has a zero and a pole.

(3) Problem II.3.J, Miranda p. 43

Let X be a Riemann surface, f be a meromorphic function on X.

Define
$$F(x) = \begin{cases} f(x), & x \text{ is not a pole,} \\ \infty, & x \text{ is a pole.} \end{cases}$$

We will show that F is holomorphic as a function from X to $\mathbb{C}_\infty$, and F is not identically ∞.

First, let us make the set $\mathbb{C}_\infty$ clear. We know that $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

There is a bijective map from S^2 to $\mathbb{C}_\infty$ given by the stereographic projection, namely

$$(a,b,c) \mapsto \begin{cases} \frac{a}{1-c} + i\frac{b}{1-c} & \text{if } c \neq 1, \\ \infty & \text{if } c = 1. \end{cases}$$

By this projection (bijection), $\mathbb{C}_\infty$ inherits the complex structure from S^2. Thus we can think of F as a function from X to S^2.

By definition, $S^2 = \{(a,b,c) \in \mathbb{R}^3 : a^2+b^2+c^2 = 1\}$ has an atlas consisting of two charts (U_+, ϕ_+) and (U_-, ϕ_-) where

$$U_+ = \{(a,b,c) \mid c \neq 1\},$$

$$U_- = \{(a,b,c) \mid c \neq -1\},$$

$$\phi_+ : U_+ \to \mathbb{C}, \quad \phi_+(a,b,c) = \frac{a}{1-c} + i\frac{b}{1-c},$$

$$\phi_- : U_- \to \mathbb{C}, \quad \phi_-(a,b,c) = \frac{a}{1+c} - i\frac{b}{1+c}.$$

The inverse maps are $\phi_+^{-1} : \mathbb{C} \to U_+$, $\phi_+^{-1}(z) = \dfrac{(2\operatorname{Re}(z), \operatorname{Im}(z), |z|^2-1)}{|z|^2+1}$,

$$\phi_-^{-1} : \mathbb{C} \to U_-, \quad \phi_-^{-1}(z) = \frac{(2\operatorname{Re}(z), -2\operatorname{Im}(z), 1-|z|^2)}{|z|^2+1}.$$

The transition maps are $\phi_+ \circ \phi_-^{-1}(z) = \phi_- \circ \phi_+^{-1}(z) = \frac{1}{z} \quad \forall z \in \mathbb{C} \setminus \{0\}$.

Then as a function from X to S^2, F is given by

$$F(x) = \begin{cases} \phi_+^{-1}(f(x)) & \text{if } f(x) \neq \infty, \\ \phi_-^{-1}\left(\frac{1}{f(x)}\right) & \text{if } f(x) \neq 0. \end{cases} \qquad (*)$$

Note that in the second line we used the convention $\frac{1}{\infty} = 0$.

We'll show that F is holomorphic and F is not identically $(0,0,1)$.

Because f is meromorphic, the set of all poles of f is discrete. Since $F(x) = (0,0,1) \Leftrightarrow f(x) = \infty \Leftrightarrow x$ is a pole of f, the set of all x's satisfying $F(x) = (0,0,1)$ is also discrete. Thus F is not identically $(0,0,1)$. If f is constant then F is also constant. Then F is holomorphic. Now we consider the case f is nonconstant only.

Then the set of zeros of f is discrete.

Take any $x_0 \in X$. We'll show that F is holomorphic at x_0.

There are two cases, namely f is holomorphic or has pole at x_0.

<u>Case 1</u> f is holomorphic at x_0.

Then there exists a chart $(U, \phi : U \to V)$ on X around x_0 such that $f \circ \phi^{-1}$ is holomorphic on V. Since f has no pole in U, $F(x) = \phi_+^{-1}(f(x))$ for every $x \in U$.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\phi"'] & \mathbb{C} \\ V & \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "F"] \arrow[d] & U_+ \arrow[d, "\phi_+"] \\ V & \mathbb{C} \end{tikzcd}$$

We have $\phi_+ \circ F \circ \phi^{-1} = \phi_+ \circ (\phi_+^{-1} \circ f) \circ \phi^{-1} = f \circ \phi^{-1}$. Thus $\phi_+ \circ F \circ \phi^{-1}$ is holomorphic on V. Thus F is holomorphic on U.

