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Math 8501: Differential Equations & Dynamical Systems.

Homework #2

(1) Consider the initial-value problem

$$\dot{u} = u(1-u), \quad u(0) = u_0. \qquad (1)$$

(i) First, we find the explicit solution $u(t;u_0)$ with $u_0 \in (0,1]$ for Problem (1). The function $f(x) = x(1-x)$ is in $C^1(\mathbb{R})$. Thus, it is locally Lipschitz. By Picard-Linderlöf theorem, Problem (1) has a unique local solution. Consequently, it has unique solution on the maximal interval of existence. If $u_0 = 1$, we can check that the constant function $u(t) \equiv 1$ solves Problem (1). It is the unique solution. Now consider the case $u_0 \in (0,1)$.

The solution u is continuous at $t=0$, so $u(t)$ remains in $(0,1)$ if t is close to 0. For this reason, we can assume $u(t) \in (0,1)$ while manipulating the differential equation. We have

$$\frac{\dot{u}}{u(1-u)} = 1.$$

Taking the antiderivative both sides with respect to t, we get

$$\int \frac{du}{u(1-u)} = \int dt.$$

Thus, $\log \frac{u}{1-u} = t + C_1$. Taking the exponential of both sides, we get

$$\frac{u}{1-u} = C_1 e^t, \text{ for some constant } C_1 > 0.$$

Thus $u(t) = \dfrac{C_1 e^t}{1 + C_1 e^t}$. The condition $u(0) = u_0$ gives $C_1 = \dfrac{u_0}{1-u_0}$. Hence,

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$$u(t) = \frac{u_0 e^t}{1-u_0+u_0 e^t} \qquad (2)$$

We see that $u(t)$ remains in $(0,1)$ for all $t \in \mathbb{R}$. Thus, u given by (2) is the unique solution to Problem (1). Because of the uniqueness of solutions, we can view u as a function of t and u_0.

$$u(t;u_0) = \frac{e^t}{C+e^t}, \quad \text{where } C = \frac{1-u_0}{u_0}. \qquad (3)$$

Next, we compute the partial derivatives of u.

$$\partial_{u_0} u(t;u_0) = \frac{-e^t \partial_{u_0} C}{(C+e^t)^2} = \frac{e^t}{u_0^2 (C+e^t)^2} = \frac{e^t}{(1-u_0+u_0 e^t)^2}, \qquad (4)$$

$$\partial_t u(t;u_0) = \frac{e^t (C+e^t) - e^t e^t}{(C+e^t)^2} = \frac{Ce^t}{(C+e^t)^2}. \qquad (5)$$

(ii) The equations (1) can be written as

$$\partial_t u(t;u_0) = u(t;u_0) - u(t;u_0)^2, \quad u(0;u_0) = u_0. \qquad (6)$$

We only consider the case $u_0 \in (0,1)$. Differentiating both sides with respect to u_0, we get

$$\partial_{u_0} \partial_t u(t;u_0) = \partial_{u_0} u(t;u_0)(1-2u(t;u_0)).$$

Put $v(t) = \partial_{u_0} u(t;u_0)$. We get $\frac{dv}{dt}(t) = v(t)(1-2u(t;u_0))$. Because $u(0;u_0) = u_0$ for all $u_0 \in (0,1)$, $\partial_{u_0} u(0,u_0) = 1$. Thus, $v(0) = 1$. The initial-value problem that v solves is

$$\dot{v} = (1-2u)v, \quad v(0) = 1, \qquad (7)$$

where u is the function given by (3).

Differentiating both sides of (6) with respect to t, we get

$$\partial_t \partial_t u(t;u_0) = \partial_t u(t;u_0)(1-2u(t;u_0)) =$$

$$= u(t;u_0)(1-u(t;u_0))(1-2u(t;u_0)).$$

Put $w(t) = \partial_t u(t;u_0)$. We get $\frac{dw}{dt}(t) = u(t;u_0)(1-u(t;u_0))(1-2u(t;u_0))$. Because $\partial_t u(0;u_0) = u(0;u_0)(1-u(0;u_0)) = u_0(1-u_0)$, w solves the initial-value problem

$$\dot{w} = u(1-u)(1-2u), \quad w(0) = u_0(1-u_0), \qquad (8)$$

where u is the function given by (3).

