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Math 8501: Differential equations & Dynamical systems

Homework #4

(2) First, we show that if $AB = BA$ then $e^{A+B} = e^A e^B$. By the definition of exponential of a matrix,

$$e^A = \lim_{n\to\infty} \underbrace{\sum_{k=0}^{n} \frac{A^k}{k!}}_{A_n}, \quad e^B = \lim_{n\to\infty} \underbrace{\sum_{j=0}^{n} \frac{B^j}{j!}}_{B_n}, \quad e^{A+B} = \lim_{m\to\infty} \underbrace{\sum_{l=0}^{m} \frac{(A+B)^l}{l!}}_{C_m}.$$

Because $AB = BA$, we have the identity $(A+B)^l = \sum_{k=0}^{l} \binom{l}{k} A^k B^{l-k}$.

Then
$$A_n B_n = \sum_{k,j=0}^{n} \frac{A^k B^j}{k!\,j!} = \sum_{l=0}^{2n} \sum_{\substack{0\le k,j\le l \\ k+j=l}} \frac{A^k B^j}{k!\,j!}$$
$$= \sum_{l=0}^{2n} \frac{1}{l!} \sum_{k=0}^{l} \binom{l}{k} A^k B^{l-k}$$
$$= \sum_{l=0}^{2n} \frac{1}{l!} (A+B)^l = C_{2n}.$$

Let $n\to\infty$, We get $(\lim A_n)(\lim B_n) = \lim C_{2n}$. In other words,
$$e^A e^B = e^{A+B}.$$

Next, we point out an example in which $e^{A+B} \neq e^A e^B$. Consider
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Because $A$ is diagonal, $e^A = \begin{pmatrix} e^1 & 0 \\ 0 & e^0 \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$.

Because $B^2 = 0$, $\quad e^B = \sum_{k=0}^{\infty} \frac{B^k}{k!} = I_2 + B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Put $C = A+B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then $C^2 = C$. Thus $C^k = C$ for all $k \geq 1$.

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$$e^C = I_2 + \sum_{k=1}^{\infty} \frac{C^k}{k!} = I_2 + C\sum_{k=1}^{\infty}\frac{1}{k!} = I_2 + (e-1)C = \begin{pmatrix} 1 & e-1 \\ 0 & 1 \end{pmatrix}.$$

We see that
$$e^A e^B = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e & e \\ 0 & 1 \end{pmatrix} \neq e^C.$$

Next, we give an example in which $e^{A+B} = e^A e^B$ and $AB \neq BA$. Consider
$$A = \begin{pmatrix} 0 & -2\pi \\ 2\pi & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -2\pi \\ 98\pi & 0 \end{pmatrix}.$$

To compute $e^A, e^B, e^{A+B}$ conveniently, we need the following observation: A real $2\times 2$ matrix $C$ can be considered as a complex matrix. Suppose $C$ has simple complex eigenvalues and that they are integer multiples of $2\pi i$. We know that $C$ can be diagonalized by an invertible matrix $P$.

$$\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} = P^{-1} C P$$

Taking the exponential of both sides, we get

$$\underbrace{\begin{pmatrix} e^{\lambda_1} & & \\ & e^{\lambda_2} & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix}}_{= I_n} = P^{-1} e^C P.$$

Thus, $e^C = I_n$. Now we apply this observation for $C = A, B, A+B$. The eigenvalues of $A$ are $\pm 2\pi i$, of $B$ are $\pm 14\pi i$, of $A+B$ are $\pm 20\pi i$. All of them are integer multiples of $2\pi i$. Thus, $e^A = e^B = e^{A+B} = I_2$. The identity $e^{A+B} = e^A e^B$ is satisfied. However,

$$AB = \begin{pmatrix} -(14\pi)^2 & 0 \\ 0 & -(2\pi)^2 \end{pmatrix} \neq BA = \begin{pmatrix} -(2\pi)^2 & 0 \\ 0 & -(14\pi)^2 \end{pmatrix}.$$

(3) Let $E$ be a finite dimensional vector space over a field $K$ which is either $\mathbb{R}$ or $\mathbb{C}$. Let $\phi: \mathbb{R}\times E \to E$ be a linear hyperbolic flow. That is $\phi_t(x_0) = e^{tA}x_0$ where $t\in\mathbb{R}$, $x_0\in E$, $A\in\mathcal{L}(E)$ such that all complex eigenvalues of $A$ have nonzero real parts. Theorem 13.4 in Amann "Ordinary Differential Equations" 1990, page 175, states that

(i) $E$ decomposes into $A$-invariant subspaces $E = E_s \oplus E_u$,

(ii) All complex eigenvalues of $A|_{E_s}$ have negative real parts,

(iii) All complex eigenvalues of $A|_{E_u}$ has positive real parts.

