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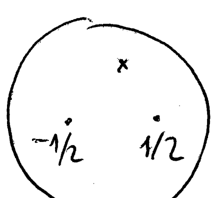
Math 8702: Complex Analysis
Take home Final

② Put $\Omega = \mathbb{D} \setminus \{-\frac{1}{2}, \frac{1}{2}\}$.

We want to find all analytic functions $f : \Omega \to \Omega$ such that for any cycle γ in Ω that is not homologous to $0 \pmod{\Omega}$, $f_*(\gamma) = f \circ \gamma$ is not homologous to $0 \pmod{\Omega}$ either.

Suppose that f is such a function. Then f is not a constant function. Indeed, if f were a constant function, it would map every cycle to a cycle whose image is a point in Ω; such a cycle of course has winding number zero with respect to both $-1/2$ and $1/2$; and so it would be homologous to $0 \pmod{\Omega}$.



Next, we will show that f can extend to an analytic function from $\mathbb{D}$ to $\mathbb{D}$. Note that currently f has two isolated singularities at $\pm 1/2$. Since f maps Ω onto Ω, it is bounded. Thus,

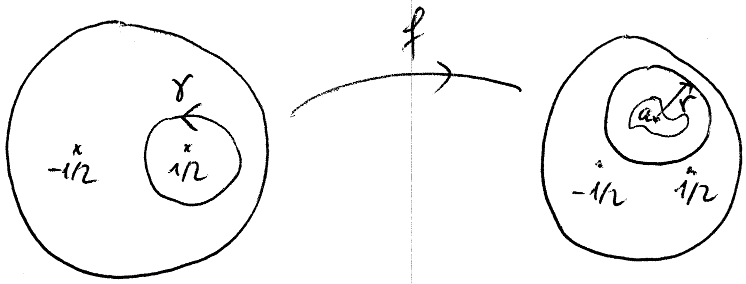
$$\lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) f(z) = 0 \quad \text{and} \quad \lim_{z \to \frac{1}{2}} \left(z - \frac{1}{2}\right) f(z) = 0.$$

This means f can extend to an analytic function on $\mathbb{D}$ by defining

$$f(-\tfrac{1}{2}) := \lim_{z\to -\frac{1}{2}} f(z), \qquad f(\tfrac{1}{2}) := \lim_{z\to\frac{1}{2}} f(z).$$

Then $f(\frac{1}{2}), f(-\frac{1}{2}) \in \overline{\mathbb{D}}$. Thus f is an analytic function from $\mathbb{D}$ to $\overline{\mathbb{D}}$. Since f is nonconstant, it is an open mapping. Thus $f(\mathbb{D})$ is an open subset of $\mathbb{C}$. Thus, $f(\mathbb{D})$ must be contained in the interior of $\overline{\mathbb{D}}$, which is $\mathbb{D}$. This means f is a map from $\mathbb{D}$ to $\mathbb{D}$.

Next, we will show that $f(-\frac{1}{2}), f(\frac{1}{2}) \in \{-\frac{1}{2}, \frac{1}{2}\}$. Suppose that this is not true. Then we can assume WLOG that $f(\frac{1}{2}) = a \in \Omega$.



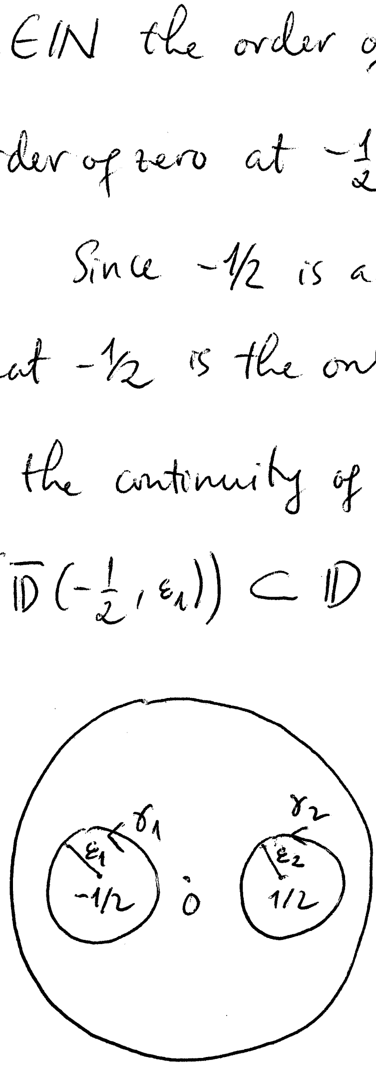
Then we can choose an open disk $\mathbb{D}(a, r)$ that lies entirely in Ω.

Since f is continuous, there exists $\varepsilon > 0$ such that $f(\mathbb{D}(\frac{1}{2}, \varepsilon)) \subset \mathbb{D}(a,r)$. Let γ be the simple closed path $\partial\mathbb{D}(\frac{1}{2}, \frac{\varepsilon}{2})$. Then γ is not homologuous to $0 \pmod{\Omega}$ because $n(\gamma, 1/2) = 1 \neq 0$. We know, however, that $f_*(\gamma)$ is a closed path lying in $\mathbb{D}(a,r)$. Thus its winding numbers with respect to $-\frac{1}{2}$ and $\frac{1}{2}$ are both zero. Then $f_*(\gamma)$ is homologous to $0 \pmod{\Omega}$. This is a contradiction.

We have proved that $f(\frac{1}{2}) = \pm\frac{1}{2}$ and $f(-\frac{1}{2}) = \pm\frac{1}{2}$. Next, we will show that $f(\frac{1}{2}) \neq f(-\frac{1}{2})$. Suppose by contradiction that

$f(\frac{1}{2}) = f(-\frac{1}{2}) = \frac{1}{2}$ (the case they are equal to $-\frac{1}{2}$ will be treated similarly). Then $\frac{1}{2}$ and $-\frac{1}{2}$ are zeros of the function $f(z) - \frac{1}{2}$. Since f is nonconstant, they are isolated zeros. Denote by $m \in \mathbb{N}$ the order of zero at $\frac{1}{2}$ of $f(z) - \frac{1}{2}$, and by $l \in \mathbb{N}$ the order of zero at $-\frac{1}{2}$ of $f(z) - \frac{1}{2}$.

Since $-1/2$ is an isolated zero of $f(z) - \frac{1}{2}$, there is $\varepsilon_1 > 0$ such that $-1/2$ is the only zero of $f(z) - \frac{1}{2}$ in $\overline{\mathbb{D}}(-\frac{1}{2}, \varepsilon_1)$. Moreover, by the continuity of f, we could have chosen $\varepsilon_1 > 0$ such that $f(\overline{\mathbb{D}}(-\frac{1}{2}, \varepsilon_1)) \subset \mathbb{D}(\frac{1}{2}, \frac{1}{4})$. Put $\gamma_1 = \partial\mathbb{D}(-\frac{1}{2}, \varepsilon_1)$ which is positively oriented. Then γ_1 has 3 following properties:



- γ_1 does not pass through any zero of $f(z) - \frac{1}{2}$,
- $-\frac{1}{2}$ is the only zero of $f(z) - \frac{1}{2}$ that is enclosed in γ_1,
- $f \circ \gamma_1$ doesn't enclose $-\frac{1}{2}$, i.e. $n(f \circ \gamma_1, -\frac{1}{2}) = 0$, because it is a loop in $\mathbb{D}(\frac{1}{2}, \frac{1}{4})$.

Similarly, since $1/2$ is an isolated zero of $f(z) - 1/2$, there is $\varepsilon_2 > 0$ such that $1/2$ is the only zero of $f(z) - 1/2$ in $\overline{\mathbb{D}}(1/2, \varepsilon_2)$. Moreover, by the continuity of f, we could have chosen $\varepsilon_2 > 0$ such that $f(\bar{\mathbb{D}}(\frac{1}{2}, \varepsilon_2)) \subset \mathbb{D}(\frac{1}{2}, \frac{1}{4})$. Put $\gamma_2 = \partial \mathbb{D}(\frac{1}{2}, \varepsilon_2)$ which is positively oriented. Then γ_2 has 3 following properties.

- γ_2 does not pass through any zero of $f(z) - \frac{1}{2}$,
- $1/2$ is the only zero of $f(z) - 1/2$ that is enclosed in γ_2,
- $f \circ \gamma_2$ does not enclose $-1/2$, i.e. $n(f \circ \gamma_2, -\frac{1}{2}) = 0$, because it is a loop in $\mathbb{D}(\frac{1}{2}, \frac{1}{4})$.

Consider a cycle $\gamma = l\gamma_1 - m\gamma_2$. It is not homologous to 0 (mod Ω) because $n(\gamma, -1/2) = l.\underbrace{n(\gamma_1, -1/2)}_{1} - m.\underbrace{n(\gamma_2, -1/2)}_{0} = l \neq 0$.

Thus $f_*(\gamma)$ must be not homologous to 0 (mod Ω). We have

$$f_*(\gamma) = l.f_*(\gamma_1) - m f_*(\gamma_2) = l(f \circ \gamma_1) - m(f \circ \gamma_2).$$

Then $n(f_*(\gamma), -\frac{1}{2}) = l.\underbrace{n(f \circ \gamma_1, -\frac{1}{2})}_{0} - m.\underbrace{n(f \circ \gamma_2, -\frac{1}{2})}_{0} = 0.$

Also, $n(f_*(\gamma), \frac{1}{2}) = l.n(f \circ \gamma_1, \frac{1}{2}) - m.n(f \circ \gamma_2, \frac{1}{2})$. $(*)$

$$n(f \circ \gamma_1, \tfrac{1}{2}) = \frac{1}{2\pi i} \int_{f \circ \gamma_1} \frac{dw}{w - \frac{1}{2}} = \frac{1}{2\pi i} \int_0^1 \frac{(f \circ \gamma_1)'(t)\, dt}{f(\gamma_1(t)) - \frac{1}{2}}$$

$$= \frac{1}{2\pi i} \int_0^1 \frac{f'(\gamma_1(t))\gamma_1'(t)}{f(\gamma_1(t)) - \frac{1}{2}} dt$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} \frac{f'(z)}{f(z) - \frac{1}{2}} dz$$

By the Argument Principle, this is the number of zeros of $f(z)-\frac{1}{2}$, counted with multiplicity, that are enclosed in γ_1. By the choice of γ_1, we get $n(f\circ\gamma_1, \frac{1}{2}) = m$.

Similarly, $n(f\circ\gamma_2, \frac{1}{2}) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f'(z)}{f(z)-\frac{1}{2}}\,dz$,

which is equal to the number of zeros of $f(z)-\frac{1}{2}$, counted with multiplicity, that are enclosed in γ_2. By the choice of γ_2, we get $n(f\circ\gamma_2, \frac{1}{2}) = \ell$. By $(*)$, $n(f_*(\gamma), \frac{1}{2}) = \ell.m - m.\ell = 0$. Thus,

$$n(f_*(\gamma), \tfrac{1}{2}) = n(f_*(\gamma), -\tfrac{1}{2}) = 0.$$

This means $f_*(\gamma)$ is homologous to $0 \pmod{\Omega}$, which is a contradiction.

We have proved that $f(\frac{1}{2}) \neq f(-\frac{1}{2})$. Therefore, there are only two possibilities, namely $\left(f(\frac{1}{2}) = \frac{1}{2}, f(-\frac{1}{2}) = -\frac{1}{2}\right)$ and $\left(f(\frac{1}{2}) = -\frac{1}{2}, f(-\frac{1}{2}) = \frac{1}{2}\right)$.

<u>Case 1</u> $f(\frac{1}{2}) = \frac{1}{2}$ and $f(-\frac{1}{2}) = -\frac{1}{2}$.

Let g be the linear fractional transformation under which

$$1 \mapsto 1,$$

$$\frac{1}{2} \mapsto 0,$$

$$2 \mapsto \infty.$$

This results in $g(z) = (z, 1, \frac{1}{2}, 2) = \dfrac{-2z+1}{z-2}$.

The inverse map is $g^{-1}(w) = \dfrac{2w+1}{w+2}$.

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Since 1 is on the unit circle (C), 2 is the reflection point of $\frac{1}{2}$, and ∞ is the reflection point of 0 with respect to (C), g maps (C) to (C) and maps the unit disk $\mathbb{D}$ conformally to itself.