<u>Case 2</u> f has a pole at x_0.

Then there exists a chart $(U, \phi: U \to V)$ on X around x_0 such that $f \circ \phi^{-1}$ is holomorphic on $V \setminus \{\phi(x_0)\}$ and has a pole at $y_0 = \phi(x_0)$. We can shrink U if necessary so that f has no zero in U. Then $F(x) = \phi_-^{-1} \circ \left(\frac{1}{f(x)}\right) \quad \forall\, x \in U \setminus \{x_0\}$.

$$\begin{tikzcd} U \ni x_0 \arrow[r] \arrow[d, "\phi"'] & \mathbb{C} \\ V \ni y_0 & \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "F"] \arrow[d, "\phi"'] & U_- \arrow[d, "\phi_-"] \\ V \ni y_0 & \mathbb{C} \end{tikzcd}$$

For every $z \in V \setminus \{y_0\}$, we have $\phi_- \circ F \circ \phi^{-1}(z) = \frac{1}{f} \circ \varphi^{-1}(z) = \frac{1}{f \circ \phi^{-1}(z)}$.

Then $\phi_- \circ F \circ \phi^{-1}$ is holomorphic on $V \setminus \{y_0\}$. Moreover,

$$\lim_{z \to y_0} \phi_- \circ F \circ \phi^{-1}(z) = \lim_{z \to y_0} \frac{1}{f \circ \phi^{-1}(z)} = 0$$

because y_0 is a pole of $f \circ \phi^{-1}$. Thus, y_0 is a removable singularity of $\phi_- \circ F \circ \phi^{-1}$. Thus this function is holomorphic on V. Thus F is holomorphic on U. Put the following sets

$$A_1 = \{\text{meromorphic function } f \text{ on } X\},$$

$$A_2 = \{\text{holomorphic function } F: X \to S^2,\ F \not\equiv (0,0,1)\}.$$

So far, we can define a map $\lambda: A_1 \longrightarrow A_2$ such that $\lambda(f) = F$, which is given at Eq. $(*)$. By the definition of F, $\lambda(f) = \lambda(g)$ implies that f and g have the same set of poles and $f(x) = g(x)$ on their domain. This means $f = g$. Thus λ is injective. Now we'll show that λ is surjective.

Take any $F \in A_2$. Since F is holomorphic and not identically $(0,0,1)$, $\cancel{F(x) = (0,0,1)}$ the set $J = \{x \in X : F(x) = (0,0,1)\}$ is discrete. Thus $X \setminus J \neq \phi$ and is open in X. We can define a map

$$f : X \setminus J \to \mathbb{C},\quad f(x) = \phi_+ \circ F(x).$$

We'll show that f is meromorphic on X.

First, take any $x_0 \in X \setminus J$. We show that f is holomorphic at x_0. Since F is holomorphic at x_0, and $F(x_0) \in U_+$, there is a chart $(U, \phi: U \to V)$ around x_0 such that $\phi_+ \circ F \circ \phi^{-1}$ is holomorphic on V.

$$\begin{tikzcd} U \arrow[r, "F"] \arrow[d, "\phi"'] & U_+ \arrow[d, "\phi_+"] \\ V & \mathbb{C} \end{tikzcd}$$

Since $f = \phi_+ \circ F$, $f \circ \phi^{-1}$ is holomorphic on V. Thus f is holomorphic on U.