Moreover, we can describe $E_s$ and $E_u$ as follows.

- For $K=\mathbb{C}$: let $p_A(z) = (z-\lambda_1)^{r_1}\cdots(z-\lambda_m)^{r_m}$ be the characteristic polynomial of $A$. We have the decomposition $E = \bigoplus_{j=1}^{m} \ker[(A-\lambda_j Id_E)^{r_j}]$. Each summand is an $A$-invariant subspace of $E$. Suppose $Re(\lambda_1),\dots,Re(\lambda_k)<0$ and $Re(\lambda_{k+1}),\dots,Re(\lambda_m)>0$. Then

$$E_s = \bigoplus_{j=1}^{k} \ker[(A-\lambda_j Id_E)^{r_j}],$$

$$E_u = \bigoplus_{j=k+1}^{m} \ker[(A-\lambda_j Id_E)^{r_j}].$$

- For $K=\mathbb{R}$: let $E_{\mathbb{C}}$ be the complexification of $E$. For example, $E_{\mathbb{C}} = \mathbb{C}^n$ if $E=\mathbb{R}^n$. Then $E_s = (E_{\mathbb{C}})_s \cap E$ and $E_u = (E_{\mathbb{C}})_u \cap E$.

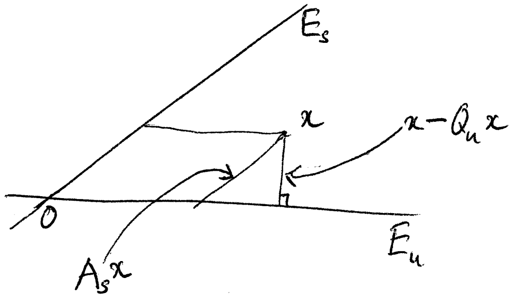
Return to the problem. We show that $E_u$ is an invariant set with respect to the flow $\phi$. Because $E_u$ is $A$-invariant, $A(E_u)\subset E_u$. Because $E_u$ is a closed vector subspace of $E$ and $e^{tA} = Id_E + \sum_{j=1}^{\infty} \frac{t^j}{j!}A^j$, we have $e^{tA}(E_u)\subset E_u$ for all $t\in\mathbb{R}$. In other words, $\phi_t(E_u)\subset E_u$ for all $t\in\mathbb{R}$.

Thus, $E_u$ is an invariant set. Similarly, so is $E_s$. Next, we show that $E_u$ is asymptotically stable with respect to the flow $\phi$. The direct sum $E = E_s \oplus E_u$ gives us two linear projection maps $P_s : E \to E_s$ and $P_u : E \to E_u$. Put $A_s = A \circ P_s$ and $A_u = A \circ P_u$. Then $A = A_s + A_u$. Because

$$A_s(E) = A(P_s(E)) = A(E_s) \subset E_s,$$

$$A_u(E) = A(P_u(E)) = A(E_u) \subset E_u,$$

we have $A_s \circ A_u = A_u \circ A_s = 0$.




Let $Q_u : E \to E_u$ be the othogonal projection onto $E_u$. We show there exists a number $C > 0$ such that

$$|A_s x| \le C|x - Q_u x| \quad \forall x \in E.$$

To do so, we show that $A_s x$ is a function of $x - Q_u x$. Suppose there are $x_1, x_2 \in E$ such that $x_1 - Q_u x_1 = x_2 - Q_u x_2$. Then $0 = x_1 - x_2 - Q_u(x_1 - x_2)$. Thus $x_1 - x_2 \in E_u$. Then $A_s(x_1 - x_2) = 0$, i.e. $A_s x_1 = A_s x_2$. Thus, $A_s x$ is a function of $x - Q_u x$. This function is also linear. Write $A_s x = T(x - Q_u x)$ for $x \in E$. Then

$$|A_s x| \le \underbrace{\|T\|}_{=C} |x - Q_u x| \qquad \forall x \in E.$$

Put $\mathcal{O} = \{x \in E : \mathrm{dist}(x, E_u) < 1\}$. We show that $\lim_{t\to\infty} \mathrm{dist}(\phi_t(x), E_u) = 0$ for all $x \in \mathcal{O}$. Fix $x_0 \in E$. Since $x_0 \in E$, $x_0 = x_1 + x_2$ for some $x_1 \in E_s$ and $x_2 \in E_u$. Then $|x_1| = |A_s x_0| \le C|x_0 - Q_u x_0| = C\,\mathrm{dist}(x_0, E_u) < C$. We have

$$\phi_t(x_0) = e^{tA} x_0 = e^{tA} x_1 + e^{tA} x_2 = x_1 + \sum_{k=1}^{\infty} \frac{t^k (A_s + A_u)^k}{k!} x_1 + \phi_t(x_2). \qquad (1)$$