Put $h = g \circ f \circ g^{-1} : \mathbb{D} \to \mathbb{D}$. Then

$$h(0) = g \circ f \circ (g^{-1}(0)) = g \circ f\left(\frac{1}{2}\right) = g\left(\frac{1}{2}\right) = 0,$$

$$h\left(-\frac{4}{5}\right) = g \circ f \circ \left(g^{-1}\left(-\frac{4}{5}\right)\right) = g \circ f\left(-\frac{1}{2}\right) = g\left(-\frac{1}{2}\right) = -\frac{4}{5}.$$

Therefore, by Schwarz's lemma, h is a rotation about the origin. Since h fixes $-4/5$, h is the identity map. Thus $f = g^{-1} \circ h \circ g$ is also the identity map.

Case 2 $\quad f\left(\frac{1}{2}\right) = -\frac{1}{2}$ and $f\left(-\frac{1}{2}\right) = \frac{1}{2}$.

Put $\tilde{f}(z) = f(-z)$. Then $\tilde{f}\left(\frac{1}{2}\right) = \frac{1}{2}$ and $\tilde{f}\left(\frac{-1}{2}\right) = -\frac{1}{2}$. We return to case 1 and conclude $\tilde{f}(z) = z$ $\forall z \in \mathbb{D}$. Thus $f(z) = -z$ $\forall z \in \mathbb{D}$.

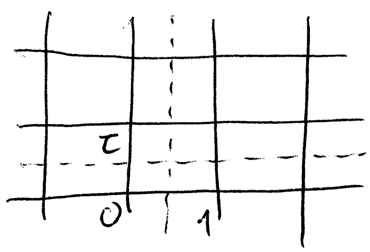
We have prove that there are at most two functions f satisfying the condition in the problem, namely $f_1(z) = z$ $\forall z \in \Omega$ and $f_2(z) = -z$ $\forall z \in \Omega$. Conversely, any cycle γ in Ω is mapped to itself under $(f_1)_*$. Thus if γ is not homologous to 0 (mod Ω), $(f_1)_*(\gamma)$ is not either. Now suppose that γ is a cycle that is not homologous to 0 (mod Ω). Then either $n(\gamma, \frac{1}{2}) \neq 0$ or $n(\gamma, -\frac{1}{2}) \neq 0$.

Since f_2 is the reflection about O, $(f_2)_*(\gamma)$ is the reflection of γ about O. Thus $n((f_2)_*(\gamma), \frac{1}{2}) = n(\gamma, -\frac{1}{2})$ and $n((f_2)_*(\gamma), -\frac{1}{2}) = n(\gamma, \frac{1}{2})$. This means $(f_2)_*(\gamma)$ is not homologous to $0 \pmod{\Omega}$.

In conclusion, all functions f satisfying the problem are $f_1(z) \equiv z$ and $f_2(z) \equiv -z$.

① Put $\tau = it$ where t is a positive real number. Consider the lattice in the complex plane generated by 1 and τ:

$$\Lambda = \{n + m\tau : m, n \in \mathbb{Z}\}$$



Let $\wp(z)$ be the Weierstrass function associated with the lattice Λ.

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(z+n+m\tau)^2} - \frac{1}{(n+m\tau)^2} \right]$$

(a) We'll show that $\wp(z)$ is real-valued on the horizontal lines $y = \frac{mt}{2}$ and on the vertical lines $x = \frac{n}{2}$. Since $\wp(z)$ is doubly periodic with periods 1 and τ, it suffices to show that $\wp(z)$ is real-valued on 4 lines, namely the imaginary axis (except at the lattice points), the real axis (except at the lattice points), the line $x = \frac{1}{2}$ and $y = \frac{t}{2}$.

Consider the imaginary axis

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Let z be a point on the imaginary axis which is not a lattice point. We have $\bar{z} = -z$ and $\bar{\tau} = -\tau$. Then

$$\overline{\wp(z)} = \frac{1}{\bar{z}^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(\bar{z}+n+m\bar{\tau})^2} - \frac{1}{(n+m\bar{\tau})^2}\right]$$

$$= \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(-z+n-m\tau)^2} - \frac{1}{(n-m\tau)^2}\right] \qquad (1)$$

Since the lattice is invariant under the transformation $(m,n)\mapsto(m,-n)$, we can replace (m,n) by $(m,-n)$ in the summands. Then (1) becomes

$$\overline{\wp(z)} = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(-z-n-m\tau)^2} - \frac{1}{(-n-m\tau)^2}\right] = \wp(z).$$

Thus $\wp(z) \in \mathbb{R}$.

<u>Consider the real axis</u>

Let z be a point on the real axis which is not a lattice point.

Then $\bar{z} = z$.

$$\overline{\wp(z)} = \frac{1}{\bar{z}^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(\bar{z}+n+m\bar{\tau})^2} - \frac{1}{(n+m\bar{\tau})^2}\right]$$

$$= \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(z+n-m\tau)^2} - \frac{1}{(n-m\tau)^2}\right] \qquad (2)$$

Since the lattice is invariant under the transformation $(m,n)\mapsto(-m,n)$, we can replace (m,n) by $(-m,n)$ in the summands. Then (2) becomes

$$\overline{\wp(z)} = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(z+n+m\tau)^2} - \frac{1}{(n+m\tau)^2}\right] = \wp(z).$$

Thus $\wp(z) \in \mathbb{R}$.

Consider the line $x = \frac{1}{2}$

Let z be a point on the vertical line $x = \frac{1}{2}$. Then $z = \frac{1}{2} + iy$ and $\bar{z} = \frac{1}{2} - iy$. Thus $z + \bar{z} = 1$, or equivalently $\bar{z} = 1 - z$.

$$\overline{\wp(z)} = \frac{1}{\bar{z}^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(\bar{z} + n + m\bar{\tau})^2} - \frac{1}{(n+m\bar{\tau})^2} \right]$$

$$= \frac{1}{(1-z)^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(-z+(n+1)-m\tau)^2} - \frac{1}{(n-m\tau)^2} \right]$$

$$= \frac{1}{(z-1)^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(z-(n+1)+m\tau)^2} - \frac{1}{(-n+m\tau)^2} \right]$$

$$= \frac{1}{(z-1)^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(z-(n+1)+m\tau)^2} - \frac{1}{(z-n+m\tau)^2} \right]$$

$$+ \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(z-n+m\tau)^2} - \frac{1}{(-n+m\tau)^2} \right] \qquad (3)$$

By replacing (m,n) with $(m,-n)$, the second sum of RHS(3) is equal to $\wp(z) - \frac{1}{z^2}$. The first sum of RHS(3) can be rewritten as follows

$$\sum_{m\in\mathbb{Z}^*} \Bigg\{ \underbrace{\sum_{n\in\mathbb{Z}} \left[\frac{1}{(z-(n+1)+m\tau)^2} - \frac{1}{(z-n+m\tau)^2} \right]}_{A} \Bigg\} + \underbrace{\sum_{n\in\mathbb{Z}^*} \left[\frac{1}{(z-(n+1))^2} - \frac{1}{(z-n)^2} \right]}_{B} \qquad (4)$$

We have
$$A = \lim_{N\to\infty} \sum_{n=-N}^{N} \left[\frac{1}{(z-(n+1)+m\tau)^2} - \frac{1}{(z-n+m\tau)^2} \right]$$

$$= \lim_{N\to\infty}\left[\frac{1}{(z-(N+1)+m\tau)^2} - \frac{1}{(z+N+m\tau)^2}\right] = 0,$$

$$B = \lim_{N\to\infty}\sum_{n=-N}^{-1}\left[\frac{1}{(z-(n+1))^2} - \frac{1}{(z-n)^2}\right] + \lim_{N\to\infty}\sum_{n=1}^{N}\left[\frac{1}{(z-(n+1))^2} - \frac{1}{(z-n)^2}\right]$$

$$= \lim_{N\to\infty}\left(\frac{1}{z^2} - \frac{1}{(z+N)^2}\right) + \lim_{N\to\infty}\left(\frac{1}{(z-(N+1))^2} - \frac{1}{(z-1)^2}\right)$$

$$= \frac{1}{z^2} - \frac{1}{(z-1)^2}.$$

Now $(4) = B = \frac{1}{z^2} - \frac{1}{(z-1)^2}$. Then (3) becomes

$$\overline{\wp(z)} = \frac{1}{(z-1)^2} + \left(\frac{1}{z^2} - \frac{1}{(z-1)^2}\right) + \left(\wp(z) - \frac{1}{z^2}\right) = \wp(z).$$

Thus $\wp(z) \in \mathbb{R}$.

<u>Consider the line $y = \frac{t}{2}$</u>

Let z be a point on the horizontal line $y = t/2$. Then $z = x + ti/2$.

Thus $\bar z = x - ti/2$ and $z - \bar z = ti = \tau$. Equivalently, $\bar z = z - \tau$.

$$\overline{\wp(z)} = \frac{1}{\bar z^2} + \sum_{(m,n)\neq(0,0)}\left[\frac{1}{(\bar z + n + m\bar\tau)^2} - \frac{1}{(n+m\bar\tau)^2}\right]$$

$$= \frac{1}{(z-\tau)^2} + \sum_{(m,n)\neq(0,0)}\left[\frac{1}{(z+n-(m+1)\tau)^2} - \frac{1}{(n-m\tau)^2}\right]$$

$$= \frac{1}{(z-\tau)^2} + \sum_{(m,n)\neq(0,0)}\left[\frac{1}{(z+n-(m+1)\tau)^2} - \frac{1}{(z+n-m\tau)^2}\right]$$

$$+ \sum_{(m,n)\neq(0,0)}\left[\frac{1}{(z+n-m\tau)^2} - \frac{1}{(n-m\tau)^2}\right] \qquad (5)$$

By replacing (m,n) with $(-m,n)$, the second sum of RHS(5) is equal to $\wp(z)-\frac{1}{z^2}$. The first sum of RHS(5) can be rewritten as follows.

$$\sum_{n\in\mathbb{Z}^+}\underbrace{\left\{\sum_{m\in\mathbb{Z}}\left[\frac{1}{(z+n-(m+1)\tau)^2}-\frac{1}{(z+n-m\tau)^2}\right]\right\}}_{C}+\underbrace{\sum_{m\in\mathbb{Z}^*}\left[\frac{1}{(z-(m+1)\tau)^2}-\frac{1}{(z-m\tau)^2}\right]}_{D} \quad (6)$$

We have

$$C=\lim_{N\to\infty}\sum_{m=-N}^{N}\left[\frac{1}{(z+n-(m+1)\tau)^2}-\frac{1}{(z+n-m\tau)^2}\right]$$

$$=\lim_{N\to\infty}\left[\frac{1}{(z+n-(N+1)\tau)^2}-\frac{1}{(z+n+N\tau)^2}\right]$$

$$=0,$$

$$D=\lim_{N\to\infty}\sum_{m=-N}^{-1}\left[\frac{1}{(z-(m+1)\tau)^2}-\frac{1}{(z-m\tau)^2}\right]+\lim_{N\to\infty}\sum_{m=1}^{N}\left[\frac{1}{(z-(m+1)\tau)^2}-\frac{1}{(z-m\tau)^2}\right]$$

$$=\lim_{N\to\infty}\left(\frac{1}{z^2}-\frac{1}{(z+N\tau)^2}\right)+\lim_{N\to\infty}\left(\frac{1}{(z-(N+1)\tau)^2}-\frac{1}{(z-\tau)^2}\right)$$

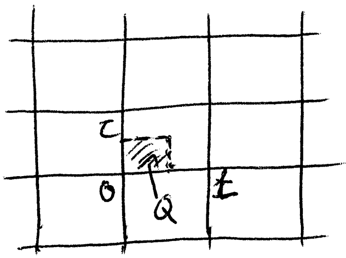
$$=\frac{1}{z^2}-\frac{1}{(z-\tau)^2}.$$

Now $(6)=D=\frac{1}{z^2}-\frac{1}{(z-\tau)^2}$. Then (5) becomes

$$\overline{\wp(z)}=\frac{1}{(z-\tau)^2}+\left(\frac{1}{z^2}-\frac{1}{(z-\tau)^2}\right)+\left(\wp(z)-\frac{1}{z^2}\right)=\wp(z).$$

Thus $\wp(z)\in\mathbb{R}$.