Next, take any $x_0 \in J$. We show that f has a pole at x_0. Since F is holomorphic at x_0 and $F(x_0) = (0,0,1) \in U_-$, there is a chart $(U, \phi : U \to V)$ around x_0 such that $\phi_- \circ F \circ \phi^{-1}$ is holomorphic on V. We can shrink U if necessary so that $(0,0,-1) \notin U$ and $U \cap J = \{x_0\}$. For each $x \in U \setminus \{x_0\}$, $F(x) \in U_+ \cap U_-$. Thus, $\phi_- \circ F(x) \neq 0$. Then $f(x) = \phi_+ \circ F(x) = \phi_+ \circ \phi_-^{-1} \circ \phi_- \circ F(x)$

$$= \frac{1}{\phi_- \circ F(x)}.$$

$$\begin{tikzcd} U \ni x_0 \arrow[r, "F"] \arrow[d, "\phi"'] & U_- \arrow[d, "\phi_-"] \\ V \ni y_0 & \mathbb{C} \end{tikzcd}$$

Put $y_0 = \phi(x_0)$. Then, for every $y \in V \setminus \{y_0\}$,

$$f \circ \phi^{-1}(y) = \frac{1}{\phi_- \circ F \circ \phi^{-1}(y)} = \left(\frac{1}{z}\right) \circ \phi_- \circ F \circ \phi^{-1}(y).$$

Since $\phi_- \circ F \circ \phi^{-1}$ is holomorphic and nonzero on $V \setminus \{y_0\}$, $f \circ \phi^{-1}$ is holomorphic on $V \setminus \{y_0\}$. We have

$$\phi_- \circ F \circ \phi^{-1}(y_0) = \phi_- \circ F(x_0) = \phi_-(0,0,1) = 0.$$

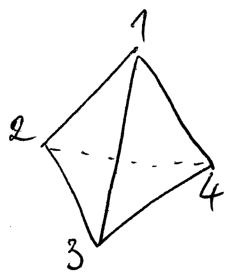
Thus,
$$\lim_{y \to y_0} f \circ \phi^{-1}(y) = \lim_{y \to y_0} \frac{1}{\phi_- \circ F \circ \phi^{-1}(y)} = \infty.$$

Thus y_0 is a pole of $f \circ \phi^{-1}$. Thus x_0 is a pole of f. We have proved that f is a meromorphic function on X. Moreover, on the way up to here, we realized that f and F are related by Eq.(*). In other words, $\lambda(f) = F$. Thus λ is surjective.

(4) Problem II.4.D, Miranda, p. 53

We will exhibit a triangulation for a sphere, disk and cylinder.

For sphere We observe that S^2 is homeomorphic to a tetrahedron.



Thus the triangulation is given as follows.

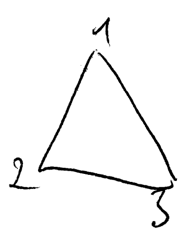
$V = \{1, 2, 3, 4\}$ (4 vertices),

$E = \{12, 13, 14, 23, 24, 34\}$ (6 edges),

$T = \{123, 124, 134, 234\}$ (4 triangles).

Thus $e(S^2) = 4 - 6 + 4 = 2$.

For disk We observe that $\overline{\mathbb{D}}$ is homeomorphic to a triangle.



Thus the triangulation is given by

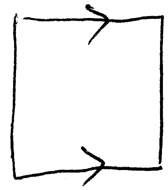
$V = \{1, 2, 3\}$ (3 vertices),

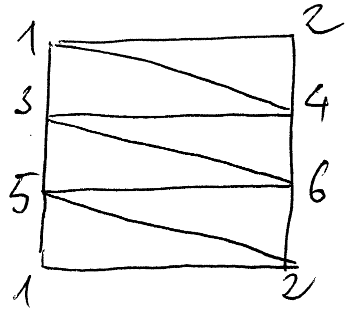
$E = \{12, 13, 23\}$ (3 edges),

$T = \{123\}$ (1 triangle).

Thus $e(\overline{\mathbb{D}}) = 3 - 3 + 1 = 1$.

<u>For cylinder</u> A cylinder is obtained when we identify one pair of opposite edges of a square. Thus one triangulation is as follows.





$V = \{1,2,3,4,5,6\}$ (6 vertices),

$E = \{12, 13, 14, 15, 24, 25, 26, 34, 35, 36, 46, 56\}$ (12 edges),

$T = \{124, 134, 125, 346, 356, 256\}$ (6 triangles).