Because $A_s A_u = A_u A_s = 0$ and $A_u x_1 = 0$, we have $(A_s + A_u)^k x_1 = A_s^k x_1$ for all $k \geq 1$. Then (1) becomes

$$\begin{aligned}\phi_t(x_0) &= x_1 + \sum_{k=1}^{\infty} \frac{t^k A_s^k}{k!} x_1 + \phi_t(x_2) \\ &= e^{tA_s} x_1 + \phi_t(x_2) \\ &= e^{t\tilde{A}_s} x_1 + \underbrace{\phi_t(x_2)}_{\in E_u \text{ since } x_2 \in E_u.}\end{aligned}$$

where $\tilde{A}_s = A_s|_{E_s} : E_s \to E_s$. Thus,

$$\operatorname{dist}(\phi_t(x_0), E_u) \leq |e^{t\tilde{A}_s} x_1| \leq \|e^{t\tilde{A}_s}\| \, |x_1| \leq C \|e^{t\tilde{A}_s}\|. \qquad (2)$$

We show that all complex eigenvalues of $\tilde{A}_s$ have negative real parts. For $y \in E_s$, $\tilde{A}_s y = A(P_s y) = Ay$. If $\lambda$ is a complex eigenvalue of $\tilde{A}_s$, it is also a complex eigenvalue of $A$; thus $\lambda \neq 0$. Let $y$ be an eigenvector of $\tilde{A}_s$ associate with $\lambda$. Then $\lambda y = \tilde{A}_s y = Ay \in E_s$. Then $y$ is an eigenvector of $A$ lying in $E_s$. Then $\operatorname{Re}(\lambda) < 0$. Hence all complex eigenvalues of $\tilde{A}_s$ have negative real parts. By Lemma 13.1, Amann "Ordinary Differential Equations" 1990, page 172, there exists a number $\mu > 0$ such that $\|e^{t\tilde{A}_s}\| \leq e^{-\mu t}$ for all $t \geq 0$. Then (2) gives us

$$0 \leq \operatorname{dist}(\phi_t(x_0), E_u) \leq C e^{-\mu t} \quad \forall t \geq 0.$$

Hence, $\lim_{t \to \infty} \operatorname{dist}(\phi_t(x_0), E_u) = 0$.

4 Consider the Lotka-Volterra model for competing species

$$u' = u(1 - u - bv)$$
$$v' = v(1 - v - au)$$

where $0 < a \leq b$. This system can be written as $(u,v)' = f(u,v)$ where

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$$f(x,y) = (x(1-x-by), y(1-y-ax)) \quad \forall x,y \in \mathbb{R}.$$

(i) We find all equilibria $u,v \geq 0$. The equation $f(u_0,v_0)=0$ is solved in 4 cases.

- $u_0 = 0,\ v_0 = 0$.
- $u_0 = 0,\ 1 - v_0 - au_0 = 0$.
- $1 - u_0 - bv_0 = 0,\ v_0 = 0$.
- $1 - u_0 - bv_0 = 0,\ 1 - v_0 - au_0 = 0$.

In the second case, $u_0 = 0$ and $v_0 = 1$. In the third case, $u_0 = 1$ and $v_0 = 0$. The ~~ford~~ fourth case is equivalent to

$$\begin{cases} u_0 + bv_0 = 1, \\ au_0 + v_0 = 1. \end{cases} \qquad (*)$$

Consider $ab \neq 1$. Then $u_0 = \dfrac{1-b}{1-ab}$ and $v_0 = \dfrac{1-a}{1-ab}$. Since we want $u_0, v_0 \geq 0$, this pair is admissible if and only if $0 < a \leq b \leq 1$ or $1 \leq a \leq b$.

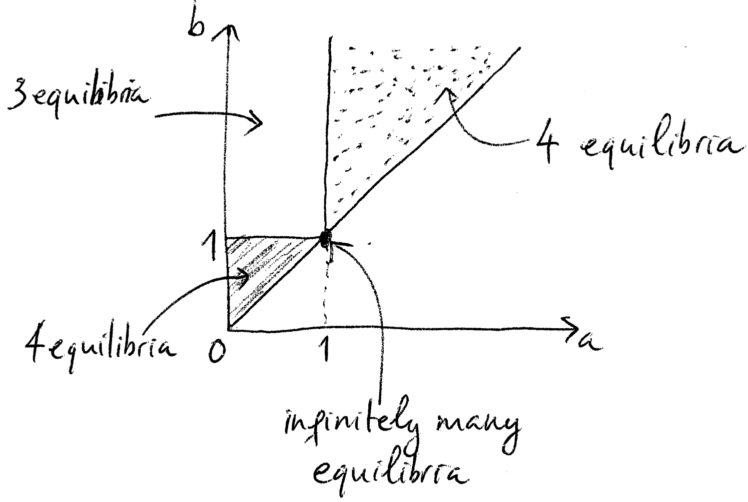
$$(u_0,v_0) = (0,1) \Leftrightarrow b = 1.$$

$$(u_0,v_0) = (1,0) \Leftrightarrow a = 1.$$

Consider $ab = 1$. Then $(*)$ has solutions if and only if $a=b=1$. In that case, $(u_0,v_0)$ is any pair such that $u_0, v_0 \geq 0$, $u_0 + v_0 = 1$.