We now solve (7). Because $v(0)=1$, $v(t)$ remains positive when t is close to 0. Thus,

$$\frac{\dot{v}}{v} = 1-2u = 1 - \frac{2e^t}{C+e^t} = \frac{C-e^t}{C+e^t}.$$

Taking the antiderivative both sides with respect to t, we get

$$\log v = \int \frac{C-e^t}{C+e^t}\,dt \stackrel{y=C+e^t}{=\!=\!=} \int \frac{2C-y}{y}\frac{dy}{y-C} = \int\left(\frac{-2}{y} + \frac{1}{y-C}\right)dy$$

$$= \log\frac{y-C}{y^2} + \text{const}.$$

Thus, $v(t) = C_2\frac{y-C}{y^2} = \frac{C_2 e^t}{(C+e^t)^2}$. The condition $v(0)=1$ yields $C_2 = (C+1)^2$.

Thus, $$v(t) = \frac{(C+1)^2 e^t}{(C+e^t)^2} = \frac{\left(\frac{1}{u_0}\right)^2 e^t}{\left(\frac{1-u_0}{u_0}+e^t\right)^2} = \frac{e^t}{(1-u_0+u_0e^t)^2}.$$

This confirms the formula (4).

Now we solve (8). By (3),

$$u(1-u)(1-2u) = \frac{e^t}{C+e^t}\,\frac{C}{C+e^t}\,\frac{C-e^t}{C+e^t} = \frac{Ce^t(C-e^t)}{(C+e^t)^3}.$$

Thus, $$w(t) = \int u(1-u)(1-2u)\,dt = \int\frac{Ce^t(C-e^t)}{(C+e^t)^3}\,dt$$

$$\stackrel{y=C+e^t}{=\!=\!=} \int \frac{C(y-C)(2C-y)}{y^3}\frac{dy}{y-C}$$

$$= \int\frac{C(2C-y)}{y^3}\,dy = \frac{C(y-C)}{y^2} + \text{const}$$

$$= \frac{Ce^t}{(C+e^t)^2} + \text{const.}$$

The condition $w(0) = u_0(1-u_0) = \frac{C}{(C+1)^2}$ yields $w(t) = \frac{Ce^t}{(C+e^t)^2}$. This confirms formula (5).

(2) Let $f: \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function such that $f(x).x \leq 0$ for all x with $|x|$ sufficiently large. We show that the initial-value problem

$$\dot{x} = f(x), \quad x(0) = x_0 \qquad (1)$$

has at least one local solution, and that every local solution can extend to a global solution (i.e. for all $t \geq 0$).

<u>Step 1</u>: Show that Problem (1) has a solution on the interval $[0,1]$.

There is a number $R > |x_0|$ such that $f(x).x \leq 0$ for all $x \in \mathbb{R}^d$, $|x| \geq R$.

For $0 < \varepsilon < 1$, we define $g(x) = f(x) - \varepsilon x$. Let $\varphi: \mathbb{R}^d \to \mathbb{R}$ be a function such that

$$\begin{cases} \varphi \in C^\infty(\mathbb{R}^d) \\ \varphi(x) \geq 0 \quad \forall x \in \mathbb{R}^d \\ \operatorname{supp} \varphi \subset \overline{B_1(0)}, \\ \int_{\mathbb{R}^d} \varphi \, dx = 1. \end{cases}$$

Such a function exists, for example $\varphi(x) = \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right) & , |x| < 1 \\ 0 & , |x| \geq 1. \end{cases}$

