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Math 8501: Differential Equations & Dynamical Systems

Homework #3

① Let X be a metric space and $\phi: \mathbb{R}\times X \to X$ be a flow. Let M be a compact subset of X and define $\omega(M) = \bigcap_{T>0} \overline{\bigcup_{t\geq T} \phi_t(M)}$.

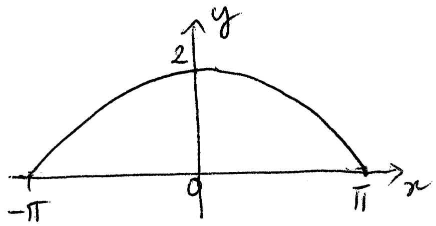
First, we show that $\bigcup_{x\in M} \omega(x) \subset \omega(M)$. That is to show for each $x_0 \in M$ that $\omega(x_0) \subset \omega(M)$. By the definition of ω-limit sets, for each $x \in \omega(x_0)$ there exists a sequence (t_k) tending to infinity such that $\lim_{k\to\infty} \phi_{t_k}(x_0) = x$. Thus, x is an accumulation point of the set $\{\phi_t(x_0): t\geq T\}$ for every $T>0$.

Thus,
$$x \in \overline{\{\phi_t(x_0): t \geq T\}} = \overline{\bigcup_{t\geq T} \{\phi_t(x_0)\}} \subset \overline{\bigcup_{t\geq T} \phi_t(M)} \quad \forall T>0.$$

Then
$$x \in \bigcap_{T>0} \overline{\bigcup_{t\geq T} \phi_t(M)} = \omega(M).$$

We have showed $\omega(x) \subset \omega(M)$.

Next, we give an example where $\omega(M) \neq \bigcup_{x\in M} \omega(x)$. The idea comes from looking at a heteroclinic orbit of a pendulum.



Take $X = \{(x,y): y = 2\cos\frac{x}{2},\ -\pi \leq x \leq \pi\} \subset \mathbb{R}^2$.

Define a flow $\phi: \mathbb{R}\times X \to X$, $\phi_t(x_0, y_0) = (x,y)$

where
$$\begin{cases} (\dot{x}, \dot{y}) = (y, -\sin x), \\ (x(0), y(0)) = (x_0, y_0). \end{cases} \quad (*)$$

Note that the above ODE has a unique global solution because $f(x,y) = (y, -\sin x)$ is Lipschitz and has linear growth. Take $M = X$, which is a compact set. For each $T>0$,

$$X \supset \overline{\bigcup_{t\geq T} \phi_t(X)} \supset \bigcup_{t\geq T} \phi_t(X) = \{\phi_t(x_0,y_0) : (x_0,y_0)\in X, t\geq T\}$$

$$\supset \{\underbrace{\phi_t \circ \phi_{-T}}_{=\phi_{t-T}}(x_1,y_1) : (x_1,y_1)\in X, t\geq T\}$$

$$\supset \{\underbrace{\phi_0}_{=id_X}(x_1,y_1) : (x_1,y_1)\in X\}$$

$$= X.$$

Thus, $\overline{\bigcup_{t\geq T} \phi_t(X)} = X$ for all $T>0$. Hence,

$$\omega(M) = \omega(X) = \bigcap_{T>0} \overline{\bigcup_{t\geq T} \phi_t(X)} = X = M.$$

Now we show that

$$\omega(x_0,y_0) = \begin{cases} \{(-\pi,0)\} & \text{if } (x_0,y_0)=(-\pi,0), \\ \{(\pi,0)\} & \text{otherwise.} \end{cases}$$

If $(x_0,y_0)=(-\pi,0)$ then the constant function $(-\pi,0)$ solves $(*)$. Thus, $\omega(x_0,y_0)=\{(-\pi,0)\}$. Consider $(x_0,y_0)\in X\setminus\{(-\pi,0)\}$. Because $\dot{x}=y\geq 0$, $x=x(t)$ is an increasing function. Because $x(t)\in[-\pi,\pi]$ for all $t\in\mathbb{R}$, $\lim_{t\to\infty} x(t) = \alpha\in[-\pi,\pi]$. In additon, $\alpha\in(-\pi,\pi]$ because $x(0)=x_0>-\pi$. Then

$$\lim_{t\to\infty} \dot{y}(t) = \lim_{t\to\infty} -\sin x(t) = -\sin\alpha.$$

Because $y(t)\in[0,2]$ for all $t\in\mathbb{R}$, $-\sin\alpha=0$. Thus, $\alpha=\pi$. Then

$$y(t) = 2\cos\frac{x(t)}{2} \to 2\cos\frac{\pi}{2} = 0 \quad \text{as } t\to\infty.$$

