Name: Tuan Pham

ID: [redacted]

Math 8501: Differential Equations & Dynamical Systems

Homework #1

① For each $\eta>0$, we denote $Y_\eta = \{u: \mathbb{R}\to\mathbb{R}^n \text{ continuous and } \sup_{t\in\mathbb{R}} |u(t)e^{-\eta|t|}|<\infty\}$.

For each $u\in Y_\eta$, put $\|u\|_{Y_\eta} = \sup_{t\in\mathbb{R}} |u(t)e^{-\eta|t|}|<\infty$.

(i) We show that Y_η is a Banach space.

First, we show that Y_η is a vector space over $\mathbb{R}$. For $u, v\in Y_\eta$,

$$\sup_{t\in\mathbb{R}} |(u(t)+v(t))e^{-\eta|t|}| = \sup_{t\in\mathbb{R}} |u(t)e^{-\eta|t|}+v(t)e^{-\eta|t|}|$$
$$\leq \underbrace{\sup_{t\in\mathbb{R}} |u(t)e^{-\eta|t|}|}_{<\infty} + \underbrace{\sup_{s\in\mathbb{R}} |v(s)e^{-\eta|s|}|}_{<\infty}$$

Thus, $u+v\in Y_\eta$ and $\|u+v\|_{Y_\eta}\leq \|u\|_{Y_\eta}+\|v\|_{Y_\eta}$. (1)

For $c\in\mathbb{R}$ and $u\in Y_\eta$, $\sup_{t\in\mathbb{R}} |cu(t)e^{-\eta|t|}| = |c|\sup_{t\in\mathbb{R}}|u(t)e^{-\eta|t|}|<\infty$.

Thus, $cu\in Y_\eta$ and $\|cu\|_{Y_\eta}=|c|\,\|u\|_{Y_\eta}$. (2)

Therefore, Y_η is a vector space. Moreover, $\|u\|_{Y_\eta}\geq 0$ for all $u\in Y_\eta$ and $\|u\|_{Y_\eta}=0$ if and only if $u\equiv 0$ in $\mathbb{R}$. Together with (1) and (2), we conclude that Y_η is a normed vector space.

Next, we show that Y_η is complete. Let (u_k) be a Cauchy sequence in Y_η. For each $\varepsilon>0$, there exists $N(\varepsilon)\in\mathbb{N}$ such that $\|u_m-u_k\|_{Y_\eta}<\varepsilon$ for all $m,k>N(\varepsilon)$. Thus,

$$|u_m(t) - u_k(t)|\, e^{-\eta|t|} < \varepsilon \qquad \forall t \in \mathbb{R},\ \forall m,k > N(\varepsilon).$$

Thus, $|u_m(t) - u_k(t)| < \varepsilon\, e^{\eta|t|}$ $\forall t \in \mathbb{R}$, $\forall m,k > N(\varepsilon)$. (3)

This implies that the sequence $(u_m(t))$ is a Cauchy sequence in $\mathbb{R}$ for each $t \in \mathbb{R}$. Denote $u(t) = \lim_{m\to\infty} u_m(t)$. We show that $u \in Y_\eta$ and $u_k \to u$ in Y_η.

For $a > 0$, we get from (3) that $\sup_{t\in[-a,a]} |u_m(t) - u_k(t)| < \varepsilon e^{\eta a}$ for $m,k > N(\varepsilon)$. Thus, $(u_m|_{[-a,a]})$ is a Cauchy sequence in $(C([-a,a],\mathbb{R}^n), \|\cdot\|_\infty)$. This is a Banach space according to a previous homework problem. Thus, $(u_m|_{[-a,a]})$ converges in $C([-a,a],\mathbb{R}^n)$. Because u is the pointwise limit of (u_m), we have

$$u_m|_{[-a,a]} \to u|_{[-a,a]} \quad \text{in } C([-a,a],\mathbb{R}^n).$$

Thus, u is continuous on $[-a,a]$. Since $a > 0$ is arbitrary, u is continuous on $\mathbb{R}$. For $t \in \mathbb{R}$, $k \in \mathbb{N}$,

$$|u_k(t) - u(t)|\, e^{-\eta|t|} \le \underbrace{|u_k(t) - u_m(t)|\, e^{-\eta|t|}}_{\le \|u_k - u_m\|_{Y_\eta}} + |u_m(t) - u(t)|\, e^{-\eta|t|}$$

$$< \varepsilon + |u_m(t) - u(t)|\, e^{-\eta|t|} \qquad \forall m,k > N(\varepsilon)$$

Let $m \to \infty$. We get $|u_k(t) - u(t)|\, e^{-\eta|t|} \le \varepsilon$ $\forall t \in \mathbb{R}$ $\forall k \in \mathbb{N}, k > N(\varepsilon)$. (4)

Consequently, $|u(t)|\, e^{-\eta|t|} \le |u_k(t)|\, e^{-\eta|t|} + |u_k(t) - u(t)|\, e^{-\eta|t|} \le \|u_k\|_{Y_\eta} + \varepsilon$ $\forall k > N(\varepsilon)$.