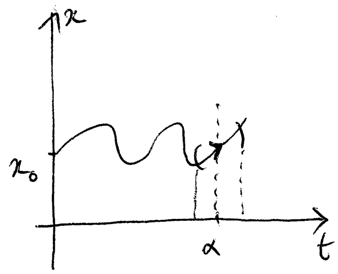
For each $n \in \mathbb{N}$, we define $\varphi_n(x) = n^d \varphi(nx)$. Then the sequence (φ_n) is an approximate identity on $\mathbb{R}^d$. Put $g_n = \varphi_n * g$. Then $g_n \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and (g_n) converges to g uniformly on every bounded subset of $\mathbb{R}^d$. There exists $N \in \mathbb{N}$ such that $|g_n(x) - g(x)| < \frac{\varepsilon R}{2} \quad \forall x \in \mathbb{R}^d, |x| < 2R, \forall n \geq N$

By shifting the indices of the sequence $g_n(x)$ if necessary, we can assume $N=1$.

Thus,
$$g_n(x)\cdot x = (g_n(x)-g(x))\cdot x + g(x)\cdot x$$
$$\leqslant |g_n(x)-g(x)||x| + f(x)\cdot x - \varepsilon|x|^2$$
$$\leqslant \frac{\varepsilon R}{2}|x| - \varepsilon|x|^2 = \varepsilon|x|\left(\frac{R}{2}-|x|\right) < 0 \quad \forall x\in\mathbb{R}^d,\ R\leq|x|<2R. \tag{2}$$

We first show that the initial-value problem
$$\dot{x} = g_n(x),\ x(0) = x_0 \tag{3}$$
has a global solution (i.e. for all $t\geqslant 0$). Because $g_n\in C^1(\mathbb{R}^d,\mathbb{R}^d)$, it is locally Lipschitz. By Picard-Linderlöf theorem, Problem (3) has a unique local solution. Let x be the unique solution in a minimal interval $t\in[0,\alpha)$ for $0<\alpha\leq\infty$. Suppose by contradiction that $\alpha<\infty$. We show that $|x(t)|\leq R$ for all $t\in[0,\alpha)$.
$$\beta := \inf\{s\in[0,\alpha) : |x(t)|\leqslant R \ \ \forall t\in[0,s]\}.$$
Suppose that $\beta<\alpha$. Then $|x(\beta)|=R$ because of the continuity of x. For every $k\in\mathbb{N}$, there exists t_k such that $\beta<t_k<\min\{\alpha,\beta+\frac{1}{k}\}$ and $|x(t_k)|>R$. Then
$$\frac{d|x|^2}{dt}(\beta) = \lim_{k\to\infty}\frac{|x(t_k)|^2-|x(\beta)|^2}{t_k-\beta} = \lim_{k\to\infty}\frac{|x(t_k)|^2-R^2}{t_k-\beta} \geqslant 0.$$
On the other hand, by taking the dot product by x both sides of (3), we get $\dot{x}x = g_n(x)\cdot x$. Thus,
$$\frac{1}{2}\frac{d|x|^2}{dt}(\beta) = g_n(x(\beta))\cdot x(\beta) \overset{(2)}{<} 0.$$
This is a contradiction. Hence, $\beta=\alpha$ and $|x(t)|\leq R$ for all $t\in[0,\alpha)$.

Let (t_k) be a sequence in $[0,\alpha)$ that converges to α. Because $(x(t_k))$ is



bounded in $\mathbb{R}^d$, it has a converging subsequence. By replacing (t_k) by this subsequence, we can assume $(x(t_k))$ converges to some $a \in \mathbb{R}^d$. We show that

$$\lim_{s\to\alpha^-} x(s) = a.$$

Denote $\tilde{M} = \max_{|x|\le R} |g_n(x)|$.