(b) The grid lines $x=\frac{n}{2}$ and $y=\frac{mt}{2}$ divide the complex plane into disjoint open rectangles. Put



$$\mathbb{H}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\},$$

$$\mathbb{H}^- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}.$$

We'll show that $\wp$ maps each of these rectangles conformally to $\mathbb{H}^+$ or $\mathbb{H}^-$. We'll prove this for the open rectangle Q whose vertices are at $0, \frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}$. The method shown in the following also applies for other rectangles.

First, we will show that $\wp$ is injective on Q. One of the problems in Homework 5 was Problem 1, Ahlfors page 276. It said that for any $z, u \in \mathbb{C} \setminus \Lambda$,

$$\wp(z) - \wp(u) = -\frac{\sigma(z-u)\,\sigma(z+u)}{\sigma(z)^2\,\sigma(u)^2},$$

where $\sigma(z) = z \prod_{\omega \in \Lambda^*} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2}$.

We see that the zeros of σ are exactly at the lattice points. Thus,

$$\wp(z) = \wp(u) \iff \sigma(z-u) = 0 \quad \text{or} \quad \sigma(z+u) = 0$$

$$\iff z - u \in \Lambda \quad \text{or} \quad z + u \in \Lambda$$

In case $z, u \in Q$, we can write $z = \alpha + \beta\tau$, $u = \gamma + \delta\tau$, where $0 < \alpha, \beta, \gamma, \delta < \frac{1}{2}$. Then $-\frac{1}{2} < \alpha - \gamma, \beta - \delta < \frac{1}{2}$ and

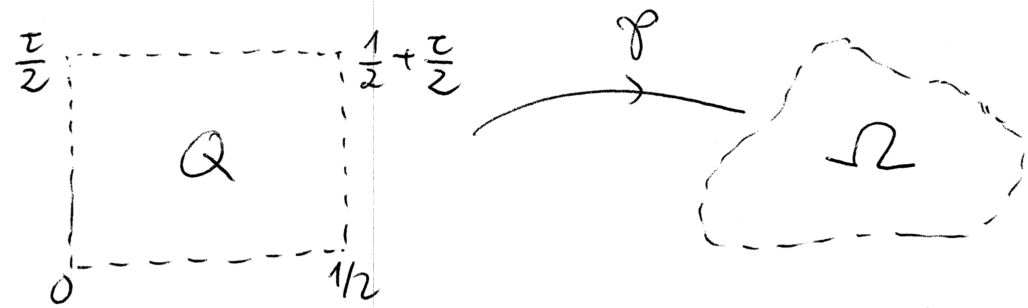
$$0 < \alpha + \gamma, \beta + \delta < 1.$$

Since $z+u=(\alpha+\gamma)+(\beta+\delta)\tau$ and $0<\alpha+\gamma, \beta+\delta<1$, it never lies on the lattice Λ. Thus

$$\begin{cases}\wp(z)=\wp(u)\\ z,u\in Q\end{cases} \iff \begin{cases}z-u\in\Lambda\\ z,u\in Q\end{cases} \iff \begin{cases}(\alpha-\gamma)+(\beta-\delta)\tau\in\Lambda,\\ 0<\alpha,\beta,\gamma,\delta<\frac{1}{2}.\end{cases}$$

This happens only if $\alpha-\gamma=0$ and $\beta-\delta=0$, i.e. $z=u$. Therefore, $\wp$ is injective on Q.

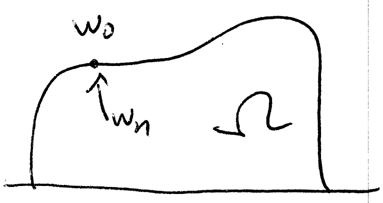
Consequently, $\wp$ maps Q conformally to a simply connected domain Ω in $\mathbb{C}$. We'll show that $\Omega=\mathbb{H}^+$ or $\Omega=\mathbb{H}^-$.



In the following, we'll use the following Theorem (Theorem 2, Ahlfors page 233):

Let f be a topological mapping of a region U to V. If (z_n) tends to the boundary of U then $(f(z_n))$ tends to the boundary of V

Let (z_n) be any sequence in Q which tends to a point $a\in(\partial Q)\setminus\{0\}$. Since a lies on the boundary of Q, $\wp(a)=\lim_{n\to\infty}\wp(z_n)$ lies on the boundary of Ω by the above Theorem. As shown in part (a), if $a \in \partial Q$ and $a \neq 0$ then $\wp(a) \in \mathbb{R}$. Thus $\partial\Omega \cap \mathbb{R} \neq \phi$.



Next, we'll show that $\partial\Omega \subset \mathbb{R}$. Suppose by contradiction that there exists a point $w_0 \in (\partial\Omega)\setminus\mathbb{R}$. Then w_0 is the limit of a sequence (w_n) in Ω. We apply the above Theorem for the map $(\wp|_Q)^{-1}$ and the sequence (w_n). Since $\lim w_n = w_0 \in \partial\Omega$, $\{\wp^{-1}(w_n)\}$ tends to the boundary of Q. Since $\overline{Q}$ is compact, there exists a subsequence $\{\wp^{-1}(w_{n_k})\}$ that converges to some $z_0 \in \partial Q$. Thus, from the beginning we could have taken (w_n) to be the subsequence (w_{n_k}) to get $\lim_{n\to\infty} \wp^{-1}(w_n) = z_0 \in \partial Q$. We consider two cases of z_0, namely $z_0 = 0$ and $z_0 \in (\partial Q)\setminus\{0\}$.

• $\underline{z_0 = 0}$ Then $\lim_{n\to\infty} \wp^{-1}(w_n) = 0$. Since 0 is a pole of $\wp$,

$$\lim_{n\to\infty} \wp(\wp^{-1}(w_n)) = \infty, \text{ i.e. } \lim_{n\to\infty} w_n = \infty.$$

This is a contradiction because (w_n) converges in $\mathbb{C}$.

• $\underline{z_0 \in (\partial Q)\setminus\{0\}}$

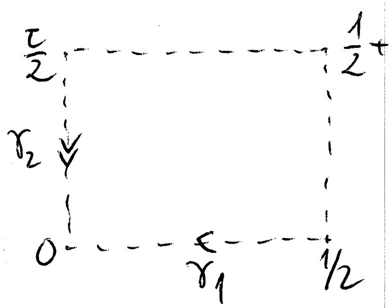
Then $\wp$ is continuous at z_0. Then $\lim \wp(\wp^{-1}(w_n)) = \wp(z_0)$. Thus, $\lim w_n = \wp(z_0) \in \mathbb{R}$. This contradicts the fact that

$\lim w_n = w_0 \notin \mathbb{R}$.

We have shown that $\partial\Omega \subset \mathbb{R}$. Next, we'll show that $\Omega = \mathbb{H}^+$ or $\Omega = \mathbb{H}^-$. Since Ω is a nonempty open subset of $\mathbb{C}$, it must intersect either $\mathbb{H}^+$ or $\mathbb{H}^-$.

Suppose that $\Omega \cap \mathbb{H}^+ \neq \emptyset$. Then $\Omega \cap \mathbb{H}^+$ is a nonempty open subset of $\mathbb{H}^+$. We'll show that it is also closed in $\mathbb{H}^+$. Let (w_n) be a sequence in $\Omega \cap \mathbb{H}^+$ that converges to some $w_0 \in \mathbb{H}^+$. We need to show that $w_0 \in \Omega \cap \mathbb{H}^+$. Suppose this is not true. Then $w_0 \notin \Omega$. Since w_0 is the limit of a sequence in Ω, $w_0 \in \partial\Omega$. Thus $w_0 \in \mathbb{R}$. This contradicts the fact that $w_0 \in \mathbb{H}^+$. Therefore, $\Omega \cap \mathbb{H}^+$ is nonempty, open and closed in $\mathbb{H}^+$. Since $\mathbb{H}^+$ is connected, $\Omega \cap \mathbb{H}^+ = \mathbb{H}^+$. Thus, $\mathbb{H}^+ \subset \Omega$.

We have proved that if $\Omega \cap \mathbb{H}^+ \neq \emptyset$ then $\mathbb{H}^+ \subset \Omega$. Similarly, we can show that if $\Omega \cap \mathbb{H}^- \neq \emptyset$ then $\mathbb{H}^- \subset \Omega$. Therefore, to show that $\Omega = \mathbb{H}^+$ or $\Omega = \mathbb{H}^-$, we only need to show that Ω cannot intersect both $\mathbb{H}^+$ and $\mathbb{H}^-$. Suppose by contradiction that $\Omega \cap \mathbb{H}^+ \neq \emptyset$ and $\Omega \cap \mathbb{H}^- \neq \emptyset$. Then $\mathbb{H}^+, \mathbb{H}^- \subset \Omega$. We showed earlier that $\partial\Omega \neq \emptyset$ and $\partial\Omega \subset \mathbb{R}$. Now we'll show that $\mathbb{R} \setminus (\partial\Omega)$ is bounded.



Let $\gamma_1(t) = \frac{1}{2} - \frac{1}{2}t$, $0<t<1$,

$\gamma_2(s) = \frac{\tau}{2} - \frac{\tau}{2}s$, $0<s<1$,

be the parametrization of the edges of Q that contain 0. We know by the Theorem mentioned earlier that $\wp\circ\gamma_1(t)$ and $\wp\circ\gamma_2(s)$ lie on $\partial\Omega$ for all $0<s,t<1$.

Since 0 is a pole of $\wp$, $\lim_{z\to 0} \wp(z) = \infty$. Thus,

$$\lim_{t\to 1^-} \wp\circ\gamma_1(t) = \pm\infty \quad \text{and} \quad \lim_{s\to 1^-} \wp\circ\gamma_2(s) = \pm\infty. \qquad (*)$$

For $s,t\in(0,1)$, we have $\gamma_1(t)+\gamma_2(s) = \frac{1-t}{2} + \frac{1-s}{2}\tau$,

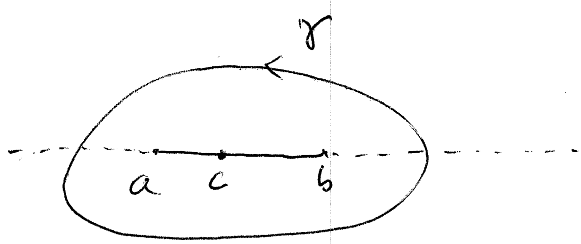
$$\gamma_1(t)-\gamma_2(s) = \frac{1-t}{2} - \frac{1-s}{2}\tau.$$

Thus, $\gamma_1(t)+\gamma_2(s)$ and $\gamma_1(t)-\gamma_2(s)$ do not belong to the lattice Λ. Thus, $\wp(\gamma_1(t)) \neq \wp(\gamma_2(s))$ for all $t,s\in(0,1)$. Since $\wp\circ\gamma_1$ and $\wp\circ\gamma_2$ are continuous, 4 possibilities in $(*)$ collapse into 2, namely

$$\lim_{t\to 1^-} \wp(\gamma_1(t)) = \infty, \quad \lim_{s\to 1^-} \wp(\gamma_2(s)) = -\infty,$$

and $\displaystyle\lim_{t\to 1^-} \wp(\gamma_1(t)) = -\infty, \quad \lim_{s\to 1^-} \wp(\gamma_2(s)) = \infty.$

Due to the continuity of $\wp\circ\gamma_1$ and $\wp\circ\gamma_2$, the set $\wp\circ\gamma_1((0,1)) \cup \wp\circ\gamma_2((0,1))$ contains a subset $(-\infty,a)\cup(b,\infty)$ for some $a,b\in\mathbb{R}$. Thus, $(-\infty,a)\cup(b,\infty) \subset (\partial\Omega)$. Consequently, $\mathbb{R}\setminus(\partial\Omega) \subset [a,b]$. We have proved that $\mathbb{R}\setminus(\partial\Omega)$ is bounded in some interval $[a,b]$ in $\mathbb{R}$. Since $(\partial\Omega)\cap\mathbb{R}\neq\emptyset$, there is $c\in(\partial\Omega)\cap[a,b]$. Let γ be any loop enclosing c but does not intersect the segment $[a,b]$. Then $\gamma\subset\Omega$ and $n(\gamma,c)\neq 0$.



This is a contradiction because Ω is simply connected.

(3) Let Ω be an open connected subset of $\mathbb{C}$ and $f:\Omega\to\mathbb{R}$ be a continuous function. We'll show that f is harmonic if and only if it satisfies the Mean-Value property.