Thus, $\omega(x_0,y_0)=\{(\pi,0)\}$. We see that

$$\bigcup_{(x_0,y_0)\in M} \omega(x_0,y_0) = \{(-\pi,0),(\pi,0)\} \neq M = \omega(M).$$

② Consider the ODE

$$\begin{cases} (u', v') = (u^2 - v u^3, 0), \\ (u(0), v(0)) = (u_0, v_0). \end{cases}$$

Because $v' = 0$, v is the constant function v_0. The given ODE reduces to

$$u' = u^2 - v_0 u^3, \quad u(0) = u_0 \qquad (*)$$

(i) Put $f(u) = u^2 - v_0 u^3 = u^2(1 - v_0 u)$. We see that $f \in C^1(\mathbb{R})$, thus it is locally Lipschitz. The local existence and uniqueness of (*) is guaranteed. The zeros of f are the equilibria. To check the sign of u', it is convenient for us to consider 3 cases: $v_0 = 0$, $v_0 < 0$ and $v_0 > 0$.

$v_0 = 0$

u		0	
$u' = f(u)$	$+$	0	$+$

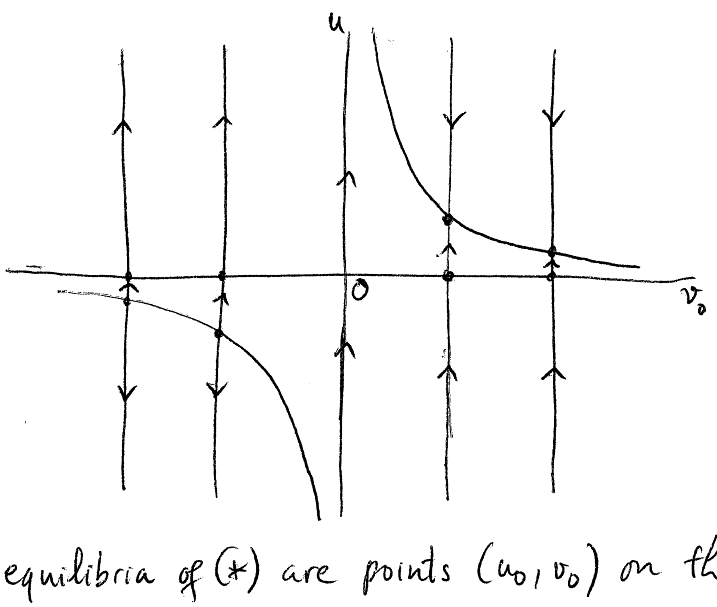
$v_0 < 0$

u		v_0^{-1}		0	
$u' = f(u)$	$-$	0	$+$	0	$+$

$v_0 > 0$

u		0		v_0^{-1}	
$u' = f(u)$	$+$	0	$+$	0	$-$

We obtain the phase portrait of (*) as follows.



The equilibria of (*) are points (u_0, v_0) on the line $u_0 = 0$ and on the hyperbola $u_0 = v_0^{-1}$.

(ii) According to the phase portrait, the solution $u = u(t)$ is bounded if $v_0 > 0$. If $v_0 < 0$ and $v_0^{-1} \leq u_0 \leq 0$, the solution satisfies $v_0^{-1} \leq u(t) \leq 0$, i.e. being trapped between two equilibria. If $v_0 = 0$ and $u_0 \leq 0$, u is also bounded. In these cases, $u = u(t)$ never blows up after a finite time. Denote by $t^+(u_0, v_0)$ the maximal positive time of existence. Then $t^+(u_0, v_0) = \infty$ if $v_0 > 0$ or $(v_0 < 0,\ v_0^{-1} \leq u_0 \leq 0)$ or $(v_0 = 0,\ u_0 \leq 0)$. To find (u_0, v_0) such that $t^+(u_0, v_0) < \infty$, we only need to consider the cases $(v_0 \leq 0,\ u_0 > 0)$ and $(v_0 < 0,\ u_0 < v_0^{-1})$.