$$x(s) = x(t_k) + \dot{x}(\xi_{s,k})(s-t_k) = x(t_k) + g_n(x(\xi_{s,k}))(s-t_k),$$

for some $\xi_{s,k}$ lying between s and t_k. Thus,

$$|x(s)-x(t_k)| = |g_n(x(\xi_{s,k}))|\,|s-t_k| \le M|s-t_k| \quad \forall k\in\mathbb{N}.$$

Letting $k\to\infty$, we get $|x(s)-a| \le M|s-\alpha|$. Hence, $\lim_{s\to\alpha^-} x(s) = a$. In addition,

$$\dot{x}(s) = g_n(x(s)) \to g_n(a) \quad \text{as } s\to\alpha^-.$$

Thus, x and $\dot{x}$ extend continuously to α. The initial-value problem

$$\dot{x} = g_n(x), \quad x(\alpha) = a$$

has a unique local solution in some interval $(\alpha-\delta, \alpha+\delta)$ by Picard-Linderlöf theorem. It leads to a continuation of the solution to Problem (3) beyond α. This is a contradiction. Thus, $\alpha=\infty$. In other words, Problem (3) has a unique global solution, say x_n. We also showed that

$$|x_n(t)| \le R \quad \forall t\ge 0 \qquad (4)$$

Next, we show that the initial-value problem

$$\dot{x} = g(x), \quad x(0) = x_0 \qquad (5)$$

has a solution on $[0,1]$. Denote $\overline{B_R} = \{x\in\mathbb{R}^d : |x|\le R\}$. Recall that $C(\overline{B_R})$ is a normed space with $\|u\|_{\overline{B_R}} = \sup_{x\in \overline{B_R}} |u(x)|$. Viewing g_n, g and f as functions on $\overline{B}_R$, we have

$$\|g_n\|_{\overline{B}_R} = \|\varphi_n * g\|_{\overline{B}_R} \leq \|g\|_{\overline{B_{R+1}}} = \|f - \varepsilon x\|_{\overline{B_{R+1}}} \leq \|f\|_{\overline{B_{R+1}}} + \varepsilon(R+1). \quad (6)$$

Put $M = \max_{|x| \leq R+1} |f(x)| + R + 1$. Then

$$|g_n(x)| \leq M \quad \forall x \in \overline{B}_R,\ n \in \mathbb{N}.$$

Then, $\quad |\dot{x}_n(t)| = |g_n(x_n(t))| \leq M \quad \forall n \in \mathbb{N},\ t \geq 0.$

For $0 \leq s,t \leq 1$,

$$|x_n(t) - x_n(s)| = |\dot{x}_n(\xi)||t-s| \leq M|t-s| \quad \forall n \in \mathbb{N}. \quad (7)$$

If we view x_n as its restriction on $[0,1]$, $x_n \in C([0,1])$. By (4), (x_n) is bounded. By (7), (x_n) is equicontinuous. By Arzela-Ascoli theorem, (x_n) has a convergent subsequence. By replacing (x_n) by this subsequence, we can assume (x_n) converges to some $x_\varepsilon \in C([0,1])$.

Recall that $C([0,1])$ is a normed space with $\|v\|_{[0,1]} = \sup_{t \in [0,1]} |v(t)|$. We have

$$\|g_n(x_n) - g(x_\varepsilon)\|_{[0,1]} \leq \|g_n(x_n) - g(x_n)\|_{[0,1]} + \|g(x_n) - g(x_\varepsilon)\|_{[0,1]}$$
$$\leq \|g_n - g\|_{\overline{B}_R} + \|g(x_n) - g(x_\varepsilon)\|_{[0,1]}. \quad (8)$$

Because g is continuous on the compact set $\overline{B}_R$, it is uniformly continuous there. For each $\delta > 0$, there exists $\delta' > 0$ such that $|g(x) - g(y)| < \delta$ for all $x, y \in \overline{B}_R$, $|x-y| < \delta'$. There exists $N \in \mathbb{N}$ such that $\|x_n - x_\varepsilon\|_{[0,1]} < \delta'$ for all $n > N$. Thus,

$$\|g(x_n) - g(x_\varepsilon)\|_{[0,1]} < \delta \quad \forall n \in \mathbb{N},\ n > N.$$