Thus, $e(\text{cylinder}) = 6 - 12 + 6 = 0$.

(5) Problem II.4.G, Miranda, p. 53

Let $f(z) = \dfrac{z^3}{1-z^2}$ be a meromorphic function on $\mathbb{C}_\infty$. We have

$\mathrm{ord}_p(f) \neq 0 \Leftrightarrow p$ is a pole or a zero of f

$f(z) = \infty \Leftrightarrow z = \pm 1$ or $z = \infty$,

$f(z) = 0 \Leftrightarrow z = 0$.

Thus the poles of f are $-1, 1, \infty$. The zeros of f is 0. Thus all points $p \in \mathbb{C}_\infty$ such that $\mathrm{ord}_p(f) \neq 0$ are $0, \pm 1$ and ∞.

In Problem (3), we showed that f associates with a holomorphic map $F: \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by

$$F(z) = \begin{cases} \dfrac{z^3}{1-z^2} & \text{if } z \notin \{\pm 1, \infty\}, \\ \infty & \text{if } z \in \{\pm 1, \infty\}. \end{cases}$$

By definition, $\deg(F) = \sum_{p \in F^{-1}(0)} \mathrm{mult}_p(F)$.

We showed earlier that $F(p) = 0$ iff $p = 0$. Thus $\deg(F) = \mathrm{mult}_0(F)$.

Since 0 is a zero of f, $\mathrm{mult}_0(F) = \mathrm{ord}_0(f)$. For $|z| < 1$, we have $f(z) = z^3 \dfrac{1}{1-z^2} = z^3(1 + z^2 + z^4 + \dots) = z^3 + z^5 + z^7 + \dots$

Thus $\mathrm{ord}_0 f = 3$. Therefore, $\deg(F) = 3$.

Next, we will determine all ramification points and branch points of F. For each $z_0 \in \mathbb{C}_\infty$, we know that

$$\mathrm{mult}_{z_0}(F) = \begin{cases} \mathrm{ord}_{z_0}(f) & \text{if } z_0 \text{ is a zero of } f, \\ -\mathrm{ord}_{z_0}(f) & \text{if } z_0 \text{ is a pole of } f, \\ \mathrm{ord}_{z_0}(f - f(z_0)) & \text{if } z_0 \text{ is otherwise}. \end{cases}$$

Let's consider the following cases.

$\underline{z_0 = \infty}$ ∞ is a pole of f. Thus $\mathrm{mult}_\infty F = -\mathrm{ord}_\infty f = -\mathrm{ord}_0 g$,

where $g(z) = f\left(\frac{1}{z}\right) = \dfrac{(1/z)^3}{1-(1/z)^2} = z^{-1}\dfrac{1}{z^2-1} = z^{-1}(-1 - z^2 - z^4 - \dots)$

Then $\mathrm{ord}_0(g) = -1$. Thus $\mathrm{mult}_\infty(F) = 1$, i.e. ∞ is not a ramification point of F.

$\underline{z_0 = 1}$ 1 is a pole of f. Then $\text{mult}_1(F) = -\text{ord}_1(f)$.

We have $f(z) = \dfrac{z^3}{1-z^2} = -(z-1)^{-1}\dfrac{z^3}{1+z}$. Since $\dfrac{z^3}{1+z}$ is holomorphic

and nonzero around $z=1$, $\text{ord}_1(f) = -1$. Then $\text{mult}_1(F) = 1$, i.e.

1 is not a ramification point of F.

$\underline{z_0 = -1}$ Similarly to the case $z_0 = 1$, -1 is not a ramification

point of F.

$\underline{z_0 = 0}$ We computed earlier that $\text{mult}_0(F) = 3$. Thus 0 is

a ramification point of F and $F(0) = 0$ is a branch point.

$\underline{z_0 \notin \{0, \pm 1, \infty\}}$ z_0 is not a zero nor a pole of f.