- $v_0 \leq 0$ and $u_0 > 0$.

Then $u(t) > 0$ for all $0 < t < t^+(u_0, v_0)$ according to the phase portrait. Equation (*) implies $u' \geq u^2$. Thus, $\frac{u'}{u^2} \geq 1$. Taking the antiderivative both sides with respect to t, we get
$$-\frac{1}{u(t)} \geq t + C.$$
Because $u(t) > 0$, $t + C < 0$. Thus, $t^+(u_0, v_0) \leq -C < \infty$.

- $v_0 < 0$ and $u_0 < v_0^{-1}$.

Then $u(t) \leq u_0$ for all $0 < t < t^+(u_0, v_0)$ according to the phase portrait. Equation (*) implies $u' = u^2(1 - v_0 u) \leq u^2(1 - v_0 u_0)$. Thus, $\frac{u'}{u^2} \leq 1 - v_0 u_0$. Taking the antiderivative both sides with respect to t, we get $-\frac{1}{u(t)} \leq (1 - v_0 u_0)t + C$.
Thus, $(v_0 u_0 - 1)t - C \leq \frac{1}{u(t)} < 0$. Hence, $t^+(u_0, v_0) \leq \frac{C}{v_0 u_0 - 1} < \infty$.

In conclusion, $t^+(u_0, v_0) < \infty$ if and only if $(v_0 \leq 0,\ u_0 > 0)$ or $(v_0 < 0,\ u_0 < v_0^{-1})$.

(iii) Let $u_0 > 0$. We know that $t^+(u_0, 0) < \infty$ and $t^+(u_0, \varepsilon) = \infty$ for every $\varepsilon > 0$.

Then $\lim\limits_{v_0 \to 0^+} t^+(u_0, v_0) = \infty \neq t^+(u_0, 0)$. Thus, $t^+(u_0, v_0)$ is not a continuous function from $\mathbb{R}^2$ to $(0, \infty]$.

(3) The Riemann sphere is a complex manifold $S^2 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ whose atlas consists of two charts (V_+, ψ_+) and (V_-, ψ_-).

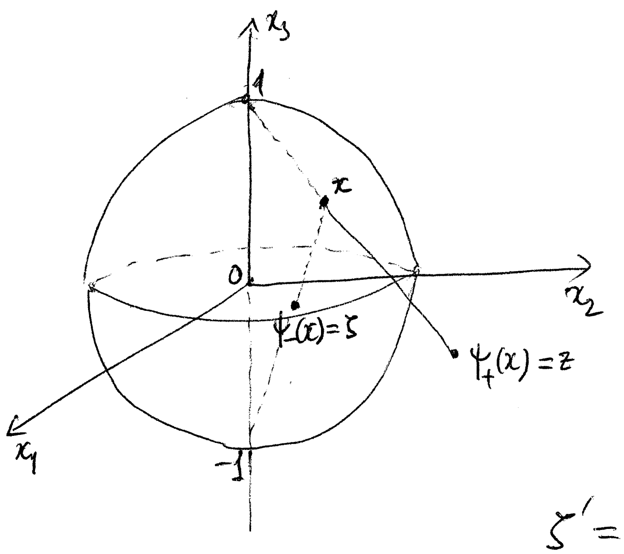
$$V_+ = \{(x_1, x_2, x_3) \in S^2 : x_3 \neq 1\}$$

$$V_- = \{(x_1, x_2, x_3) \in S^2 : x_3 \neq -1\}$$

$$\psi_+ : V_+ \to \mathbb{C}, \quad \psi_+(x_1, x_2, x_3) = \frac{x_1 + i x_2}{1 - x_3}$$

$$\psi_- : V_- \to \mathbb{C}, \quad \psi_-(x_1, x_2, x_3) = \frac{x_1 - i x_2}{1 + x_3}.$$

The transition maps are $\psi_+ \circ \psi_-^{-1}(z) = \psi_- \circ \psi_+^{-1}(z) = z^{-1}$ $\forall z \in \mathbb{C} \setminus \{0\}$.



