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Math 8501: Differential Equations & Dynamical Systems

Homework #0

(1) Put $Y = C^0([0,1])$ and $X = C^1([0,1]) \cap \{u \mid u(0) = 0\}$.

(a) We show that Y is a Banach space with norm $\|u\|_Y = \sup_{t\in[0,1]} |u(t)|$.

We see that Y is a vector space over $\mathbb{R}$. The map $\|.\|_Y : Y \to \mathbb{R}$ is well-defined because every continuous function on $[0,1]$ is bounded (in fact attains minimum and maximum values). This is a norm on Y because

$$\|u\|_Y = 0 \iff u(t) = 0 \quad \forall t \in [0,1],$$

$$\|\lambda u\|_Y = \sup_{t\in[0,1]} |\lambda u(t)| = |\lambda| \sup_{t\in[0,1]} |u(t)| = |\lambda| \|u\|_Y \quad \forall \lambda \in \mathbb{R}\ \forall u \in Y,$$

$$\|u+v\|_Y = \sup_{t\in[0,1]} |u(t)+v(t)| \le \sup_{t\in[0,1]} |u(t)| + \sup_{s\in[0,1]} |v(s)| = \|u\|_Y + \|v\|_Y \quad \forall u,v \in Y.$$

To show that Y is a Banach space, we take an arbitrary Cauchy sequence (u_n) in Y and show that it converges in Y. For each $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that $\|u_m - u_n\|_Y < \varepsilon$ for all $m,n > N(\varepsilon)$. Thus, for each $t \in [0,1]$,

$$|u_n(t) - u_m(t)| < \varepsilon \quad \forall m,n > N(\varepsilon).$$

This means $(u_n(t))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. It converges because $\mathbb{R}$ is a complete metric space. Define

$$u(t) := \lim_{n\to\infty} u_n(t) \quad \forall t \in [0,1].$$

We show that $u \in Y$ and $u_n \to u$ in Y. For each $t \in [0,1]$,

$$|u_n(t) - u(t)| \leq |u_n(t) - u_m(t)| + |u_m(t) - u(t)|$$

$$< \varepsilon + |u_m(t) - u(t)| \quad \forall m,n > N(\varepsilon).$$

Let $m \to \infty$. We get $|u_n(t) - u(t)| \leq \varepsilon$ for all $n > N(\varepsilon)$. Thus,

$$\sup_{t\in[0,1]} |u_n(t) - u(t)| \leq \varepsilon \quad \forall n > N(\varepsilon). \qquad (1)$$

Take $t_0 \in [0,1]$ arbitrarily.

$$|u(t) - u(t_0)| \leq |u_n(t) - u(t)| + |u_n(t) - u_n(t_0)| + |u_n(t_0) - u(t_0)|$$

$$\overset{(1)}{\leq} 2\varepsilon + |u_n(t) - u_n(t_0)| \quad \forall n > N(\varepsilon).$$

Put $n_0 = N(\varepsilon) + 1$. Then $|u(t) - u(t_0)| \leq 2\varepsilon + |u_{n_0}(t) - u_{n_0}(t_0)|$. Because u_{n_0} is continuous in $[0,1]$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|u_{n_0}(t) - u_{n_0}(t_0)| < \varepsilon \quad \forall t \in [0,1],\ |t - t_0| < \delta.$$

Thus, $|u(t) - u(t_0)| < 3\varepsilon \quad \forall t \in [0,1],\ |t - t_0| < \delta$.

This means u is continuous at t_0. Since t_0 was taken arbitrarily in $[0,1]$, $u \in Y$. Then (1) becomes $\|u_n - u\| \leq \varepsilon$ for all $n > N(\varepsilon)$. Therefore $u_n \to u$ in Y.

(b) We show that X is a Banach space with norm $\|u\|_X = \sup_{t\in[0,1]} (|u'(t)| + |u(t)|)$.

X is a vector space over $\mathbb{R}$. The map $\|.\|_X : X \to \mathbb{R}$ is well-defined because every continuous function in $[0,1]$ is bounded. This is a norm on X because

$$\|u\|_X = 0 \Leftrightarrow u(t) = 0 \quad \forall t \in [0,1],$$

$$\|\lambda u\|_X = \sup_{t\in[0,1]} \left(|\lambda u'(t)| + |\lambda u(t)|\right) = |\lambda| \sup_{t\in[0,1]} \left(|u'(t)|+|u(t)|\right) = |\lambda|\, \|u\|_X,$$

$$\|u+v\|_X = \sup_{t\in[0,1]} \left(|u'(t)+v'(t)| + |u(t)+v(t)|\right) \le \sup_{t\in[0,1]} \left(|u'(t)|+|v'(t)|+|u(t)|+|v(t)|\right)$$

$$\le \sup_{t\in[0,1]}\left(|u'(t)|+|u(t)|\right) + \sup_{s\in[0,1]}\left(|v'(s)|+|v(s)|\right)$$

$$= \|u\|_X + \|v\|_X.$$

To show that X is a Banach space, we take an arbitrary Cauchy sequence (u_n) in X and show that it converges in X. For each $\varepsilon>0$, there exists $N(\varepsilon)\in\mathbb{N}$ such that $\|u_m-u_n\|_X<\varepsilon$ for all $m,n>N(\varepsilon)$. Thus,

$$\|u_m'-u_n'\|_Y = \sup_{t\in[0,1]} |u_m'(t)-u_n'(t)| \le \|u_m-u_n\|_X < \varepsilon \quad \forall m,n>N(\varepsilon),$$

$$\|u_m-u_n\|_Y = \sup_{t\in[0,1]} |u_m(t)-u_n(t)| \le \|u_m-u_n\|_X < \varepsilon \quad \forall m,n>N(\varepsilon).$$