Then $\text{mult}_{z_0}(F) = \text{ord}(f - f(z_0)) = 1 + \text{ord}_{z_0}\dfrac{df}{dz}$. We have

$$\frac{df}{dz} = \frac{d}{dz}\left(\frac{z^3}{1-z^2}\right) = \frac{3z^2(1-z^2) - z^3(-2z)}{(1-z^2)^2} = \frac{z^2(3-z^2)}{(1-z^2)^2} = \frac{z^2(\sqrt{3}-z)(\sqrt{3}+z)}{(1-z^2)^2}.$$

Thus df/dz has simple zeros at $\pm\sqrt{3}$. Then

$\text{ord}_{z_0}\dfrac{df}{dz} \geq 1 \iff z_0 = \pm\sqrt{3}$ (since $z_0 \neq 0$).

Moreover, $\text{ord}_{\sqrt{3}}\dfrac{df}{dz} = \text{ord}_{-\sqrt{3}}\dfrac{df}{dz} = 1$.

Thus $\pm\sqrt{3}$ are ramification points of F. The branch points are

$$f(\sqrt{3}) = \frac{3\sqrt{3}}{1-\sqrt{3}^2} = -\frac{3}{2}\sqrt{3},$$

$$f(-\sqrt{3}) = \frac{-3\sqrt{3}}{1-\sqrt{3}^2} = \frac{3}{2}\sqrt{3}.$$

To sum up, F has three ~~b~~ramification points $0, -\sqrt{3}, \sqrt{3}$.

$$\text{mult}_0(F) = 3, \quad \text{mult}_{\sqrt{3}}(F) = \text{mult}_{-\sqrt{3}}(F) = 2.$$

The corresponding branch points are $0, -\frac{3}{2}\sqrt{3}, \frac{3}{2}\sqrt{3}$.

Because $\mathbb{C}_\infty \simeq S^2$, $g(\mathbb{C}_\infty) = g(S^2) = 0$. Then the Hurwitz's formula for F is

$$2g(\mathbb{C}_\infty) - 2 = \deg(F)(2g(\mathbb{C}_\infty) - 2) + \sum_{p\in\mathbb{C}_\infty}[\text{mult}_p(F) - 1]$$

$$\Leftrightarrow \quad -2 = 3(-2) + (\text{mult}_0(F) - 1) + (\text{mult}_{\sqrt{3}}(F) - 1) + (\text{mult}_{-\sqrt{3}}(F) - 1)$$

$$\Leftrightarrow \quad -2 = -6 + (3-1) + (2-1) + (2-1)$$

$$\Leftrightarrow \quad -2 = -2,$$

which is true.

(6) Problem II. 4. K, Miranda, p. 54

Let f and g be two polynomial functions on $\mathbb{C}^2$ defined by

$$f(x,t) = x^2 - (3 + 10t^4 + 3t^8),$$

$$g(z,w) = w^2 - (z^6 - 1).$$

Let U and V be the loci of roots of f and g respectively. First we will show that U and V are smooth affine plane curves.

Suppose by contradiction that f is singular at a root (x_0, t_0). Then

$$f(x_0,t_0) = \frac{\partial f}{\partial x}(x_0,t_0) = \frac{\partial f}{\partial t}(x_0,t_0) = 0.$$

Then we have 3 equations:

$$\begin{cases} x_0^2 - (3+10t_0^4+3t_0^8) = 0, \\ 2x_0 = 0, \\ -40t_0^3 - 24t_0^7 = 0. \end{cases} \iff \begin{cases} x_0 = 0, \\ 3+10t_0^4+3t_0^8 = 0, & (1) \\ t_0^3(40+27t_0^4) = 0. & (2) \end{cases}$$

By (1), $t_0 \neq 0$. Then by (2), $40+27t_0^4 = 0$, i.e. $t_0^4 = -\frac{40}{24}$. Then (1) becomes $3 + 10\left(-\frac{40}{24}\right) + 3\left(-\frac{40}{24}\right)^2 = 0$, which is not true. This means f has no singular point on U. Thus U is a smooth affine curve.