Since (u_k) is a Cauchy sequence in Y_η, it is bounded. There is a number $M > 0$ such that $\|u_k\|_{Y_\eta} < M$ for all $k \in \mathbb{N}$. Thus, $|u(t)|\, e^{-\eta|t|} < M + \varepsilon$ for all $t \in \mathbb{R}$.

Hence, $u \in Y_\eta$. Then (4) implies that $\|u_k - u\|_{Y_\eta} \le \varepsilon$ for all $k > N(\varepsilon)$. Therefore, $u_k \to u$ in Y_η.

(ii) Let $u_0 \in \mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz function with Lipschitz constant $L > 0$. Put

$$(Tu)(t) = u_0 + \int_0^t f(u(s))\,ds \qquad \forall u \in Y_\eta, \ \forall t \in \mathbb{R}.$$

We show that T is a map from Y_η to Y_η. First, $Tu : \mathbb{R} \to \mathbb{R}^n$ is well-defined because the integrand is a continuous function in $\mathbb{R}$. Also for the same reason, Tu is continuous. Put $a = f(0) \in \mathbb{R}^n$.

$$(Tu)(t) = u_0 + ta + \int_0^t (f(u(s)) - f(0))\,ds.$$

Thus,
$$\begin{aligned}
|(Tu)(t)| &\le |u_0| + |t||a| + \left| \int_0^t |f(u(s)) - f(0)|\,ds \right| \\
&\le |u_0| + |t||a| + L \underbrace{\left| \int_0^t |u(s)|\,ds \right|}_{= \int_0^t |u(s)| e^{-\eta|s|} e^{\eta|s|} ds \ \le \left(\sup_{s \in \mathbb{R}} |u(s)| e^{-\eta|s|}\right) \left| \int_0^t e^{\eta|s|} ds \right|} \\
&\le |u_0| + |t||a| + L\|u\|_{Y_\eta} \int_0^{|t|} e^{\eta s}\,ds \\
&= |u_0| + |t||a| + L\eta^{-1} \|u\|_{Y_\eta} (e^{\eta|t|} - 1).
\end{aligned}$$

Hence,
$$|Tu(t)| e^{-\eta|t|} \le \underbrace{(|u_0| + |t||a|) e^{-\eta|t|}}_{\text{bounded in } \mathbb{R}} + \underbrace{L\eta^{-1} \|u\|_{Y_\eta} (1 - e^{-\eta|t|})}_{\le L\eta^{-1}\|u\|_{Y_\eta}}.$$

Therefore, $\sup_{t \in \mathbb{R}} |(Tu)(t)| e^{-\eta|t|} < \infty$ and $Tu \in Y_\eta$.

(iii) We show that $T : Y_\eta \to Y_\eta$ is a contraction map if η is large enough.

For $u, v \in Y_\eta$, we have

$$\begin{aligned}
|(Tu)(t) - (Tv)(t)| = \left| \int_0^t (f(u(s)) - f(v(s)))\,ds \right| &\le \left| \int_0^t |f(u(s)) - f(v(s))|\,ds \right| \\
&\le L \left| \int_0^t |u(s) - v(s)|\,ds \right|
\end{aligned}$$

$$= L \left| \int_0^t \underbrace{|u(s)-v(s)|\, e^{-\eta|s|}}_{\leq \|u-v\|_{Y_\eta}} e^{\eta|s|} ds \right| .$$

Thus,
$$|(Tu)(t)-(Tv)(t)| \leq L\|u-v\|_{Y_\eta} \left| \int_0^t e^{\eta|s|} ds \right|$$
$$= L\|u-v\|_{Y_\eta} \frac{1}{\eta}(e^{\eta|t|}-1)$$
$$\leq L\eta^{-1} \|u-v\|_{Y_\eta} e^{\eta|t|} .$$

This is true for all $t \in \mathbb{R}$. Thus, $\|Tu-Tv\|_{Y_\eta} \leq L\eta^{-1}\|u-v\|_{Y_\eta}$.

If $\eta > L$ then $L\eta^{-1} < 1$ and hence T is a contraction map.

(2) Consider the differential system
$$\begin{cases} \dot{x} = y^2, \\ \dot{y} = -yx^3 \end{cases} \qquad (1)$$
with the initial condition (x_0, y_0). We can write (1) as $\dot{Y} = F(Y)$ where $Y = \binom{x}{y}$ and $F\binom{x}{y} = \binom{y^2}{-yx^3}$. Because F is locally Lipschitz, by Picard-Linderlof's theorem, the problem $\dot{Y} = F(Y)$, $Y(0) = Y_0$ has a unique local solution. In fact, the solution is unique on the maximal interval of existence.