Therefore, (u_n') and (u_n) are Cauchy sequences in Y. According to Part (a), Y is a Banach space. Thus, these sequences converge in Y. Denote

$$u = \lim u_n \in Y, \qquad v = \lim u_n' \in Y \qquad (2),$$

We show that $v=u'$. We have

$$u_n(t) = u_n(0) + \int_0^t u_n'(s)\,ds = \int_0^t u_n'(s)\,ds = \int_0^t v(s)\,ds + \int_0^t (u_n'(s)-v(s))\,ds.$$

Thus, $$\left|u_n(t) - \int_0^t v(s)\,ds\right| \le \int_0^t |u_n'(s)-v(s)|\,ds \le \int_0^t \|u_n'-v\|_Y\,ds \le \|u_n'-v\|_Y.$$

Let $n\to\infty$: $$\left|u_n(t) - \int_0^t v(s)\,ds\right| \to 0 \qquad \forall t\in[0,1].$$

Thus, $\lim_{n\to\infty} u_n(t) = \int_0^t v(s)\,ds$. This means $u(t) = \int_0^t v(s)\,ds$. Thus $u'=v$ and

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$u \in X$. By (2), $\|u_n - u\|_Y \to 0$ and $\|u_n' - u'\|_Y \to 0$. Hence,

$$\|u_n - u\|_X \leq \|u_n - u\|_Y + \|u_n' - u'\|_Y \to 0 \quad \text{as } n \to \infty.$$

Therefore, $u_n \to u$ in X.

(c) Define a map $L : X \to Y$, $L(u) = u'$. This is a linear map. Moreover,

$$\|L(u)\|_Y = \sup_{t \in [0,1]} |u'(t)| \leq \sup_{t \in [0,1]} (|u'(s)| + |u(s)|) = \|u\|_X \quad \forall u \in X.$$

Thus, L is bounded.

(d) We show that L is invertible. Define a map $\tilde{L} : Y \to X$, $\tilde{L}(v) = \int_0^t v(s)\, ds$.

Note that $\tilde{L}$ is well-defined. It is linear, and bounded because

$$\|\tilde{L}(v)\|_X = \sup_{t \in [0,1]} \left(|(\tilde{L}v)'(t)| + |(\tilde{L}v)(t)| \right) = \sup_{t \in [0,1]} \left(|v(t)| + \left| \int_0^t v(s)\, ds \right| \right)$$

$$\leq \sup_{t \in [0,1]} \left(|v(t)| + \int_0^t \|v\|_Y ds \right)$$

$$\leq \sup_{t \in [0,1]} \left(|v(t)| + \|v\|_Y \right) = 2\|v\|_Y \quad \forall v \in Y.$$

We have $(\tilde{L} \circ L(u))(t) = (\tilde{L}(u'))(t) = \int_0^t u'(s)\, ds = u(t) - u(0) = u(t)$ $\forall t \in [0,1]$ $\forall u \in X$.

Thus $\tilde{L} \circ L = id_X$. Also,

$$(L \circ \tilde{L}(v))(t) = L\left(t \mapsto \int_0^t v(s)\, ds\right) = \left(\int_0^t v(s)\, ds\right)' = v(t) \quad \forall t \in [0,1] \ \forall v \in Y.$$

Thus, $L \circ \tilde{L} = id_Y$. Therefore, L is invertible and its inverse is $\tilde{L}$.

② We'll find the solution to the problem $\begin{cases} x' = Ax \in \mathbb{R}^2, \\ x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{cases}$

where $A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$ with $\mu \in \mathbb{R}$.

Write $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ for $t \in \mathbb{R}$. The equation $x' = Ax$ becomes

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + \mu x_2 \\ x_2 \end{pmatrix}.$$

Thus, $\begin{cases} x_1' = x_1 + \mu x_2, & (1) \\ x_2' = x_2. & (2) \end{cases}$

From (2), we get $x_2(t) = Ce^t$ for some constant $C \in \mathbb{R}$. The initial condition yields $x_2(0) = 1$. Thus, $C = 1$ and $x_2(t) = e^t$. Then (1) becomes

$$x_1' - x_1 = \mu e^t.$$

Multiplying both sides by e^{-t}, we get $(e^{-t}x_1)' = \mu$. Thus, $e^{-t}x_1 = \mu t + C_1$. Equivalently, $x_1(t) = e^t(\mu t + C_1)$. The inital condition yields $x_1(0) = 0$. Thus, $C_1 = 0$ and $x_1(t) = \mu t e^t$. Therefore,

$$x(t) = \begin{pmatrix} \mu t e^t \\ e^t \end{pmatrix} \quad \forall t \in \mathbb{R}.$$