Similarly, suppose by contradiction that g is singular at a root (z_0, w_0). Then $g(z_0, t_0) = \frac{\partial g}{\partial z}(z_0,t_0) = \frac{\partial g}{\partial t}(z_0,t_0) = 0$. Then we have 3 equations

$$\begin{cases} w_0^2 - (z_0^6 - 1) = 0, \\ -6z_0^5 = 0, \\ 2w_0 = 0. \end{cases} \iff \begin{cases} z_0 = w_0 = 0, \\ 0 - (0-1) = 0. \end{cases}$$

This system in fact has no solution. This means g has no singular point on V. Thus V is a smooth affine curve.

Next, we will show that U and V are Riemann surfaces. This is equivalent to showing that f and g are irreducible.

We can view $f(x,t)$ as a monic polynomial $\tilde{f}(x)$ in $\mathbb{C}[t]$. Put $h(t) = 3 + 10t^4 + 3t^8$. Then $\tilde{f}(x) = x^2 - h(t)$. We have $h'(t) = t^3(40+24t)$. As shown earlier, h and h' has no ~~common~~ common root in $\mathbb{C}$. Thus, $h(t)$ has eight distinct roots. Let α be a root of $h(t)$. Then $t-\alpha$ is a prime of $\mathbb{C}[t]$, $(t-\alpha) \mid h(t)$, and $(t-\alpha)^2 \nmid h(t)$. By the Eisenstein's Criterion, $\tilde{f}(x)$ is irreducible in $\mathbb{C}[t][x]$. Thus f is irreducible in $\mathbb{C}[x,t]$.

Similarly, we can view $g(z,w)$ as a monic polynomial $\tilde{g}(w)$ in $\mathbb{C}[z]$. Put $k(z) = z^6 - 1 = (z-1)(z^5+z^4+z^3+z^2+1)$. Then $(z-1) \mid k(z)$ but $(z-1)^2 \nmid k(z)$. By the Eisenstein's criterion, $\tilde{g}(w) = w^2 - k(z)$ is irreducible in $\mathbb{C}[z][w]$. Thus g is irreducible in $\mathbb{C}[z,w]$.

Suppose that $(x,t) \in U$. Then $t = \pm 1$ implies $x^2 = 16$, i.e. $x = \pm 4$. Thus $\{(x,t) \in U : t \neq \pm 1\} = U \setminus \{(\pm 4, \pm 1)\}$. Consider the map

$$F : U \setminus \{(\pm 4, \pm 1)\} \longrightarrow V$$

$$F(x,t) = \Big(\underbrace{\frac{1+t^2}{1-t^2}}_{z}, \underbrace{\frac{2tx}{(1-t^2)^3}}_{w} \Big).$$

First we show that F is well-defined. Since $(x,t) \in U$, $x^2 = 3 + 10t^4 + 3t^8$. Then

$$w^2 = \frac{4t^2x^2}{(1-t^2)^6} = \frac{4t^2(3+10t^4+3t^8)}{(1-t^2)^6} = \frac{(1+t^2)^6-(1-t^2)^6}{(1-t^2)^6}$$

$$= \left(\frac{1+t^2}{1-t^2}\right)^6 - 1 = z^6 - 1.$$

Thus $(z,w) \in V$, i.e. F is well-defined.

Next, we'll show that F is holomorphic. We have

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial t} = -t^3(40+24t),$$

$$\frac{\partial g}{\partial z} = -6z^5, \quad \frac{\partial g}{\partial w} = 2w.$$

Pick any $(x_0, t_0) \in U \setminus \{(\pm 4, \pm 1)\}$. We will show that F is holomorphic at (x_0, t_0). Put $(z_0, w_0) = F(x_0, t_0) = \left(\frac{1+t_0^2}{1-t_0^2}, \frac{2t_0x_0}{(1-t_0^2)^3}\right)$.