(i) We show that a global solution to (1) always converges as $t \to \infty$. Note that if a global solution exists, it is unique.

Because $\dot{x} = y^2 \geq 0$, $x = x(t)$ is an increasing function. Multiplying the first equation of (1) by x^3, the second by y and adding the new equations together, we get $\dot{x}x^3 + \dot{y}y = 0$. Thus,
$$\frac{d}{dt}\left(\frac{x^4}{4} + \frac{y^2}{2} \right) = 0.$$

Thus, $\frac{x^4}{4}+\frac{y^2}{2} = \text{const}$. Applying the initial condition, we get

$$\frac{x^4}{4}+\frac{y^2}{2} = \frac{x_0^4}{4}+\frac{y_0^2}{2} \qquad (*)$$

Thus, $x=x(t)$ is a bounded function. Since $x(t)$ is increasing and bounded, the limit as $t\to\infty$ exists. By $(*)$, $\lim_{t\to\infty}(y^2)$ also exists.

Put $\beta = \lim_{t\to\infty}(y^2) \geq 0$. Suppose that $\beta>0$. Then there exists $t_0>0$ such that $y^2(t) > \beta/2$ for all $t>t_0$. For $t>t_0$,

$$\frac{x(t)-x(t_0)}{t-t_0} = \dot{x}(\xi) = y^2(\xi) > \frac{\beta}{2} \qquad (\text{where } \xi\in(t_0,t)).$$

Thus, $\quad x(t) > \frac{\beta}{2}(t-t_0)+x(t_0) \quad \forall t>t_0.$

This contradicts the fact that $x=x(t)$ is bounded. Therefore, $\beta=0$. We get $\lim_{t\to\infty} y(t) = 0$. Then $(*)$ implies $\lim_{t\to\infty} x(t)^4 = x_0^4+2y_0^2$. Thus,

$$\lim_{t\to\infty} x(t) = \pm\sqrt[4]{x_0^4+2y_0^2}.$$

Since $x(t)$ is increasing, $\lim_{t\to\infty} x(t) = \sup_{t\geq 0} x(t) \geq x(0) = x_0$. The case $\lim_{t\to\infty} x(t) = -\sqrt[4]{x_0^4+2y_0^2}$ happens only if $-\sqrt[4]{x_0^4+2y_0^2} \geq x_0$, which implies $y_0=0$ and $x_0\leq 0$. In such a case, $\lim_{t\to\infty} x(t) = x_0$ and thus $x(t)=x_0$ for all $t\in\mathbb{R}$. The equation $\dot{x}=y^2$ then forces $y(t)=0$ for all $t\in\mathbb{R}$. Therefore,

$$\lim_{t\to\infty} x(t) = \begin{cases} -\sqrt[4]{x_0^4+2y_0^2} & \text{if } x_0\leq 0,\ y_0=0, \\ \sqrt[4]{x_0^4+2y_0^2} & \text{otherwise,} \end{cases}$$

$$\lim_{t\to\infty} y(t) = 0.$$

(ii) We find an Euler multiplier so that the system becomes Hamiltonian.

The system (1) is of the form
$$\begin{cases} \dot{x} = g(x,y) \\ \dot{y} = -f(x,y) \end{cases} \qquad (2)$$

We want to bring it to the exact differential form $dI(x,y)=0$. To do so, we want to find a pair of functions (I,λ) satisfying
$$\begin{cases} \dfrac{\partial I}{\partial y} = \lambda(x,y)\, g(x,y) \\ \dfrac{\partial I}{\partial x} = \lambda(x,y)\, f(x,y) \end{cases} \qquad (3)$$

in which λ is called an Euler multiplier and I the Hamiltonian. In the following, we find (I,λ) by a method introduced in Neudert-Wahl "On the global existence of Euler's multiplier", 2000 (Theorem 2.7, page 7).

Formally, from (2) we get
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}} = -\frac{f(x,y)}{g(x,y)} = -\frac{f}{g}(x,y).$$

Consider the ODE
$$\begin{cases} u'(x) = -\dfrac{f}{g}(x, u(x)) \\ u(x_0) = y_0 \end{cases} \qquad (4)$$

Suppose that it has a unique global solution $u: \mathbb{R} \to \mathbb{R}$ for any choice of initial data (x_0, y_0) which doesn't vanish g. Denote $u = u(x; x_0, y_0)$. Define
$$I(x_0,y_0) = u(0; x_0, y_0) \qquad \forall (x_0,y_0) \qquad (5)$$

We show that $\nabla I(x_0,y_0)$ is parallel to vector $(f(x_0,y_0), g(x_0,y_0))$. It

Suffices to show that $\nabla I(x_0, y_0)$ is perpendicular to $(g(x_0,y_0), -f(x_0,y_0))$.