($\Rightarrow$) Suppose that f is harmonic. Consider a closed disk $\overline{\mathbb{D}}(z_0,r)\subset\Omega$. We want to show that $f(z_0)=\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta$.

By the compactness of $\overline{\mathbb{D}}(z_0,r)$, there is $r'>r$ such that $\overline{\mathbb{D}}(z_0,r)\subset\mathbb{D}(z_0,r')\subset\Omega$. Since f is harmonic in $\mathbb{D}(z_0,r')$, which is a simply connected domain, there exists an analytic function $F:\mathbb{D}(z_0,r')\to\mathbb{C}$ such that $f(z)=\operatorname{Re}F(z)$, $\forall z\in\mathbb{D}(z_0,r')$. Since F is analytic, it satisfies the Mean-Value property (which is a consequence of Cauchy Integral Theorem):

$$F(z_0) = \frac{1}{2\pi} \int_0^{2\pi} F(z_0 + re^{i\theta})\, d\theta$$

By taking the real parts of both sides, we get

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

($\Leftarrow$) Suppose that f satisfies the Mean Value property. We'll show that f is harmonic. First, we'll ~~show~~ prove the following lemma:

[If $u : \overline{\mathbb{D}}(z_0, r) \longrightarrow \mathbb{R}$ is a continuous function satisfying the Mean Value property, then u attains maximum and minimum on $\partial\mathbb{D}(z_0, r)$.]

Proof of the lemma

Since $\overline{\mathbb{D}}(z_0, r)$ is compact, u attains maximum and minimum in $\overline{\mathbb{D}}(z_0, r)$. Suppose that u attains maximum at some point in $\mathbb{D}(z_0, r)$, and the maximum value is a. Then the set

$$S = \{z \in \mathbb{D}(z_0, r) : u(z) = a\}$$

is nonempty. Since u is continuous, S is closed in $\mathbb{D}(z_0, r)$. Take any $z_1 \in \mathbb{D}(z_0, r)$ and any $r_1 > 0$ such that $\overline{\mathbb{D}}(z_1, r_1) \subset \mathbb{D}(z_0, r)$. Since u satisfies the Mean Value property, we have

$$a = u(z_1) = \frac{1}{2\pi} \int_0^{2\pi} u(z_1 + r_1 e^{i\theta})\, d\theta$$

Since $a = \max_{\overline{\mathbb{D}}(z_0, r)} u$, $a \geq u(z_1 + r_1 e^{i\theta})$ for all $\theta \in [0, 2\pi]$. Thus

$$a \geq \frac{1}{2\pi} \int_0^{2\pi} u(z_1 + r_1 e^{i\theta})\, d\theta$$

Since the equality actually happens and u is continuous, we have $u(z_1 + r_1 e^{i\theta}) = a$ for all $\theta \in [0, 2\pi]$. Since r_1 was chosen arbitrarily ~~at~~ as long as $\overline{\mathbb{D}}(z_1, r_1) \subset \mathbb{D}(z_0, r)$, $u(z) = a$ for all z in a neighborhood of z_1 in $\mathbb{D}(z_0, r)$. Thus S is open in $\mathbb{D}(z_0, r)$. Now that S is nonempty, open and closed in $\mathbb{D}(z_0, r)$, and that $\mathbb{D}(z_0, r)$ is connected, $S = \mathbb{D}(z_0, r)$. Thus u is a constant function. Thus, it attains maximum on $\partial\mathbb{D}(z_0, r)$.

For the case u attains minimum in $\mathbb{D}(z_0, r)$, $-u$ attains maximum in $\mathbb{D}(z_0, r)$. By the above arguments, $-u$ a~~ttains~~ is a constant map. Thus u attains minimum on $\partial\mathbb{D}(z_0, r)$.

Return to the problem. To show that f is harmonic, we take any closed disk $\overline{\mathbb{D}}(z_0, r) \subset \Omega$ and show that f is harmonic in $\mathbb{D}(z_0, r)$. Since f is continuous on $\partial D(z_0, r)$, the Schwarz's theorem says that the function $g : \mathbb{D}(z_0, r) \to \mathbb{R}$ defined by

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, \mathrm{Re}\left(\frac{re^{i\theta} + z - z_0}{re^{i\theta} - z + z_0} \right) d\theta$$

is harmonic, and that g extends continuously to $\overline{\mathbb{D}}(z_0, r)$, and that $g = f$ on $\partial\mathbb{D}(z_0, r)$.

Since g is harmonic in $\mathbb{D}(z_0, r)$, it satisfies Mean Value property as shown in the first part of this problem. Thus $u = g - f$ also satisfies the Mean Value property. Since u is continuous in $\overline{\mathbb{D}}(z_0, r)$, by the above lemma, u attains ~~minum~~ minimum and maximum on $\partial\mathbb{D}(z_0, r)$. On the other hand, $u = 0$ on $\partial\mathbb{D}(z_0, r)$. Thus u is identically zero in $\overline{\mathbb{D}}(z_0, r)$. Thus $g = f$ in $\overline{\mathbb{D}}(z_0, r)$. Thus f is harmonic on $\mathbb{D}(z_0, r)$.

(4) Let ω be a meromorphic 1-form on the Riemann sphere $\mathbb{C}_\infty$. We'll show that $\sum_{p \in \mathbb{C}_\infty} \operatorname{Res}_p(\omega) = 0$ $(*)$.

We know that $\mathbb{C}_\infty$ has two charts: the finite chart, which is commonly identified with the complex plane, and the infinite chart, which contains ∞. The transition map is $z \mapsto \frac{1}{z}$. Also, we know that every meromorphic function on $\mathbb{C}_\infty$ has the representation in $\mathbb{C}$ as a rational function, namely $r(z)$. Thus $\omega = r(z)\,dz$ in $\mathbb{C}$. The representation of ω in the infinite chart is then obtained by replacing z with $\frac{1}{z}$ and then applying the chain rule. Specifically,

$$\omega = r\left(\frac{1}{z}\right) d\left(\frac{1}{z}\right) = -\frac{1}{z^2} r\left(\frac{1}{z}\right) dz.$$

This means any meromorphic 1-form on $\mathbb{C}_\infty$ is determined by its representation in the finite chart. Thus, we can just write $\omega = r(z)dz$ to indicate a 1-form in $\mathbb{C}_\infty$.

Every rational function $r(z)$ on $\mathbb{C}$ admits a partial fraction decomposition
$$r(z) = Q(z) + \sum_{j=1}^{n} \sum_{k=1}^{e_j} \frac{p_{jk}(z)}{(z-z_j)^k},$$
where $Q(z)$, $p_{jk}(z)$ are polynomials and $\deg p_{jk} < k$. Thus ω is a sum of simple meromorphic 1-forms.
$$\omega = Q(z)dz + \sum_{j=1}^{n} \sum_{k=1}^{e_j} \frac{p_{jk}(z)\,dz}{(z-z_j)^k}.$$
We know that Residues satisfy the linear laws
$$\mathrm{Res}_p(\omega_1+\omega_2) = \mathrm{Res}_p(\omega_1) + \mathrm{Res}_p(\omega_2),$$
$$\mathrm{Res}_p(c\omega_1) = c\,\mathrm{Res}_p(\omega_1) \quad \forall c \in \mathbb{C}.$$
Thus it suffices to prove (*) for ω of the following forms:

- $z^k dz$, where $k \geq 0$,
- $\dfrac{z^k}{(z-a)^j}$, where $a \in \mathbb{C}$, $0 \leq k < j$.

<u>Consider the case $\omega = z^k dz$, $k \geq 0$</u>

Then ω has no pole in the finite chart. In the infinite chart,
$$\omega = \left(\frac{1}{z}\right)^k d\left(\frac{1}{z}\right) = -z^{-k-2}\,dz.$$

Then $\mathrm{Res}_\infty(\omega) = \mathrm{Res}_{z=0}(-z^{-k-2}) = 0$ because $-k-2 \le -2$.

Thus $\mathrm{Res}_p(\omega) = 0$ for all $p \in \mathbb{C}_\infty$. Therefore $(*)$ is satisfied.

Consider the case $\omega = \dfrac{z^k dz}{(z-a)^j}$, $a \in \mathbb{C}$, $0 \le k < j$

In the finite chart, ω has a pole at $z=a$. In the infinite chart,

$$\omega = \frac{\left(\frac{1}{z}\right)^k}{\left(\frac{1}{z}-a\right)^j} d\left(\frac{1}{z}\right) = -\frac{1}{z^{k+2}} \frac{z^j}{(1-az)^j} dz = -\frac{z^{j-k-2}}{(1-az)^j} dz.$$

Then

$$\sum_{p \in \mathbb{C}_\infty} \mathrm{Res}_p(\omega) = \mathrm{Res}_a(\omega) + \mathrm{Res}_\infty(\omega)$$

$$= \underbrace{\mathrm{Res}_{z=a}\left(\frac{z^k}{(z-a)^j}\right)}_{A} + \underbrace{\mathrm{Res}_{z=0}\left(\frac{-z^{j-k-2}}{(1-az)^j}\right)}_{B}$$

Now we consider two cases of a, namely $a=0$ and $a \ne 0$.

- $a=0$

$$A = \mathrm{Res}_{z=0}\left(\frac{z^k}{z^j}\right) = \mathrm{Res}_{z=0}(z^{k-j}) = \begin{cases} 1 & \text{if } k = j-1, \\ 0 & \text{if } k < j-1. \end{cases} \quad (1)$$

$$B = \mathrm{Res}_{z=0}(-z^{j-k-2}) = \begin{cases} -1 & \text{if } k = j-1 \\ 0 & \text{if } k < j-1 \end{cases} \quad (2)$$

By (1) and (2), $A+B$ is always zero.

- $a \ne 0$ we have $z^k = [(z-a)+a]^k = \sum_{l=0}^{k} C_k^l (z-a)^l a^{k-l}$.

Thus, $A = \mathrm{Res}_{z=a}\left[\sum_{l=0}^{k} C_k^l (z-a)^{l-j} a^{k-l}\right]$.

Since $l \le k < j$, we have $l - j \le -1$. The equality occurs only if $l = k = j-1$. Thus,

$$A = \begin{cases} C_k^{j-1} a^{k-(j-1)} & \text{if } k = j-1, \\ 0 & \text{if } k < j-1 \end{cases}$$

$$= \begin{cases} 1 & \text{if } k = j-1, \\ 0 & \text{if } k < j-1 \end{cases} \qquad (3)$$

We have $\dfrac{1}{(1-az)^j} = \sum_{l=0}^{\infty} (az)^l$ for all z in some neighborhood of 0.

Thus, $$B = \operatorname{Res}_{z=0}\left[-z^{j-k-2} \sum_{l=0}^{\infty} (az)^l\right] = \operatorname{Res}_{z=0}\left[\sum_{l=0}^{\infty} -a^l z^{j+l-k-2}\right].$$

Since $j-1 \ge k$ and $l \ge 0$, $j+l-k-2 \ge -1$. The equality occurs only if $l = 0$ and $k = j-1$. Thus,

$$B = \begin{cases} -a^0 & \text{if } k = j-1, \\ 0 & \text{if } k < j-1 \end{cases} = \begin{cases} -1 & \text{if } k = j-1, \\ 0 & \text{if } k < j-1. \end{cases} \qquad (4)$$

By (3) and (4), $A+B$ is always zero.

(5) Let a and b be two nonnegative integers. Consider a polynomial over $\mathbb{C}^2$ given by $f(z,w) = z^{2a} - 2w^b z^a + 1$. Let Y be the locus of roots of $f(z,w)$. To have the connectedness of Y, f is necessarily irreducible. If $a = 0$ then $f(z,w) = -2w^b + 1$, which is reducible because $w = \sqrt[b]{1/2}$ is a root. If $b = 0$ then $f(z,w) = z^{2a} - 2z^a + 1$

$$= (z^a - 1)^2,$$

which is reducible. Therefore, a and b must be positive integers.

Next, we'll show that Y is actually a Riemann surface. To do so, we only need to show that f is irreducible and nonsingular on Y.