Then (8) implies $\|g_n(x_n) - g(x_\varepsilon)\|_{[0,1]} < \|g_n - g\|_{\overline{B}_R} + \delta \quad \forall n > N.$

Because $\|g_n - g\|_{\overline{B}_R} \to 0$ as $n \to \infty$, $g_n(x_n) \to g(x_\varepsilon)$ in $C([0,1])$. Thus, $\dot{x}_n \to g(x_\varepsilon)$

in $C([0,1])$.

$$x_n(t) = x_0 + \int_0^t \dot{x}_n(s)\,ds \to x_0 + \int_0^t g(x_\varepsilon(s))\,ds.$$

Thus, $\quad x_\varepsilon(t) = x_0 + \int_0^t g(x_\varepsilon(s))\,ds.$

This implies x_ε is a solution to Problem (5). We also showed that

$$\|x_\varepsilon\|_{[0,1]} \leq R \quad \forall \varepsilon \in (0,1).$$

Next, we show that Problem (1) has a solution on $[0,1]$. We have

$$|x_\varepsilon(t) - x_\varepsilon(s)| = |\dot{x}_\varepsilon(\xi)|\,|t-s| = |g(x_\varepsilon(\xi))|\,|t-s| \overset{(6)}{\leq} M\,|t-s| \quad \forall \varepsilon \in (0,1)$$
$$\forall t, s \in [0,1].$$

Thus, the set $\{x_\varepsilon : \varepsilon \in (0,1)\}$ is bounded and equicontinuous in $C([0,1])$. By Arzela-Ascoli theorem, there is a convergent sequence (x_{ε_n}) in $C([0,1])$. Denote $x = \lim_{n\to\infty} x_{\varepsilon_n}$. Then $\|g(x_{\varepsilon_n}) - f(x_{\varepsilon_n})\|_{[0,1]} = \varepsilon_n \underbrace{\|x_{\varepsilon_n}\|_{[0,1]}}_{\text{bounded}} \to 0$ as $n \to \infty$.

Because f is uniformly continuous on $\overline{B}_R$, we have

$$\|f(x_{\varepsilon_n}) - f(x)\|_{[0,1]} \to 0 \quad \text{as } n \to \infty.$$

Thus, $\|g(x_{\varepsilon_n}) - f(x)\|_{[0,1]} \to 0$ as $n \to \infty$. Equivalently, $\|\dot{x}_{\varepsilon_n} - f(x)\|_{[0,1]} \to 0$ as $n \to \infty$.

$$x_{\varepsilon_n}(t) = x_0 + \int_0^t \dot{x}_{\varepsilon_n}(s)\,ds \longrightarrow x_0 + \int_0^t f(x(s))\,ds, \quad \forall t \in [0,1]$$

Hence, $\quad x(t) = x_0 + \int_0^t f(x(s))\,ds, \quad \forall t \in [0,1].$

This means x is a solution on the interval $[0,1]$ to Problem (1).

Step 2: Suppose x is a solution on an interval $[0,\delta]$, $\delta > 0$, to Problem (1). We show that x can extend to a solution on $[0,\infty)$.

To do so, it suffices to show that x can extend to a solution on $[0,\delta+1]$.

Indeed, by applying this result repetitively, we can extend x to a solution on $[0,\delta+2]$, then $[0,\delta+3]$, and so on. By Step 1, there is a solution y to the initial value problem

$$\dot{y} = f(y), \quad y(0) = x(\delta)$$

on the interval $[0,1]$. By defining $x(t) := y(t-\delta)$ for $t \in [\delta, \delta+1]$, we get

$$x \in C([0,\delta+1]) \quad \text{and} \quad \dot{x} = f(x) \quad \forall t \in [0,\delta+1] \setminus \{\delta\}.$$

Because $\lim_{t\to\delta} \dot{x}(t) = \lim_{t\to\delta} f(x(t)) = f(x(\delta))$, x is differentiable at $t=\delta$ and $\dot{x}(\delta) = f(x(\delta))$. Therefore, $\dot{x} = f(x)$ for all $t \in [0,\delta+1]$.