Another way to say that (4) has a unique global solution is that $u(0; x, u(x; x_0, y_0)) = u(0; x_0, y_0)$ for all $x \in \mathbb{R}$. Thus,

$$\frac{d}{dx}\bigg|_{x=x_0} \left[u(0; x, u(x; x_0, y_0))\right] = 0 \qquad (6)$$

By the chain rule,

$$\begin{aligned} LHS(6) &= \frac{\partial I}{\partial x}(x_0, u(x_0; x_0, y_0)) + \frac{\partial I}{\partial y}(x_0, u(x_0; x_0, y_0)) \frac{d}{dx}\bigg|_{x=x_0}[u(x; x_0, y_0)] \\ &= \frac{\partial I}{\partial x}(x_0, y_0) + \frac{\partial I}{\partial y}(x_0, y_0)\left(-\frac{f}{g}(x_0, y_0)\right) \\ &= \frac{1}{g(x_0, y_0)} \nabla I(x_0, y_0) \cdot (g(x_0, y_0), -f(x_0, y_0)). \end{aligned}$$

Thus, $\nabla I(x_0, y_0)$ is perpendicular to $(g(x_0, y_0), -f(x_0, y_0))$. The function $\lambda(x,y)$ in (3) is then simply the ratio between two parallel vectors. The key step of this method is solving (4).

Return to our problem. Comparing (1) and (2), we get $g(x,y) = y^2$, $f(x,y) = yx^3$. Then (4) becomes

$$\begin{cases} u'(x) = -\dfrac{x^3}{u(x)} \\ u(x_0) = y_0 \end{cases}$$

The first equation implies $u'(x)u(x) = -x^3$. Thus, $\frac{d}{dx}\left(\frac{u^2}{2}\right) = -x^3$. Integrating both sides with respect to x, we get $\frac{u^2}{2} = C - \frac{x^4}{4}$. Thus, $u(x) = \pm\sqrt{C - \frac{x^4}{2}}$.

The sign is determined by y_0. Let us simply take $u(x) = \sqrt{C - \frac{x^4}{2}}$.

Applying the initial condition, we get $u(x)=\sqrt{\frac{x_0^4}{2}+y_0^2-\frac{x^4}{2}}$ $\left(=u(x; x_0, y_0)\right)$.

Thus, $I(x_0, y_0) = u(0; x_0, y_0) = \sqrt{\frac{x_0^4}{2}+y_0^2}$.

Then $$\lambda(x_0, y_0) = \frac{\frac{\partial I}{\partial y}(x_0, y_0)}{g(x_0, y_0)} = \frac{2y_0}{2y_0^2\sqrt{\frac{x_0^4}{2}+y_0^2}} = \frac{1}{y_0\sqrt{\frac{x_0^4}{2}+y_0^2}}.$$

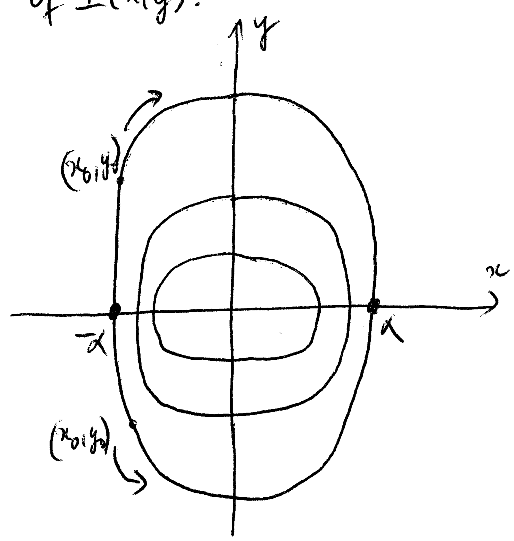
In conclusion, we have found an Euler multiplier

$$\lambda(x, y) = \frac{1}{y\sqrt{\frac{x^4}{2}+y^2}}$$

and a Hamiltonian $I(x, y) = \sqrt{\frac{x^4}{2}+y^2}$.

(iii) We analyze the level sets of $I(x,y)$ found above.

Using the command <u>contourplot</u> in Maple, we can draw some level sets of $I(x,y)$.



$\alpha = \sqrt[4]{x_0^4+2y_0^2}$

The solution $(x(t), y(t))$ always runs on the level set passing through (x_0, y_0). If the solution starts from $(-\alpha, 0)$, it will stay there forever. If the solution starts from anywhere else on the level set, it will run along the curve on the direction that keeps x increasing. It will come arbitrarily close to the point $(\alpha, 0)$ because $\lim_{t\to\infty}(x(t), y(t)) = (\alpha, 0)$.