<u>Show that $f(z,w)$ is irreducible</u>

We can view $f(z,w)$ as a polynomial $\tilde{f}(w)=(-2z^a)w^b+(z^{2a}+1)$ in $(\mathbb{C}[z])[w]$. Let $\zeta\in\mathbb{C}$ be a root of $z^{2a}+1$. Then $z-\zeta$ is a prime in $\mathbb{C}[z]$ and $(z-\zeta)\mid(z^{2a}+1)$. Since $\zeta\neq 0$, $(z-\zeta)\nmid(-2z^a)$. For the same reason, ζ is not a root of the derivative of $z^{2a}+1$. Thus, $(z-\zeta)^2\nmid(z^{2a}+1)$. By the Eisenstein's Criterion of irreducibility, $\tilde{f}(w)$ is irreducible over the field of fraction of $\mathbb{C}[z]$. On the other hand, $\operatorname{cont}(\tilde{f})=\gcd(-2z^a, z^{2a}+1)=1$. Thus, $\tilde{f}(w)$ is a primitive polynomial. Thus $\tilde{f}(w)$ is also irreducible over $\mathbb{C}[z]$. This means $f(z,w)$ is ~~irrecible~~ irreducible in $\mathbb{C}[z,w]$.

<u>Show that $f(z,w)$ is nonsingular on Y</u>

Suppose by contradiction that f is singular at $(z_0,w_0)\in Y$. Then

$$f(z_0,w_0)=\frac{\partial f}{\partial z}(z_0,w_0)=\frac{\partial f}{\partial w}(z_0,w_0)=0$$

Equivalently,
$$\begin{cases} z_0^{2a}-2w_0^b z_0^a+1=0 & (1)\\ 2a z_0^{2a-1}-2a w_0^b=0 & (2)\\ 2bw_0^{b-1}z_0^a=0 & (3)\end{cases}$$

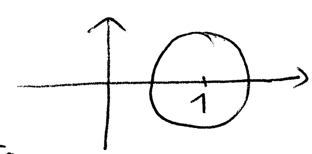
(3) implies $w_0 = 0$ or $z_0 = 0$. Then (2) implies $z_0 = w_0 = 0$. However, $(z_0, w_0) = (0,0)$ does not satisfy (1). This is a contradiction.

We have proved that Y is a Riemann surface. However, Y is not compact because it is an unbounded subset of $\mathbb{C}^2$. We'll compactify Y by adding two points $(z,w) = (0,\infty)$ and $(z,w) = (\infty,\infty)$. This process must be done carefully because we want the resulting space X to be a Riemann surface. Thus we need to resolve the singularities $(0,\infty)$ and (∞,∞) on $Y \cup \{(0,\infty),(\infty,\infty)\}$.

<u>Resolving the singularity $(0,\infty)$</u>

For each $(z,w) \in Y$, $\quad w^b = \frac{1}{2} z^{-a}(1+z^{2a}) \quad (4)$.

For $|z| < \frac{1}{2}$, $1+z^{2a}$ lies on the right half plane.



Thus $1+z^{2a}$ has an analytic b'th root, i.e. there exists an analytic function $g : \mathbb{D}(0,\frac{1}{2}) \to \mathbb{C}$ such that $g(z)^b = 1+z^{2a}$.

Then $(4) \Leftrightarrow w^b = \frac{1}{2} z^{-a} g(z)^b \quad (5)$

We consider 3 following cases;

- <u>$\gcd(a,b) = 1$</u> Then there exist $m,n \in \mathbb{Z}$ such that $-an+bm = 1$.

Define $r(t) := (z,w) = \left(t^b, \frac{1}{\sqrt[b]{2}} t^{-a} g(t^b)\right) \in Y$ for all $t \in \mathbb{D}\left(0, \frac{1}{\sqrt[b]{2}}\right) \setminus \{0\}$.

Then r is holomorphic and has an inverse map;

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$$s(z,w) = t = z^m \left(\frac{w\sqrt[b]{2}}{g(z)}\right)^n.$$

s is also holomorphic and thus gives us a hole chart on Y around the point $(0,\infty)$. By plugging the hole, we have resolved the singularity. Then the coordinate representation of the projection map $\pi : X \to \mathbb{C}_\infty$, $\pi(z,w) = z$ around the new point p is $t \mapsto z = t^b$. Thus $\mathrm{mult}_p \pi = b$.

- $\underline{a = b}$

Then $(5) \Leftrightarrow w^b = \frac{1}{2} z^{-b} g(z)^b \Leftrightarrow \prod_{j=0}^{b-1} \left(w - \zeta^j \frac{1}{\sqrt[b]{2}} z^{-1} g(z)\right) = 0,$

where ζ is a primitive b'th root of unity. Each factor defines a smooth curve which passes through the point $(0,\infty)$. Thus, removing $(0,\infty)$ then gives a space which decomposes into b smooth curves, each with a hole in it. For the j'th curve, we define

$$r(t) = (z,w) = \left(t, \zeta^j \frac{1}{\sqrt[b]{2}} \frac{g(t)}{t}\right) \quad \forall t \in D(0, \tfrac{1}{2}) \setminus \{0\}$$

Then r is holomorphic and has an inverse $s(z,w) = t = z$. Since s is also holomorphic, it gives us a hole chart on the j'th curve. Plugging the hole resolves the singularity in this case. (This means we plugged n holes total, one for each smooth cure). The coordinate representation of the map $\pi : X \to \mathbb{C}_\infty$ around each new point p is $t \mapsto z = t$. Thus $\mathrm{mult}_p \pi = 1$.

- $a \neq b$ and $\gcd(a,b) = k > 1$

Then $a = ka_1$ and $b = kb_1$ with $\gcd(a_1, b_1) = 1$. Then

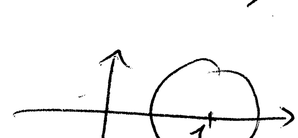
$$(5) \Leftrightarrow \left(w^{b_1}\right)^k = \frac{1}{2}\left(z^{-a_1}\right)^k \left(g(z)^{b_1}\right)^k$$

$$\Leftrightarrow \prod_{j=0}^{k-1} \left(w^{b_1} - \frac{1}{\sqrt[k]{2}} z^{-a_1} g(z)^{b_1}\right) = 0.$$

Then we have k factors, each of which we know ~~have~~ how to resolve (this was our first case). We have also seen that there is one hole to plug for each factor. There are k plugged holes and the multiplicity of π at each new point p is $\mathrm{mult}_p \pi = b_1$. We see that the last statement is true for all 3 cases.

Resolving the singularity (∞, ∞)

We will use almost the same argument as we dealt with the singularity $(0, \infty)$. For each $(z,w) \in Y$, $w^b = \frac{1}{2} z^a (1 + z^{-2a})$ (6)

For $|z| > 2$, $1 + z^{-2a}$ lies on the right half plane 

Thus, $1 + z^{-2a}$ has an analytic b'th root, namely $h(z) : \mathbb{C} \setminus \overline{D}(0,2) \to \mathbb{C}$.

Then $(6) \Leftrightarrow w^b = \frac{1}{2} z^a h(z)^b$.

From now, the procedure is just repeated as in the case we resolved $(0, \infty)$. The only difference is that a is replaced by $-a$, and g is replaced by h. We get the similar conclusion: with $\gcd(a,b)=k$ and $b=b_1k$, there are k plugged holes and the ~~multi~~ multiplicity of π at each new point p is $\operatorname{mult}_p\pi = b_1$.

Now we can compute $\deg(\pi)$. Note that $\pi^{-1}(0)$ has k elements, which are k plugged holes of type $(0,\infty)$. Then

$$\deg(\pi) = d_0(\pi) = \sum_{p\in\pi^{-1}(0)} \operatorname{mult}_p(\pi) = kb_1 = b.$$

So far, we have determined 2 branch points of π, which are not in $\pi(Y)$, namely $z=0$ and $z=\infty$. Next, we will determine all branch points in $\pi(Y)$. For each $(z,w)\in Y$, we have $w^b = \frac{1}{2}(z^a+z^{-a})$. Thus $z\in\mathbb{C}\setminus\{0\}$ is a branch point of π if and only if $\begin{cases} w^b = 0. \\ b\geq 2.\end{cases}$

$$w^b = 0 \Leftrightarrow z^a + z^{-a} = 0 \Leftrightarrow z^{2a} = -1$$

Thus if $b\geq 2$, there~~are~~ are exactly $2a$ branch points in $\mathbb{C}\setminus\{0\}$. They are the $(2a)$'s roots of -1, namely $z_1, z_2, \dots, z_{2a}$. Since the equation $w^b=0$ has a zero at 0 with multiplicity b, we have $\operatorname{mult}_{(z_j,0)}\pi = b$ for all $j=1,2,\dots,2a$. Therefore, we have a conclusion as follows.

Put $b_1 = \dfrac{b}{\gcd(a,b)}$. If $b=1$ then π has no branch point.

If $b\geq 2$:

- If $b_1 = 1$: there are $2a$ branch points $\underbrace{z_1, z_2, \dots, z_{2a}}_{(2a)\text{'th roots of } -1}$.
- If $b_1 \geq 2$: there are $2a+2$ branch points $z_1, z_2, \dots, z_{2a}, 0, \infty$.

Now applying Hurwitz formula for the map π, we have

$$2g(X) - 2 = \deg(\pi)(2g(\mathbb{C}_\infty) - 2) + \sum_{j=1}^{2a} (\underbrace{\text{mult}_{(z_j,0)}\pi}_{b} - 1) +$$

$$+ \sum_{p \in \pi^{-1}(0)} (\underbrace{\text{mult}_p \pi}_{b_1} - 1) + \sum_{p \in \pi^{-1}(\infty)} (\underbrace{\text{mult}_p \pi}_{b_1} - 1)$$

$\Leftrightarrow \quad 2g(X) - 2 = b(-2) + 2a(b-1) + k(b_1 - 1) + k(b_1 - 1)$

$\Leftrightarrow \quad g(X) = ab - a - b + k(b_1 - 1) + 1$

$\Leftrightarrow \quad g(X) = ab - a - k + 1$

$\Leftrightarrow \quad g(X) = ab - a - \gcd(a,b) + 1$.

⑥ Let X be the hyperelliptic surface defined by $y^2 = x^5 - x$. Put $\pi_1, \pi_2 : X \to \mathbb{C}_\infty$, $\pi_1(x,y) = x$ and $\pi_1(x,y) = y$. We want to find $\text{div}(\pi_1)$ and $\text{div}(\pi_2)$. By definition, $\text{div}(\pi_1) = \sum_{p \in X} \text{ord}_p(\pi_1) \cdot p$ and $\text{div}(\pi_2) = \sum_{p \in X} \text{ord}_p(\pi_2) \cdot p$.

Then the problem becomes finding the order of π_1 and π_2 at each point on X.

Let $X_1 = \{(x,y) \in \mathbb{C}^2 : y^2 = x^5 - x\}$,

$X_2 = \{(z,w) \in \mathbb{C}^2 : w^2 = -z^5 + z\}$.

Then there is an isomorphism $\phi: X_1 \setminus \{(0,0)\} \to X_2 \setminus \{(0,0)\}$

$$\phi(x,y) = (z,w) = \left(\frac{1}{x}, \frac{y}{x^3}\right).$$

By the definition of hyperelliptic surfaces, $X = (X_1 \amalg X_2)/\phi$. We'll determine the coordinate of X around $(x,y) = (0,0)$ and around $(x,y) = (\infty, \infty)$. Put

$$f(x,y) = y^2 - x^5 + x, \quad g(z,w) = w^2 + z^5 - z.$$

Let $p_1 \in X$ be the point corresponding to $(x,y) = (0,0)$, and $p_2 \in X$ be the point corresponding to $(x,y) = (\infty, \infty)$. Note that p_2 corresponds to $(z,w) = (0,0)$.

Since $\frac{\partial f}{\partial x}(p_1) = \frac{\partial f}{\partial x}(0,0) = 1 \neq 0$, y is a coordinate of X around p_1.

Since $\frac{\partial g}{\partial z}(p_2) = \frac{\partial g}{\partial z}(0,0) = -1 \neq 0$, w is a coordinate of X around p_2.