(3) Theorem 7.6 in Amann "Ordinary Differential Equations" 1990 (page 100) gives us a criterion for global existence of solutions to an ODE. We state a consequence which is enough for our purpose.

$\Bigg[$ Let $f: \mathbb{R}^d \to \mathbb{R}^d$ be a locally Lipschitz function and $u_0 \in \mathbb{R}^d$. Let $(\alpha,\beta) \subset \mathbb{R}$ be the maximal interval of existence for the initial-value problem

$$\dot{u} = f(u), \quad u(0) = u_0.$$

If $\alpha > -\infty$ then $\lim_{t\to\alpha^+} |u(t)| = \infty$. If $\beta < \infty$ then $\lim_{t\to\beta^-} |u(t)| = \infty$. $\Bigg]$

In the following, we study whether each given initial-value problem has a global solution (i.e. for all $t \in \mathbb{R}$). Denote by (α,β) the maximal interval of existence. Because initial conditions are specified at $t=0$, $\alpha < 0 < \beta$.

(i) $\dot{x} = \sin x, \quad x(0) = x_0. \qquad (1)$

In this case, $f:\mathbb{R}\to\mathbb{R}$, $f(x) = \sin x$. Because $f \in C^1(\mathbb{R})$, it is locally Lipschitz.

$$x(t) = x(0) + t\dot{x}(\xi_t) = x_0 + t\sin\xi_t \quad \forall t \in (\alpha,\beta).$$

Thus, $|x(t)| \leq |x_0| + |t|$ for all $t \in (\alpha, \beta)$. This implies that x does not blow up in finite time. Therefore, Problem (1) has a global solution.

(ii) $\dot{x} = x - x^3,\ x(0) = 2.$ (2)

$f(x) = x - x^3$ is locally Lipschitz because it is in $C^1(\mathbb{R})$. Suppose there is $t_0 \in (\alpha, \beta)$ such that $x(t_0) = 1$. Then the constant function $y_1(t) \equiv 1$ solves the problem $\dot{y} = y - y^3,\ y(0) = x(t_0)$. On the other hand, the function $y_2(t) = x(t+t_0)$ also solves this problem. By the uniqueness of solutions, $x(t+t_0) = 1$ for all t. This is a contradiction because $x(0) = 2$. Therefore, $x(t) \neq 1$ for all $t \in (\alpha, \beta)$.

Because x is continuous on (α, β) and $x(0) > 1$, $x(t) > 1$ for all $t \in (\alpha, \beta)$. Then $\dot{x}(t) = x(t) - x(t)^3 < 0$. Thus, x is decreasing on (α, β). Then $x(t) \geqslant x(0) = 2$ for all $t \in (\alpha, 0)$. Then

$$\dot{x}(t) = x(t) - x(t)^3 \leqslant x(t)^2 - x(t)^3 = -x(t)^2 + x(t)^2 \underbrace{(2 - x(t))}_{\leqslant 0} \leqslant -x(t)^2 \quad \forall t \in (\alpha, 0).$$

Thus, $$\int_t^0 \frac{\dot{x}(s)}{x(s)^2}\, ds \leq \int_t^0 -ds \quad \forall t \in (\alpha, 0),$$

which gives $$-\frac{1}{x(0)} + \frac{1}{x(t)} \leq t \quad \forall t \in (\alpha, 0).$$

Because $x(0) = 2$, $$\frac{1}{x(t)} \leq t + \frac{1}{2} \quad \forall t \in (\alpha, 0). \qquad (3)$$

If $\alpha < -\frac{1}{2}$ then $-\frac{1}{2} \in (\alpha, 0)$. Substituting $t = -\frac{1}{2}$ into (3), we get $x(-\frac{1}{2}) < 0$. This is a contradiction. Therefore, $\alpha \geqslant -\frac{1}{2}$. It implies that the local solution to Problem (2) cannot extend beyond $t = -\frac{1}{2}$.