Let z and ζ be the coordinates in V_+ and V_- respectively. We have $z = \zeta^{-1}$. Consider an ODE on S^2 satisfying $z' = z^2$ in V_+. In V^-, the equation is expressed in terms of ζ as follows.

$$\zeta' = (z^{-1})' = \frac{-z'}{z^2} = -1.$$

Thus, the ODE on S^2 are

$$\begin{cases} z' = z^2 \text{ in } V_+, & \zeta' = -1 \text{ in } V_- \\ z(0) = z_0, & \zeta(0) = \zeta_0 = z_0^{-1} \end{cases} \qquad (I)$$

where $z_0 \in \mathbb{C} \cup \{\infty\}$.

(i) We solve (I) with the initial data $z_0 = \infty$, which corresponds to the north pole $(0,0,1)$ of S^2.

Because a solution to (I) starts from a point in V_-, it remains in V_- after some positive time. During this period of time, (I) is given by

$$\zeta' = -1, \quad \zeta(0) = 0.$$

Thus, $\zeta(t) = \zeta(0) + \int_0^t \zeta'(s)\,ds = 0 - \int_0^t ds = -t \quad \forall t \in \mathbb{R}$.

We see that the solution is unique and does not blow up after a finite time. On the sphere, the solution curve stays in V_- for all times.

(ii) Part (i) gives us the solution for $z_0 = \infty$. Now we solve (I) for $z_0 \in \mathbb{C}$.

Consider the case $z_0 \in \mathbb{C}\setminus\{0\}$. The initial data of (I) corresponds to a point in V_-. A solution should stay in V_- after some positive time. During this period of time, (I) is given by

$$\zeta' = -1, \quad \zeta(0) = \zeta_0 = z_0^{-1}.$$

Then $\zeta(t) = \zeta(0) + \int_0^t \zeta'(s)\,ds = \zeta_0 - \int_0^t ds = \zeta_0 - t \quad \forall t \in \mathbb{R}$.

We see that the solution is unique and does not blow up after a finite time. On the sphere, the solution curve stays in V_- for all times. It is a circle crossing the south pole. We can write the solution in terms of coordinate z.

$$z(t) = \frac{1}{\zeta(t)} = \frac{1}{\zeta_0 - t} = \frac{1}{z_0^{-1} - t} \quad \forall t \in \mathbb{R}$$

with the convention $\frac{1}{0} = \infty$. It corresponds to the time when the solution curve goes out of V_+.

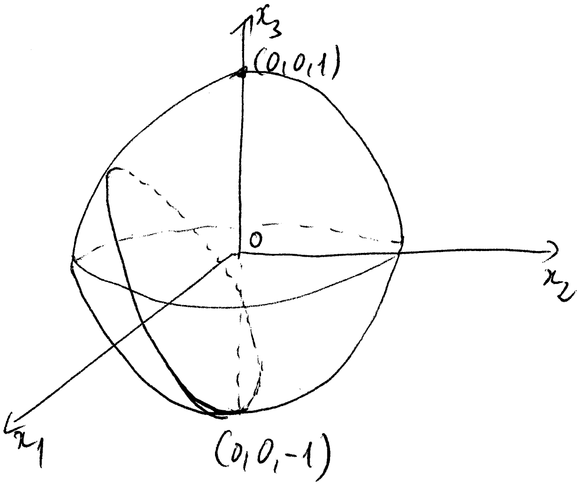
Consider the case $z_0 = 0$. The initial data of (I) corresponds to a point in V_+. A solution should stay there if t is near 0. During that period of time, (I) is given by

$$z' = z^2, \quad z(0) = 0.$$

The constant function 0 is a solution. We show that it is the unique solution. Write $z = x + iy$ with $x, y \in \mathbb{R}$. Then the equation $z' = z^2$ becomes

$$(\dot{x}, \dot{y}) = (x^2 - y^2, 2xy). \qquad (*)$$

The function $f(x,y) = (x^2-y^2, 2xy)$ is in $C^1(\mathbb{R}^2)$, so it is locally Lipschitz. With any given initial data $(x(0), y(0))$, $(*)$ has a unique solution on a maximal interval of existence. Therefore, $(x,y) \equiv (0,0)$ is the unique solution.