Consider π_1

We concern only the zeros and the poles of π_1 because for any other point $p \in X$, $\mathrm{ord}_p(\pi_1) = 0$. We have

$$\pi_1(x,y) = 0 \iff x = 0 \iff \begin{cases} x = 0, \\ y = 0. \end{cases}$$

$$\pi_1(x,y) = \infty \iff x = \infty \iff \begin{cases} x = \infty, \\ y = \infty. \end{cases}$$

Thus p_1 is the only zero of π_1, and p_2 is the only pole of π_1. Since y is a coordinate of X around p_1, the coordinate representation of π_1 is

$\hat{\pi}_1 : y \mapsto x = x(y)$. We have $y^2 = x^5 - x = x(y)^5 - x(y)$.

Differentiate both sides with respect to y, we get

$$2y = 5x^4 \frac{dx}{dy} - \frac{dx}{dy} = (5x^4 - 1)\frac{dx}{dy}$$

Thus $\dfrac{dx}{dy} = \dfrac{2y}{4x^5-1}$ $(*)$

As a consequence, $\dfrac{dx}{dy}(0) = 0$. Differentiate both sides of $(*)$ w.r.t. y,

$$\frac{d^2x}{dy^2} = \frac{2(4x^5-1) - 2y \cdot 20x^4 \frac{dx}{dy}}{(4x^5-1)}$$

Thus, $\dfrac{d^2x}{dy^2}(0) = \dfrac{-2}{1} = -2 \neq 0$.

This means $x(y) = -y^2 + O(y^3)$. Thus, $\pi_1(y) = x(y) = -y^2 +$ higher orders

Therefore, $\mathrm{ord}_{p_1}(\pi_1) = 2$.

Since w is a coordinate of X around p_2, the coordinate representation of π_1 is $\tilde{\pi}_1 : w \mapsto x = \dfrac{1}{z} = \dfrac{1}{z(w)}$. We have

$$w^2 = -z^5 + z = -z(w)^5 + z(w)$$

Taking derivative both sides w.r.t. w, we get

$$2w = -5z^4 \frac{dz}{dw} + \frac{dz}{dw} = (1 - 5z^4)\frac{dz}{dw}$$

Then $\dfrac{dz}{dw} = \dfrac{2w}{1-5z^4}$ $(**)$

Consequently, $\dfrac{dz}{dw}(0) = 0$. Now take the derivative w.r.t. w both

Sides of $(**)$: $$\frac{d^2z}{dw^2} = \frac{2(1-5z^4) - 2w(-20)z^3\frac{dz}{dz}}{(1-5z^4)^2}.$$

Thus, $\frac{d^2z}{dw^2}(0) = \frac{2}{1} = 2 \neq 0.$

this means $z(w) = w^2 + O(w^3)$. Thus
$$\tilde{\pi}_1(w) = \frac{1}{z(w)} = w^{-2} + \text{higher order terms}.$$

Thus $\operatorname{ord}_{p_2}(\pi_1) = -2$. Therefore, $\operatorname{div}(\pi_1) = 2 \cdot p_1 - 2 \cdot p_2$.

<u>Consider π_2</u>

$$\pi_2(x,y) = 0 \Leftrightarrow y = 0 \Leftrightarrow \begin{cases} x^5 - x = 0 \\ y = 0 \end{cases} \Leftrightarrow (x,y) \in \{(0,0), (\zeta,0), (\zeta^2,0), (\zeta^3,0), (1,0)\}$$

where $\zeta = i$. Let $q_j \in X$ be the point on X having coordinate $(x,y) = (\zeta^j, 0)$ for $1 \leq j \leq 4$.

$$\pi_2(x,y) = \infty \Leftrightarrow y = \infty \Leftrightarrow \begin{cases} x = \infty \\ y = \infty \end{cases}$$

thus, p_1, q_1, q_2, q_3, q_4 are all zeros of π_2, and p_2 is the only pole of π_2. Since y is a coordinate of X around p_1, the coordinate representation of π_2 is $\hat{\pi}_2 : y \mapsto y$. Thus $\operatorname{ord}_{p_1}(\pi_2) = 1$.

For each $j = 1,2,3,4$, we have $\frac{\partial f}{\partial x}(q_j) = \frac{\partial f}{\partial x}(\zeta^j, 0) = -5(\zeta^j)^4 + 1 = -4 \neq 0$. Thus y is a cordinate of X around q_j. Then the coordinate repesentation of π_2 is $\hat{\pi}_2 : y \mapsto y$. Thus $\operatorname{ord}_{q_j}(\pi_2) = 1$.

Since w is a coordinate of X around p_2, the coordinate representation of π_2 is $\tilde{\pi}_2 : w \mapsto y = wx^3 = \frac{w}{z^3}$.

We showed earlier in the case of π_1 that $z(w) = w^2 + O(w^3)$. Thus,

$$\tilde{\pi}_2(y) = \frac{w}{z^3} = \frac{w}{w^6 + O(w^7)} = w^{-5} + \text{higher-order terms}$$

Thus, $\mathrm{ord}_{p_2}(\pi_2) = -5$. Therefore,

$$\mathrm{div}(\pi_2) = 1.p_1 + 1.q_1 + 1.q_2 + 1.q_3 + 1.q_4 - 5.p_2.$$

(7) Consider a homogeneous polynomial of 3 complex variables

$$F(x,y,z) = xy^3 + yz^3 + zx^3$$

To show that F determines a smooth projective plane curve which is also a Riemann surface, we need to show that F is irreducible in $\mathbb{C}[x,y,z]$ and nonsingular on its locus of roots. We can write

$$F(x,y,z) = \tilde{F}(x) = zx^3 + y^3x + yz^3 \in (\mathbb{C}[y,z])[x].$$

We know that y is a prime in $\mathbb{C}[y,z]$ since it's irreducible. Also, $y \mid yz^3$, $y^2 \nmid yz^3$, $y \nmid z$. then by the Eisenstein's criterion, $\tilde{F}(x)$ is irreducible over the field of fractions of $\mathbb{C}[y,z]$. On the other hand,

$$\gcd(z, y^3, yz^3) = 1$$

Thus $\tilde{F}(x)$ is a primitive polynomial. Thus it is also irreducible over $\mathbb{C}[y,z]$. Thus, $F(x,y,z)$ is irreducible in $\mathbb{C}[x,y,z]$.

Next, suppose by contradiction that F is singular at a point $(x_0, y_0, z_0) \neq (0,0,0)$ on its locus of roots. Then

$$F(x_0,y_0,z_0)=\frac{\partial F}{\partial x}(x_0,y_0,z_0)=\frac{\partial F}{\partial y}(x_0,y_0,z_0)=\frac{\partial F}{\partial z}(x_0,y_0,z_0)=0.$$

Equivalently,

$$\begin{cases} x_0y_0^3+y_0z_0^3+z_0x_0^3=0 \\ y_0^3+3z_0x_0^2=0 \\ z_0^3+3x_0y_0^2=0 \\ x_0^3+3y_0z_0^2=0 \end{cases} \Longleftrightarrow \begin{cases} x_0y_0^3+y_0z_0^3+z_0x_0^3=0 & (1) \\ y_0^3=-3z_0x_0^2 & (2) \\ z_0^3=-3x_0y_0^2 & (3) \\ x_0^3=-3y_0z_0^2 & (4) \end{cases}$$

Multiplying (2), (3), (4) together, we get $x_0^3y_0^3z_0^3=-27x_0^3y_0^3z_0^3$. Thus one of x_0,y_0,z_0 must be zero. We can assume that $x_0=0$. Then (2) and (3) lead to $y_0=z_0=0$. Thus $(x_0,y_0,z_0)=(0,0,0)$, which is a contradiction.

Therefore, F determines a smooth projective plane curve, which is also a Riemann surface $X=\{[x:y:z]\mid F(x,y,z)=0\}$. Also, as a closed subset of $\mathbb{P}^2$, which is compact, X is compact. Since F is a polynomial of degree 4, by Plücker's formula, we get the genus of X,

$$g(X)=\frac{(4-1)(4-2)}{2}=3.$$

Then by Hurwitz's theorem, $|\mathrm{Aut}(X)|\le 84(g(X)-1)=84\times 2=168$.

Note that $168=3\times 7\times 2^3$. We'll show that $|\mathrm{Aut}(X)|=168$ by showing that $\mathrm{Aut}(X)$ has an element of order 3, an element of order 7 and ~~an element~~ a subgroup of order 8. We have to do some preparation to prove this.

We recall that the projective plane is $\mathbb{P}^2 = \{[x:y:z] \mid (x,y,z) \in \mathbb{C}^3 \setminus \{0\}\}$, where $[x:y:z]$ stands for the orbit of (x,y,z) under the action of $\mathbb{C}^*$ upon $\mathbb{C}^3\setminus\{0\}$. As a matter of notation, we now write an element in $\mathbb{C}^3\setminus\{0\}$ as a column vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and the orbit of its under $\mathbb{C}^*$ as $\left\{ \begin{matrix} x \\ y \\ z \end{matrix} \right\}$.

For each matrix $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in GL(3,\mathbb{C})$, and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{C}^3\setminus\{0\}$, we have
$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1x + a_2y + a_3z \\ b_1x + b_2y + b_3z \\ c_1x + c_2y + c_3z \end{pmatrix} \neq 0.$$
This allows us to define a map $\pi_A : \mathbb{P}^2 \to \mathbb{P}^2$, $\pi_A \left\{ \begin{matrix} x \\ y \\ z \end{matrix} \right\} = \left\{ A\begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\}$.

For $A, B \in GL(3,\mathbb{C})$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{C}^3\setminus\{0\}$, we have
$$\pi_A \circ \pi_B \left\{ \begin{matrix} x \\ y \\ z \end{matrix} \right\} = \pi_A \left\{ B\begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\} = \left\{ A\left(B\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) \right\} = \left\{ (AB)\begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\} = \pi_{AB} \left\{ \begin{matrix} x \\ y \\ z \end{matrix} \right\}.$$
Thus $\pi_A \circ \pi_B = \pi_{AB}$. In particular, π_A is bijective and $(\pi_A)^{-1} = \pi_{A^{-1}}$.

Now suppose that we have a matrix $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in GL(3,\mathbb{C})$ such that $\pi_A(X) \subset X$ and $\pi_{A^{-1}}(X) \subset X$. We will show that $\tilde{\pi}_A := \pi_A|_X$ is in $\mathrm{Aut}(X)$ – the group of all holomorphic automorphisms of X.

Indeed, since $\pi_A(X) \subset X$ and $\pi_{A^{-1}}(X) \subset X$, $\pi_A(X) = X$. Thus $\tilde{\pi}_A$ is a bijection from X to itself. The inverse of $\tilde{\pi}_A$ is therefore $\tilde{\pi}_{A^{-1}}$. Thus, we only need to show that $\tilde{\pi}_A$ is holomorphic.

Recall that $\mathbb{P}^2$ has 3 complex charts:

$$U_1 = \left\{ \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \in \mathbb{P}^2 \mid x \neq 0 \right\},$$

$$U_2 = \left\{ \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \in \mathbb{P}^2 \mid y \neq 0 \right\},$$

$$U_3 = \left\{ \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \in \mathbb{P}^2 \mid z \neq 0 \right\}.$$

Put $X_i = X \cap U_i$ for $i = 1, 2, 3$. Then $\{X_1, X_2, X_3\}$ is an open cover of X.

$$X_1 = \left\{ \begin{Bmatrix} 1 \\ a \\ b \end{Bmatrix} \mid F(1, a, b) = 0 \right\},$$

$$X_2 = \left\{ \begin{Bmatrix} a \\ 1 \\ b \end{Bmatrix} \mid F(a, 1, b) = 0 \right\},$$

$$X_3 = \left\{ \begin{Bmatrix} a \\ b \\ 1 \end{Bmatrix} \mid F(a, b, 1) = 0 \right\}.$$

Thus X_1, X_2, X_3 are smooth affine curves. Now take $p = \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} \in X$.

We'll show that $\tilde{\pi}_A$ is smooth at p. We will prove this for the cases $p \in X_1$, $\tilde{\pi}_A(p) \in X_1$ and $p \in X_1$, $\tilde{\pi}_A(p) \in X_2$ only. Other cases will be done similarly. In both cases, on some neighborhood U of p, either $u = \frac{y}{x}$ or $v = \frac{z}{x}$ is a coordinate.

- $\underline{\tilde{\pi}_A(p) \in X_1}$ This means $\tilde{\pi}_A(p) = \begin{Bmatrix} a_1x_0 + a_2y_0 + a_3z_0 \\ b_1x_0 + b_2y_0 + b_3z_0 \\ c_1x_0 + c_2y_0 + c_3z_0 \end{Bmatrix} \in X_1$.