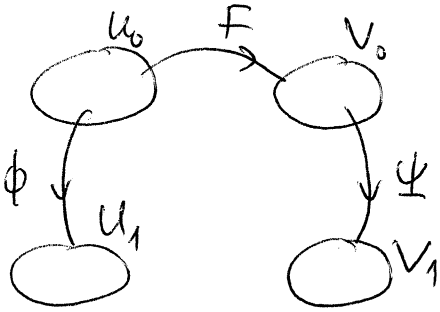
We will consider two cases, namely $\frac{\partial f}{\partial x}(x_0, t_0) \neq 0$ and $\frac{\partial f}{\partial x}(x_0, t_0) = 0$.

<u>Case 1</u> $\frac{\partial f}{\partial x}(x_0, t_0) \neq 0$

Then $x_0 \neq 0$. There is a chart $(U_0, \phi: U_0 \to U_1)$ around (x_0, t_0) in $U \setminus \{(\pm 4, \pm 1)\}$ such that $\phi(x,t) = t$. In addition, there is a holomorphic function $r: U_1 \to \mathbb{C}$ such that $x = r(t)$. To determine a chart around (z_0, w_0) in V, we have to consider separately two cases as follows.

- <u>$t_0 \neq 0$</u> Then $\frac{\partial g}{\partial w}(z_0, w_0) = 2w_0 = \frac{2t_0x_0}{(1-t_0^2)^3} \neq 0.$

Thus, there is a chart $(V_0, \Psi: V_0 \to V_1)$ around (z_0, w_0) in V such that $\Psi(z,w) = z$, $\forall (z,w) \in V_0$. ~~In addition~~,

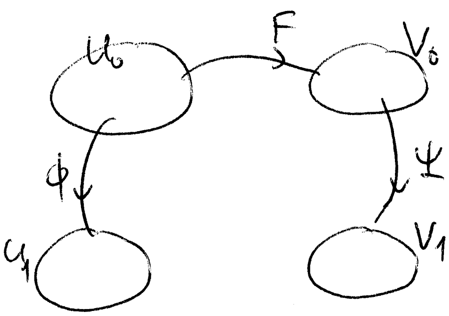


$$\begin{tikzcd} (x,t) \arrow[r, "F"] \arrow[d, "\phi"'] & (z,w) \arrow[d, "\Psi"] \\ t \arrow[r, dashed] & z \end{tikzcd}$$

Thus $\Psi \circ F \circ \phi^{-1}(t) = z = \dfrac{1+t^2}{1-t^2}$, which is holomorphic at t_0 since $t_0 \neq \pm 1$. Thus F is holomorphic at (x_0, t_0).

▸ <u>$t_0 = 0$</u> Then $\dfrac{\partial g}{\partial w}(z_0, w_0) = 2w_0 = 0$.

Thus $\dfrac{\partial g}{\partial z}(z_0, w_0) \neq 0$. Thus there is a chart $(V_0, \Psi: V_0 \to V_1)$ around (z_0, w_0) in V such that $\Psi(z,w) = w$.



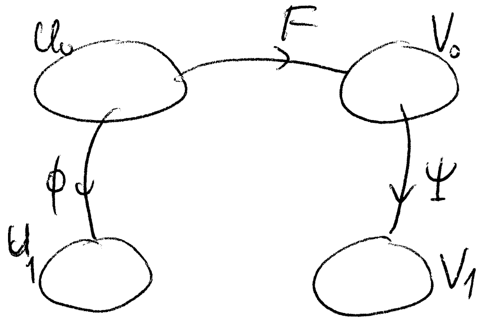
$$\begin{tikzcd} (x,t) \arrow[r, "F"] \arrow[d, "\phi"'] & (z,w) \arrow[d, "\Psi"] \\ t \arrow[r, dashed] & w \end{tikzcd}$$

Then $\Psi \circ F \circ \phi^{-1}(t) = w = \dfrac{2tx}{(1\mp t^2)^3} = \dfrac{2t\, r(t)}{(1-t^2)^3}$, which is holomorphic at $t_0 = 0$. Thus F is holomorphic at (x_0, t_0).

<u>Case 2</u> $\dfrac{\partial f}{\partial x}(x_0, t_0) = 0$

Then $x_0 = 0$, and $\dfrac{\partial g}{\partial w}(z_0, w_0) = 2w_0 = 0$.