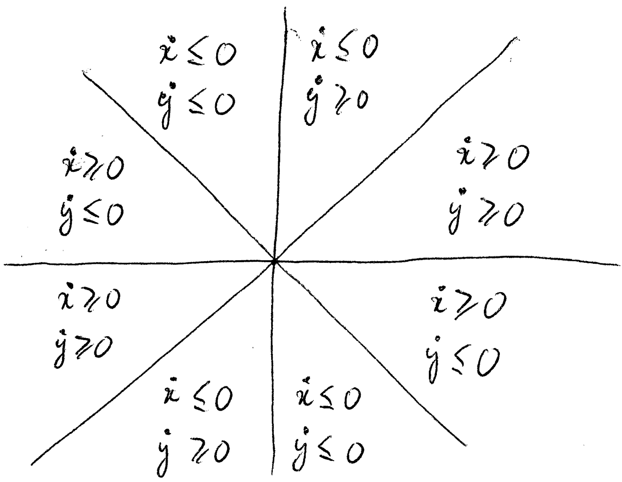


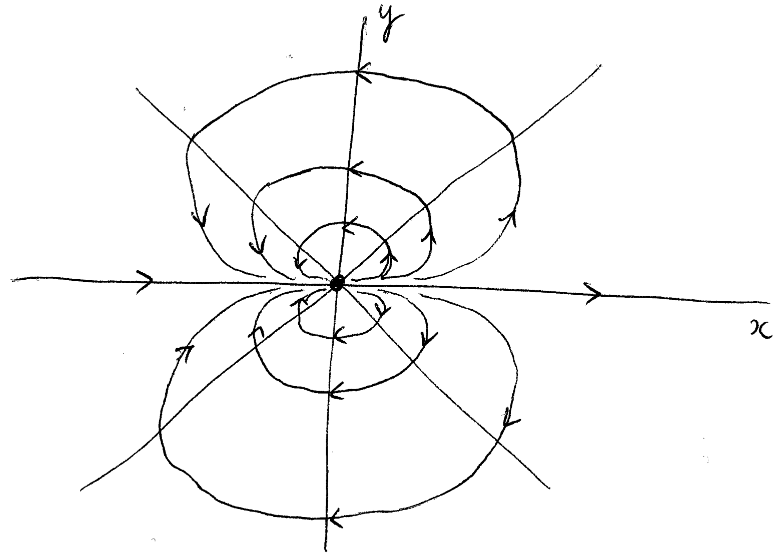
In conclusion, the solution to (I) with different initial data $z_0 \in \mathbb{C} \cup \{\infty\}$ are different circles passing through the south pole. The south pole is excluded unless $z_0 = 0$. The circles degenerates to a single point when $z_0 = 0$.

We now draw a phase portrait for (I) in z-coordinate. That is to draw a phase portrait for $(*)$. The zeros of f are the equilibria.

$$f(x,y) = 0 \iff \begin{cases} x^2 - y^2 = 0 \\ 2xy = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = 0. \end{cases}$$

Thus, $(0,0)$ is the only equilibrium. To decide the signs of $\dot{x}$ and $\dot{y}$, we need to compare $|x|$ with $|y|$, and xy with 0. Dividing the plan (x,y) into octants is helpful.





(iii) We show that all solutions according to the phase portrait converges to the origin as $t \to \pm\infty$. It is equivalent to showing that in z-coordinate every solution to (I) converges to 0.

If $z_0 = 0$, $z(t) = 0$ for all $t \in \mathbb{R}$. Consider $z_0 \neq 0$. The solution to (I) in z-coordinate is $z(t) = \dfrac{1}{z_0^{-1} - t}$ $\forall t \in \mathbb{R}$, with the convention $\frac{1}{0} = \infty$.

Thus, $\lim\limits_{t \to \pm\infty} z(t) = 0$.

(iv) We show that the origin in the phase portrait is not Lyapunov stable.

In z-coordinate, the flow is $\phi_t(z_0) = \dfrac{1}{z_0^{-1} - t}$.

By the convention $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$, the above formula can represent a flow on the whole sphere.