Then on some neighborhood V of $\tilde{\pi}_A(p)$, either

$$w = \frac{b_1x + b_2y + b_3z}{a_1x + a_2y + a_3z} = \frac{b_1 + b_2u + b_3v}{a_1 + a_2u + a_3v}$$

or

$$t = \frac{c_1x + c_2y + c_3z}{a_1x + a_2y + a_3z} = \frac{c_1 + c_2u + c_3v}{a_1 + a_2u + a_3v}$$

is a coordinate.

Thus the coordinate representation of $\tilde{\pi}_A$ on U is one of the following types: $u \mapsto w$, $u \mapsto t$, $v \mapsto w$, $v \mapsto t$. If u is a coordinate on U then v is a holomorphic function of u. Then w and t are holomorphic functions of u. If v is a coordinate on U, then u is a holomorphic function of v. Then w and t are holomorphic functions of v. Thus, in both cases, $\tilde{\pi}_A$ is always holomorphic at p.

- $\underline{\tilde{\pi}_A(p) \in X_2}$ This means $\tilde{\pi}_A(p) = \begin{Bmatrix} a_1x_0 + a_2y_0 + a_3z_0 \\ b_1x_0 + b_2y_0 + b_3z_0 \\ c_1x_0 + c_2y_0 + c_3z_0 \end{Bmatrix} \in X_2$.

Then on some neighborhood V of $\tilde{\pi}_A(p)$, either

$$w = \frac{a_1x + a_2y + a_3z}{b_1x + b_2y + b_3z} = \frac{a_1 + a_2u + a_3v}{b_1 + b_2u + b_3v}$$

or

$$t = \frac{c_1x + c_2y + c_3z}{b_1x + b_2y + b_3z} = \frac{c_1 + c_2u + c_3v}{b_1 + b_2u + b_3v}$$

is a coordinate. Thus, the coordinate representation of $\tilde{\pi}_A$ on U is one of the following types: $u \mapsto w$, $u \mapsto t$, $v \mapsto w$, $v \mapsto t$. If u is a coordinate on U then v is a holomorphic function of u. Then w and t are holomorphic functions of u. If v is a coordinate on U then u is a holomorphic function of v. Then w and t are holomorphic functions of v. In both cases, $\tilde{\pi}_A$ is always holomorphic at p.

So far, we have proved that if $\pi_A(X) \subset X$ and $\pi_{A^{-1}}(X) \subset X$ then $\tilde{\pi}_A \in \mathrm{Aut}(X)$. This gives us a way to construct an element in $\mathrm{Aut}(X)$. We want to relate the order of $\tilde{\pi}_A$ to ~~to~~ the order of A. Put $\mathcal{E} = \{\lambda I_3 \mid \lambda \in \mathbb{C}^*\} \subset GL(3, \mathbb{C})$. Suppose $n \in \mathbb{N}$ is the smallest number such that $A^n \in \mathcal{E}$. We'll show that $\mathrm{ord}(\tilde{\pi}_A) = n$. First, for each $\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \in X$, we have

$$\tilde{\pi}_{A^n}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \left\{A^n\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right\} = \left\{\lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right\} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}.$$

Thus $\tilde{\pi}_{A^n} = \mathrm{id}_X$. Thus, $\tilde{\pi}_A^{\,n} = \mathrm{id}_X$. Suppose by contradiction that there is $1 \le m < n$ such that $\tilde{\pi}_A^{\,m} = \mathrm{id}_X$. Then $\tilde{\pi}_{A^m} = \mathrm{id}_X$. Then

$$\left\{A^m\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right\} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \quad \forall \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \in X.$$

In other words, for any $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{C}^3 \setminus \{0\}$ that satisfies

$$xy^3+yz^3+zx^3=0, \qquad (*)$$

there exists $\lambda_{x,y,z}\in\mathbb{C}^*$ such that $A^m\begin{pmatrix}x\\y\\z\end{pmatrix}=\lambda_{x,y,z}\begin{pmatrix}x\\y\\z\end{pmatrix}$.

Note that $\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}$ satisfy $(*)$. Put $\lambda_1=\lambda_{1,0,0}$, $\lambda_2=\lambda_{0,1,0}$, $\lambda_3=\lambda_{0,0,1}$.

We have
$$A^m\begin{pmatrix}1\\0\\0\end{pmatrix}=\lambda_1\begin{pmatrix}1\\0\\0\end{pmatrix},\quad A^m\begin{pmatrix}0\\1\\0\end{pmatrix}=\lambda_2\begin{pmatrix}0\\1\\0\end{pmatrix},\quad A^m\begin{pmatrix}0\\0\\1\end{pmatrix}=\lambda_3\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Thus
$$A^m=\begin{pmatrix}\lambda_1 & & \mathbb{O}\\ & \lambda_2 & \\ \mathbb{O} & & \lambda_3\end{pmatrix}.$$

For $x=y=1$, $(*)$ becomes $z^3+z+1=0$, which has a root $z_0\in\mathbb{C}^*$. Thus $\begin{pmatrix}1\\1\\z_0\end{pmatrix}$ satisfies $(*)$. Thus
$$A^m\begin{pmatrix}1\\1\\z_0\end{pmatrix}=\underbrace{\lambda_{1,1,z_0}}_{\lambda'}\begin{pmatrix}1\\1\\z_0\end{pmatrix}.$$

Then $\lambda_1=\lambda'$, $\lambda_2=\lambda'$, $\lambda_3 z_0=\lambda' z_0$. Thus $\lambda_1=\lambda_2=\lambda_3$. Therefore $A^m\in\mathcal{E}$. This contradicts the minimality of n. Thus $n=\operatorname{ord}(\tilde{\pi}_A)$.

So far, we showed that if $A\in GL(3,\mathbb{C})$ and $n\in\mathbb{N}$ are such that $\pi_A(X)\subset X$, $\pi_{A^{-1}}(X)\subset X$ and n is the smallest number such that A^n is ~~diagonal~~ in $\mathcal{E}$, then $\tilde{\pi}_A\in \operatorname{Aut}(X)$ and $\operatorname{ord}(\tilde{\pi}_A)=n$. We will use this method to get some elements of $\operatorname{Aut}(X)$ together with their orders. Put
$$A=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\in GL(3,\mathbb{C}).$$

Then $A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $A^3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = I_3$.

Thus 3 is the smallest n such that A^n is in $\mathcal{E}$. Moreover, $A^{-1} = A^2$.

For $\left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right\} \in X$, we have $\left\{A\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right\} = \left\{\begin{pmatrix} y \\ z \\ x \end{pmatrix}\right\} \in X$ because $yz^3 + zx^3 + xy^3 = 0$.

Thus $\pi_A(X) \subset X$. Moreove, $\pi_{A^{-1}}(X) = \pi_{A^2}(X) = \pi_A(\pi_A(X)) \subset \pi_A(X) \subset X$.

Therefore $\tilde{\pi}_A \in \mathrm{Aut}(X)$ and $\mathrm{ord}(\tilde{\pi}_A) = 3$.

Put $B = \begin{pmatrix} 1 & & \\ & r & \\ & & r^3 \end{pmatrix} \in GL(3, \mathbb{C})$, where r is a primitive 7^{th} root of unity.

Then $B^j = \begin{pmatrix} 1 & & \\ & r^j & \\ & & r^{3j} \end{pmatrix}$ for any $j \in \mathbb{N}$. Then $B^j \in \mathcal{E}$ if and only if $r^j = r^{3j} = 1$, which occurs only if $7 | j$. Thus the smallest $j \in \mathbb{N}$ such that $B^j \in \mathcal{E}$ is 7. Moreover, $B^7 = I_3$ and so $B^{-1} = B^6$.

For $\left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right\} \in X$, we have $\left\{B\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right\} = \left\{\begin{pmatrix} x \\ ry \\ r^3 z \end{pmatrix}\right\} \in X$ because

$$x(ry)^3 + (ry)(r^3 z)^3 + (r^3 z)(x)^3 = r^3(xy^3 + yz^3 + zx^3) = 0.$$

Thus $\pi_B(X) \subset X$. Moreover, $\pi_{B^{-1}}(X) = \pi_{B^6}(X) = \pi_B \circ \pi_B \circ \cdots \circ \pi_B(X) \subset X$.

Therefore, $\tilde{\pi}_B \in \mathrm{Aut}(X)$ and $\mathrm{ord}(\tilde{\pi}_B) = 7$.

Put $C = \frac{i}{\sqrt{7}} \begin{pmatrix} t_1 & t_4 & t_2 \\ t_4 & t_2 & t_1 \\ t_2 & t_1 & t_4 \end{pmatrix}$ where $t_j = r^j - r^{-j}$, and $r = \exp\left(\frac{2\pi i}{7}\right)$.

$$\det(C) = \left(\frac{i}{\sqrt{7}}\right)^3 \left[3 t_1 t_2 t_4 - (t_1^3 + t_2^3 + t_4^3)\right] \qquad (1)$$

We will do some calculations about t_1, t_2, t_4 as follows.

$$t_1^2 = (r - r^{-1})^2 = r^2 + r^{-2} - 2,$$

$$t_2^2 = (r^2 - r^{-2})^2 = r^4 + r^{-4} - 2,$$

$$t_4^2 = (r^4 - r^{-4})^2 = r + r^{-1} - 2.$$

Thus, $t_1^2 + t_2^2 + t_4^2 = (r + r^2 + \cdots + r^6) - 6$. Since $1, r, \ldots, r^6$ are all roots of $X^7 - 1$, their sum is 0 by Viete's theorem. Thus,

$$t_1^2 + t_2^2 + t_4^2 = -7 \qquad (2)$$

We have $t_1 t_2 = (r - r^{-1})(r^2 - r^{-2}) = (r^4 + r^{-4}) - (r + r^{-1}),$

$$t_2 t_4 = (r^2 - r^{-2})(r^4 - r^{-4}) = (r + r^{-1}) - (r^2 + r^{-2}),$$

$$t_4 t_1 = (r^4 - r^{-4})(r - r^{-1}) = (r^2 + r^{-2}) - (r^4 + r^{-4}).$$

Thus, $t_1 t_2 + t_2 t_4 + t_4 t_1 = 0 \qquad (3)$

We have $\operatorname{Im}(t_j) = \operatorname{Im}(r^j) - \operatorname{Im}(r^{-j}) = \sin\left(\frac{2\pi j}{7}\right) - \sin\left(-\frac{2\pi j}{7}\right)$

$$= 2\sin\left(\frac{2\pi j}{7}\right)$$

Thus, $\operatorname{Im}(t_1 + t_2 + t_4) = 2\left(\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}\right) > 0. \qquad (4)$

We have $(t_1 + t_2 + t_4)^2 = (t_1^2 + t_2^2 + t_4^2) + 2(t_1 t_2 + t_2 t_4 + t_4 t_1)$

$$\underset{(2)\text{ and }(3)}{=} -7.$$

Then because of (4), $t_1 + t_2 + t_4 = i\sqrt{7} \qquad (5)$

We have $t_1 t_2 t_4 = (r-r^{-1})(r^2-r^{-2})(r^4-r^{-4})$

$$= r^7 + r - r^5 - r^3 - r^{-1} - r^{-7} + r^{-3} + r^{-5}$$

$$= (r-r^{-1}) + (r^2-r^{-2}) + (r^4-r^{-4})$$

$$= t_1 + t_2 + t_4, \qquad (6)$$

$$t_1^3 = (r-r^{-1})^3 = r^3 - r^{-3} - 3rr^{-1}(r-r^{-1}) = -t_4 - 3t_1,$$

$$t_2^3 = (r^2-r^{-2})^3 = r^6 - r^{-6} - 3r^2r^{-2}(r^2-r^{-2}) = -t_1 - 3t_2,$$

$$t_4^3 = (r^4-r^{-4})^3 = r^{12} - r^{-12} - 3r^4r^{-4}(r^4-r^{-4}) = -t_2 - 3t_4.$$

Thus, $t_1^3 + t_2^3 + t_4^3 = -4(t_1+t_2+t_4) \qquad (7).$

Thus, $\det(C_\alpha) \overset{(6)\text{ and }(7)}{=\!=\!=} \left(\frac{i}{\sqrt{7}}\right)^3 7(t_1+t_2+t_4) = \frac{-i}{7\sqrt{7}} 7(i\sqrt{7}) = 1.$

Thus, $C \in GL(3,\mathbb{C})$. Moreover, by (2) and (3), we have $C^2 = I_3$.