Thus, $\frac{\partial f}{\partial t}(x_0,t_0) \neq 0$ and $\frac{\partial g}{\partial z}(z_0,w_0) \neq 0$. Then there is a chart $(U_0, \phi : U_0 \to U_1)$ around (x_0,t_0) in $U \setminus \{(\pm 4, \pm 1)\}$ such that $\phi(x,t) = x$ $\forall (x,t) \in U_0$. In addition, there is a holomorphic function $r : U_1 \to \mathbb{C}$ such that $t = r(x)$ $\forall x \in U_1$. Also, there is a chart $(V_0, \psi : V_0 \to V_1)$ around (z_0,w_0) in V such that $\psi(z,w) = w$ $\forall (z,w) \in V_0$.



$$\begin{array}{ccc} (x,t) & \xrightarrow{F} & (z,w) \\ \phi \downarrow & & \downarrow \Psi \\ x & \dashrightarrow & w \end{array}$$

Then $\psi \circ F \circ \phi^{-1}(x) = w = \frac{2tx}{(1-t^2)^3} = \frac{2x\,r(x)}{(1-r(x)^2)^3}$, which is holomorphic at x_0 because $r(x_0) = t_0 \neq \pm 1$. Thus F is holomorphic at (x_0,t_0).

Therefore we have proved that $F : U \setminus \{(\pm 4, \pm 1)\} \to V$ is holomorphic.

Next, we will show that F has no ramification point. In the following, we will denote the coordinate representation of F, namely $\psi \circ F \circ \phi^{-1}$, by $\hat{F}$. Pick any point $(x_0,t_0) \in U \setminus \{(\pm 4, \pm 1)\}$. For each case mentioned above, we will show that the derivative of $\hat{F}$ does not vanish at (x_0,t_0).

<u>Case 1</u> $\quad \frac{\partial f}{\partial x}(x_0,t_0) \neq 0$

• If $t_0 \neq 0$ then $\hat{F}(t) = \frac{1+t^2}{1-t^2}$ then $\frac{d\hat{F}}{dt} = \frac{d}{dt}\left(\frac{1+t^2}{1-t^2}\right) = \frac{4t}{(1-t^2)^2}$

Thus $\frac{d\hat{F}}{dt}(t_0) = \frac{4t_0}{(1-t_0^2)^2} \neq 0$.

• If $t_0 = 0$ then $\hat{F}(t) = \frac{2t\,r(t)}{(1-t^2)^3}$, where $r : U_1 \to \mathbb{C}$ is a holomorphic function with $r(0) = x_0 \neq 0$. We have

$$\frac{d\hat{F}}{dt} = \frac{(2r(t) + 2t r'(t))(1-t^2)^3 - 2t(r(t))3(-2t)(1-t^2)^2}{(1-t^2)^6}$$

Thus, $\frac{d\hat{F}}{dt}(0) = 2r(0) = 2x_0 \neq 0$.

Case 2 $\frac{\partial f}{\partial x}(x_0, t_0) = 0$.

In this case, $x_0 = 0$ and $r(0) = t_0 \neq 0$. Then $\hat{F}(x) = \frac{2x\, r(x)}{(1-r(x)^2)^3}$, where $r : U_1 \to \mathbb{C}$ is holomorphic. ~~and~~ We have

$$\frac{d\hat{F}}{dx}(x) = \frac{(2r(x) + 2x r'(x))(1-r(x)^2)^2 - 2x\,r(x)\,3\,(-2r'(x)r(x))(1-r(x)^2)^2}{(1-r(x)^2)^6}$$

Thus, $\frac{d\hat{F}}{dx}(0) = \frac{2r(0)(1-r(0)^2)^2}{(1-r(0)^2)^6} = \frac{2t_0}{(1-t_0^2)^4} \neq 0$.

Therefore, in all cases, the derivative of $\hat{F}$ doesn't vanish. Thus $\hat{F}$ is nonramified.