Because $\phi_t(0) = 0$ for all $t \in \mathbb{R}$, $\{0\}$ is an invariant set. Suppose by contradiction that 0 is Lyapunov stable. Then there exists $\delta > 0$ such that

$\mathrm{dist}(\phi_t(z_0), \{0\}) < 1$ for all $t \geq 0$ and $z_0 \in \mathbb{C}$, $\mathrm{dist}(z_0, \{0\}) < \delta$. In other words,

$$|\phi_t(z_0)| = \frac{1}{|z_0^{-1} - t|} < 1 \quad \forall z_0 \in \mathbb{C}, |z_0| < \delta, \forall t \geq 0.$$

With $z_0 = \frac{\delta}{2}$, we get $\dfrac{1}{\frac{2}{\delta} - t} < 1 \quad \forall\, 0 < t < \frac{2}{\delta}$.

This is a contradiction because the left-hand side is arbitrarily large.

(4) Consider the ODE $x'' = -V'(x)$ where $V(x) = x^6 - 0.03x^5 - 4x^4 + 4x^2 + 0.2x$.

Put $y = x'$. We get a system of ODE

$$x' = y, \quad y' = -V'(x).$$

This system has first integral $I(x,y) = \frac{y^2}{2} + V(x)$ because

$$\frac{dI}{dt} = y'y + x'V'(x) = -yV'(x) + yV'(x) = 0.$$

We want to sketch the phase portrait on the phase space $(x,y) \in \mathbb{R}^2$. Each solution curve $(x(t), y(t))$ is contained in a level set $\frac{y^2}{2} + V(x) = C$ where the value of constant C is determined by the initial data $(x(0), y(0))$. The signs of x' and y' are the signs of y and $-V'(x)$ respectively.

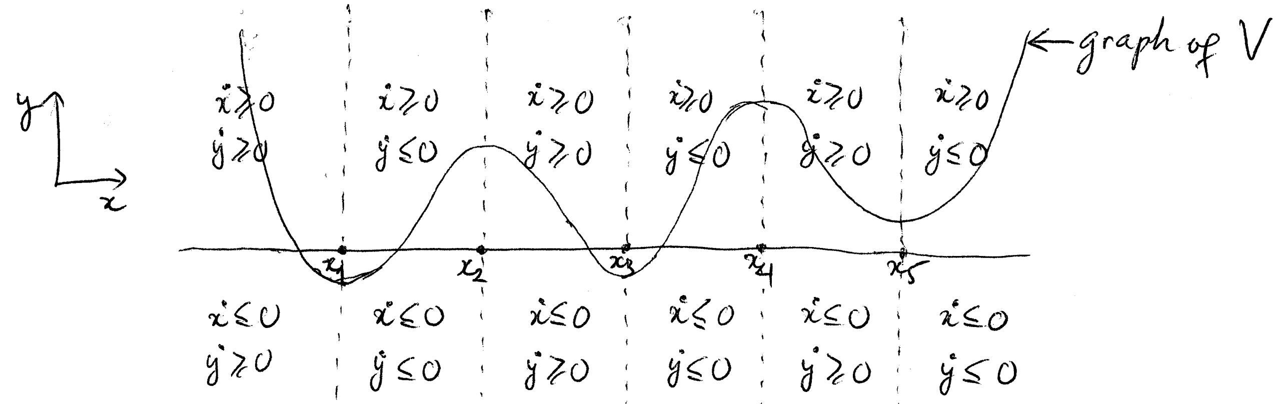
$$V'(x) = 6x^5 - 0.15x^4 - 16x^3 + 8x + 0.2$$