(iii) $\dot{x} = x - x^3,\ x(0) = \frac{1}{2}.$ (4)

The function f is the same as in Problem (2). By the same argument as in Part (ii), we can show that $x(t) \notin \{0,1\}$ for all $t \in (\alpha,\beta)$. Because x is continuous on (α,β) and $x(0) \in (0,1)$, $x(t) \in (0,1)$ for all $t \in (\alpha,\beta)$. The boundedness of $x(t)$ implies $\alpha = -\infty$ and $\beta = \infty$. In other words, Problem (4) has a global solution.

(iv) $$\dot{x} = \frac{4x^3+y}{x^2+y^2+1}, \quad \dot{y} = \frac{y^5-x^5}{x^4+y^4+1}, \quad x(0)=x_0, \quad y(0)=y_0. \qquad (5)$$

In this case, $f: \mathbb{R}^2 \to \mathbb{R}^2$, $$f(x,y) = \left(\frac{4x^3+y}{x^2+y^2+1}, \frac{y^5-x^5}{x^4+y^4+1} \right).$$

f is locally Lipschitz because it is in $C^1(\mathbb{R}^2,\mathbb{R}^2)$. Denote the maximal interval of existence by J (instead of (α,β) to avoid confusion of notations). Put $u=(x,y)$ and $u_0=(x_0,y_0)$. Problem (5) is equivalent to

$$\dot{u} = f(u), \quad u(0) = u_0. \qquad (6)$$

Then $$u(t) = u_0 + \int_0^t \dot{u}(s)\,ds = u_0 + \int_0^t f(u(s))\,ds \qquad \forall t \in J.$$

Thus, $$|u(t)| \leq |u_0| + \left| \int_0^t |f(u(s))|\,ds \right| \qquad \forall t \in J. \qquad (7)$$

We now estimate $|f(u)|$.

$$|4x^3+y| \leq 4|x|^3+|y| \leq 4|u|^3+|u| = |u|(4|u|^2+1) \leq 4|u|(|u|^2+1)$$

$$|y^5-x^5| \leq |y|^5+|x|^5 \leq 2|u|^5,$$

$$x^4+y^4+1 \geq \frac{1}{2}(x^2+y^2)^2+1 = \frac{1}{2}|u|^4+1.$$

Hence, $$|f(u)|^2 = \frac{|4x^3+y|^2}{(x^2+y^2+1)^2} + \frac{|y^5-x^5|^2}{(x^4+y^4+1)^2} \leq \frac{16|u|^2(|u|^2+1)^2}{(|u|^2+1)^2} + \frac{4|u|^{10}}{(\frac{1}{2}|u|^4+1)^2}$$

$$\leq 16|u|^2 + \frac{16|u|^2\left(\frac{1}{2}|u|^4\right)^2}{\left(\frac{1}{2}|u|^4+1\right)^2} \leq 16|u|^2 + 16|u|^2.$$

Thus, $|f(u)| \leq 4\sqrt{2}\,|u|$. Substituting this estimate into (7), we get

$$|u(t)| \leq |u_0| + \left|\int_0^t 4\sqrt{2}\,|u(s)|\,ds\right| \qquad \forall t \in J.$$

Gronwall's inequality then gives us an explicit estimate for $|u(t)|$:

$$|u(t)| \leq |u_0| + 4\sqrt{2}\,|u_0| \left|\int_0^t e^{4\sqrt{2}|t-s|}\,ds\right| \qquad \forall t \in J.$$

This implies that $u(t)$ does not blow up in finite time. Therefore, Problem (6) has a global solution (i.e. for all $t \in \mathbb{R}$).