Thus 2 is the smallest number n such that $C^n \in \mathcal{E}$. Moreover, $C^{-1} = C$.

Now we'll only need to show that $\pi_C(X) \subset Y$. We have

$$\left\{C\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right\} = \left\{\begin{matrix} t_1 x + t_4 y + t_2 z \\ t_4 x + t_2 y + t_1 z \\ t_2 x + t_1 y + t_4 z \end{matrix}\right\}$$

Given that $(x,y,z) \in \mathbb{C}^3 \setminus \{0\}$, satisfies

$$xy^3 + yz^3 + zx^3 = 0, \qquad (**)$$

we'll show that $\alpha\beta^3 + \beta\gamma^3 + \gamma\alpha^3 = 0$, where

$$\alpha = t_1 x + t_4 y + t_2 z,$$
$$\beta = t_4 x + t_2 y + t_1 z,$$
$$\gamma = t_2 x + t_1 y + t_4 z$$

We have

$$t_1 t_2^2 = (r - r^{-1})(r^4 + r^{-4} - 2) = r^5 + r^{-3} - 2r - r^3 - r^{-5} + 2r^{-1}$$
$$= -2t_1 + t_2 + t_4,$$

$$t_1^2 t_2 = (r^2 + r^{-2} - 2)(r^2 - r^{-2}) = r^4 + 1 - 2r^2 - 1 - r^{-4} + 2r^{-2}$$
$$= t_4 - 2t_2,$$

$$t_2 t_4^2 = (r^2 - r^{-2})(r + r^{-1} - 2) = r^3 + r - 2r^2 - r^{-1} - r^{-3} + 2r^{-2}$$
$$= t_1 - 2t_2 - t_4,$$

$$t_2^2 t_4 = (r^4 + r^{-4} - 2)(r^4 - r^{-4}) = r + 1 - 2r^4 - 1 - r^{-1} + 2r^{-4}$$
$$= t_1 - 2t_4,$$

$$t_4 t_1^2 = (r^4 - r^{-4})(r^2 + r^{-2} - 2) = r^{-1} + r^2 - 2r^4 - r^{-2} - r + 2r^4$$
$$= -t_1 + t_2 - 2t_4,$$

$$t_4^2 t_1 = (r + r^{-1} - 2)(r - r^{-1}) = r^2 + 1 - 2r - 1 - r^{-2} + 2r^{-1}$$
$$= -2t_1 + t_2.$$

Also, we computed earlier that

$$t_1^3 = -t_4 - 3t_1,$$
$$t_2^3 = -t_1 - 3t_2,$$
$$t_4^3 = -t_2 - 3t_4,$$
$$t_1 t_2 t_4 = t_1 + t_2 + t_4.$$

With these identities, we now can compute $\alpha\beta^3+\beta\gamma^3+\gamma\alpha^3$.

$$\alpha^3 = (t_1^3x^3+t_4^3y^3+t_2^3z^3)+3t_4t_1^2x^2y+3t_4^2t_1xy^2$$
$$+3t_1t_2^2xz^2+3t_1^2t_2x^2z$$
$$+3t_2t_4^2zy^2+3t_2^2t_4z^2y+6t_1t_2t_4xyz$$
$$=(-t_4-3t_1)x^3+(-t_2-3t_4)y^3+(-t_1-3t_2)z^3$$
$$+3(-t_1+t_2-2t_4)x^2y+3(-2t_1+t_2)xy^2$$
$$+3(-2t_1-t_2+t_4)xz^2+3(t_4-2t_2)x^2z$$
$$+3(t_1-2t_2-t_4)zy^2+3(t_1-2t_4)z^2y+6(t_1+t_2+t_4)xyz.$$

Then

$$\gamma\alpha^3 = (-t_2t_4-3t_2t_1)x^4+(-t_2t_1-3t_1t_4)y^4+(-3t_2t_4-t_1t_4)z^4$$
$$+(-t_2^2-3t_2t_4-6t_1^2+3t_2t_1)xy^3+(3t_1t_4-6t_4^2-t_1^2-3t_2t_1)yz^3$$
$$+(3t_2t_4-6t_2^2-t_4^2-3t_1t_4)zx^3+(-6t_2t_4-t_1t_4-3t_2t_1-3t_1^2+3t_2^2)x^3y$$
$$+(-t_2t_4+3t_1^2-3t_4^2-6t_2t_1-3t_1t_4)y^3z+(-3t_2^2-3t_2t_4-6t_1t_4+3t_4^2-t_1t_2)z^3x$$
$$+(-3t_2t_1+3t_2^2-3t_1^2-6t_1t_4)x^2y^2+(-3t_1t_4+3t_1^2-6t_2t_4-3t_4^2)y^2z^2$$
$$+(-6t_1t_2-3t_2^2-3t_2t_4+3t_4^2)z^2x^2+(6t_2^2+9t_2t_4-6t_4^2)yzx^2$$
$$+(9t_1t_2-6t_2^2+6t_1^2)xy^2z+(-6t_1^2+6t_4^2+9t_1t_4)xyz^2.$$

The formulae for $\alpha\beta^3$ and $\beta\gamma^3$ are then obtained by replacing the triple (t_1,t_2,t_4) by (t_2,t_4,t_1) and (t_4,t_1,t_2) respectively in the formula of $\gamma\alpha^3$. Using the fact that $t_1t_2+t_2t_4+t_4t_1=0$,

we get
$$\alpha\beta^3+\beta\gamma^3+\gamma\alpha^3=(-7t_4^2-7t_2^2-7t_3^2)\,xy^3+(-7t_4^2-7t_2^2-7t_3^2)\,yz^3+(-7t_1^2-7t_2^2-7t_3^2)\,zx^3$$
$$\underline{\underline{\text{by (2)}}}\ 49(xy^3+yz^3+zx^3)$$
$$=0.$$

We have proved that $\pi_C(X)\subset X$. Thus, $\tilde{\pi}_C\in \mathrm{Aut}(X)$ and $\mathrm{ord}(\tilde{\pi}_C)=2$.

So far, we have found 3 elements of $\mathrm{Aut}(X)$; $\tilde{\pi}_A$ with order 3, $\tilde{\pi}_B$ with order 7, and $\tilde{\pi}_C$ with order 2.

$$\tilde{\pi}_A\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{Bmatrix}y\\z\\x\end{Bmatrix},$$

$$\tilde{\pi}_B\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{Bmatrix}x\\ry\\r^3z\end{Bmatrix}\quad\text{with } r=\exp\left(\frac{2\pi i}{7}\right),$$

$$\tilde{\pi}_C\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{Bmatrix}t_1x+t_4y+t_2z\\t_4x+t_2y+t_1z\\t_2x+t_1y+t_4z\end{Bmatrix}\quad\text{with } t_j=r^j-r^{-j}.$$

Therefore, $|\mathrm{Aut}(X)|$ is divisible by $3\times 7\times 2=42$. Since $|\mathrm{Aut}(X)|\leq 168$, there are only 4 possibilities, namely $|\mathrm{Aut}(X)|\in\{42, 84, 126, 168\}$.

We will show that only the case $|\mathrm{Aut}(X)|=168$ can happen. Denote
$$G=\mathrm{Aut}(X).$$

In all of these cases, $7 \mid |G|$ but $7^2 \nmid |G|$. Thus all 7-Sylow subgroups of G are of order 7. Moreover, by Sylow's theorem, all of these 7-subgroups are conjugate and the number of them is ~~Conju~~ congruent to 1 (mod 7). Denote $H = \langle \tilde{\pi}_B \rangle$ - the cyclic group generated by $\tilde{\pi}_B$. Let N_H be the normalizer of H in G. Then we also know that the number of subgroups of G that are conjugate to H is $(G:N_H)$. Thus $(G:N_H) \equiv 1 \pmod 7$. Moreover, since $H \subset N_H$, $(G:N_H)$ divides $(G:H) = \frac{|G|}{7}$. Thus we have

$$\begin{cases} (G:N_H) \equiv 1 \pmod 7 \\ (G:N_H) \mid \frac{|G|}{7} \end{cases} \qquad (***)$$

If $|G| = 42$ then

$$(***) \Leftrightarrow \begin{cases} (G:N_H) \equiv 1 \pmod 7 \\ (G:N_H) \mid 6 \end{cases}$$

$$\Leftrightarrow (G:N_H) = 1$$

$$\Leftrightarrow H \text{ is normal in } G.$$

If $|G| = 84$ then

$$(***) \Leftrightarrow \begin{cases} (G:N_H) \equiv 1 \pmod 7 \\ (G:N_H) \mid 12 \end{cases}$$

$$\Leftrightarrow (G:N_H) = 1$$

$$\Leftrightarrow H \text{ is normal in } G$$

If $|G|=126$ then

$$(***) \iff \begin{cases} (G:N_H) \equiv 1 \pmod 7 \\ (G:N_H) \mid 18 \end{cases}$$

$$\iff (G:N_H)=1$$

$$\iff H \text{ is } \cancel{\text{a sub}}\text{normal in } G.$$

Therefore, if $|G| \neq 168$ then $H=\langle \tilde{\pi}_B \rangle$ must be a normal subgroup of G. However, we'll show that H is actually not normal in G.

Suppose that H is normal in G. Then $\tilde{\pi}_C \tilde{\pi}_B \tilde{\pi}_C \in H$, i.e. $\tilde{\pi}_{CBC} \in H$. Then there exists $k \in \mathbb{N}$ such that $\tilde{\pi}_{CBC} = \tilde{\pi}_{B^k}$. Thus CBC and B^k differ by a factor $\lambda \in \mathbb{C}^*$. In particular, CBC must be diagonal because B^k is diagonal. We have

$$CBC = -\frac{1}{7}\begin{pmatrix} t_1 & t_4 & t_2 \\ t_4 & t_2 & t_1 \\ t_2 & t_1 & t_4 \end{pmatrix}\begin{pmatrix} 1 & & \\ & r & \\ & & r^3 \end{pmatrix}\begin{pmatrix} t_1 & t_4 & t_2 \\ t_4 & t_2 & t_1 \\ t_2 & t_1 & t_4 \end{pmatrix}$$

$$= -\frac{1}{7}\begin{pmatrix} t_1 & rt_4 & r^3t_2 \\ t_4 & rt_2 & r^3t_1 \\ t_2 & rt_1 & r^3t_4 \end{pmatrix}\begin{pmatrix} t_1 & t_4 & t_2 \\ t_4 & t_2 & t_1 \\ t_2 & t_1 & t_4 \end{pmatrix}$$

The coefficient at row 2, column 1, is supposed to be zero. However,

$$t_1t_4 + rt_2t_4 + r^3t_1t_2 = (r-r^{-1})(r^4-r^{-4}) + r(r^2-r^{-2})(r^4-r^{-4}) + r^3(r-r^{-1})(r^2-r^{-2})$$

$$= (r^2+r^{-2}-r^4-r^{-4}) + r(r+r^{-1}-r^2-r^{-2}) + r^3(r^3+r^{-3}-r-r^{-1})$$

$$= r^2+r^{-2}+2-2r^4-2r^{-4}$$

$$= r^{-4}(r^6+2r^4+r^2-2r-2) \qquad (8)$$

Since $\Phi_7(X) = X^6+X^5+X^4+X^3+X^2+X+1$ is the irreducible polynomial of r over $\mathbb{Q}$, r is not a root of $X^6+2X^4+X^2-2X-2$. Thus (8) implies $t_1t_4 + rt_2t_4 + r^3t_1t_2 \neq 0$. This is a contradiction.

The above arguments also show that every subgroup of G containing $\tilde{\pi}_A, \tilde{\pi}_B, \tilde{\pi}_C$ must be of order 168, and thus must be G. Therefore, $G = \langle \tilde{\pi}_A, \tilde{\pi}_B, \tilde{\pi}_C \rangle$, which implies

$$G \simeq \underbrace{\langle A, B, C \rangle}_{\text{Subgroup of } GL(3,\mathbb{C})} / \mathcal{E}$$

(recall that $\mathcal{E} = \{\lambda I_3 : \lambda \in \mathbb{C}^*\}$)