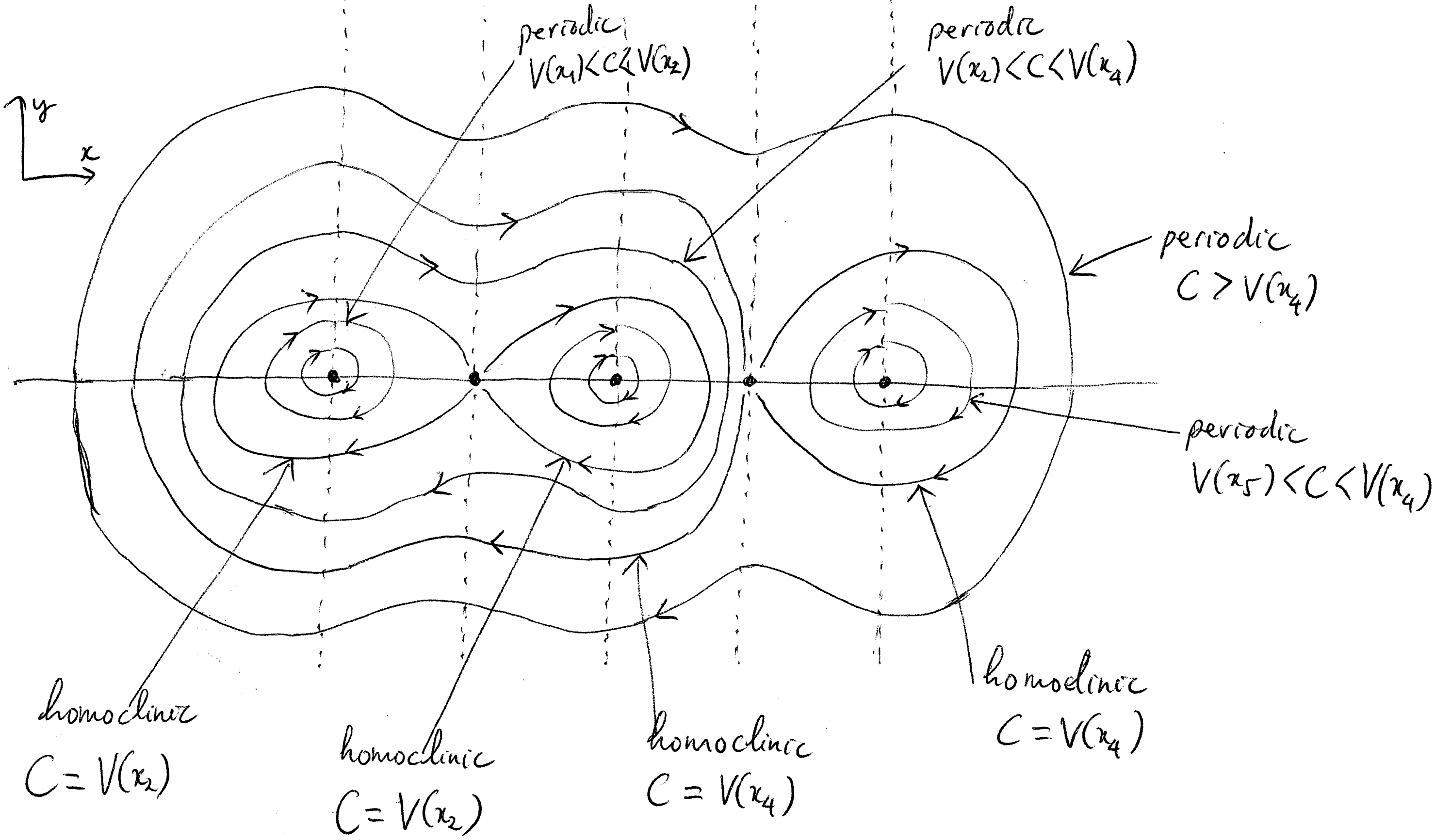
has five roots $x_1 \approx -1.40$, $x_2 \approx -0.80$, $x_3 \approx -0.03$, $x_4 \approx 0.83$, $x_5 \approx 1.43$.

The signs of y' are

x		x_1		x_2		x_3		x_4		x_5	
$y' = -V'(x)$	$+$	0	$-$	0	$+$	0	$-$	0	$+$	0	$-$

The equilibria are the zeros of $f(x,y) = (y, -V'(x))$. They are $(0, x_1)$, $(0, x_2)$, $(0, x_3)$, $(0, x_4)$ and $(0, x_5)$. The values of C that help us determine the homoclinic, heteroclinic and periodic orbits are $V(x_1), V(x_2), \dots, V(x_5)$. We have $V(x_1) < V(x_3) < V(x_5) < V(x_2) < V(x_4)$.





There is no heteroclinic orbit because $V(x_2) < V(x_4)$.