We conclude by a survey of equilibria according to $a,b$ $(0 < a \leq b)$.

| Cases of $a,b$ | Equilibria $(u_0,v_0)$ with $u_0,v_0 \geq 0$ |
|---|---|
| $0 < a \leq b < 1$ or $1 < a \leq b$ | $(0,0)$, $(0,1)$, $(1,0)$, $\left(\frac{1-b}{1-ab}, \frac{1-a}{1-ab}\right)$ |
| $a=b=1$ | $(0,0)$, $(u_0, 1-u_0)$ with $0 \leq u_0 \leq 1$ |
| Otherwise | $(0,0)$, $(0,1)$, $(1,0)$ |




(ii) The equilibria on $u_0$-axis or $v_0$-axis are $(0,0)$, $(0,1)$ and $(1,0)$. Put

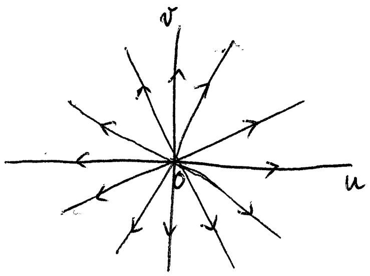
$$A = \nabla f(u_0, v_0) = \begin{pmatrix} \frac{\partial}{\partial u_0}[u_0(1-u_0-bv_0)] & \frac{\partial}{\partial v_0}[u_0(1-u_0-bv_0)] \\ \frac{\partial}{\partial u_0}[v_0(1-v_0-au_0)] & \frac{\partial}{\partial v_0}[v_0(1-v_0-au_0)] \end{pmatrix}$$

$$= \begin{pmatrix} 1-2u_0-bv_0 & -bu_0 \\ -av_0 & 1-2v_0-au_0 \end{pmatrix}.$$

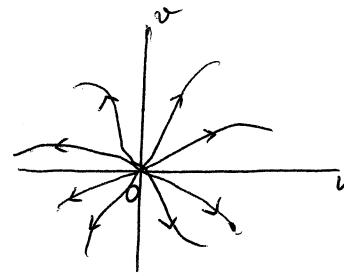
Grobman–Hartman's theorem states that if $(u_0,v_0)$ is an equilibrium and $A$ is a hyperbolic matrix (i.e. all complex eigenvalues have nonzero real parts) then the nonlinear flow $(u,v)' = f(u,v)$ in a neighborhood of $(u_0,v_0)$ is homeomorphic to the linear flow $\begin{pmatrix} u \\ v \end{pmatrix}' = A\begin{pmatrix} u \\ v \end{pmatrix}$ in a neighborhood of $(0,0)$. We consider each case of $(u_0,v_0)$ to determine the shape of the linear flow.

- $(u_0,v_0) = (0,0)$:

Then $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The linear ODE $u' = u$, $v' = v$ has a first integral $I(u,v) = \frac{u}{v}$. Indeed, $\frac{dI}{dt} = \frac{u'v - uv'}{v^2} = \frac{uv - uv}{v^2} = 0$.




Phase portrait around $(0,0)$ of $\begin{pmatrix}u\\v\end{pmatrix}' = A\begin{pmatrix}u\\v\end{pmatrix}$.




Phase portrait around $(0,0)$ of $(u,v)' = f(u,v)$ looks like this.

- $\underline{(u_0, v_0) = (0,1)}$:

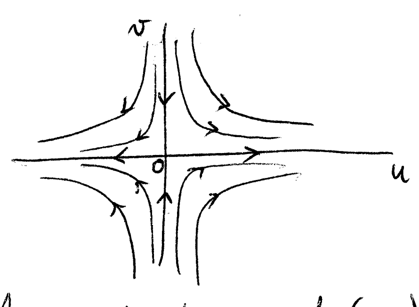
Then $A = \begin{pmatrix} 1-b & 0 \\ -a & -1 \end{pmatrix}$. It has two eigenvalues $\lambda_1 = 1-b$, $\lambda_2 = -1$.

If $b=1$ then $A$ is not hyperbolic. Thus we only consider $b \neq 1$.

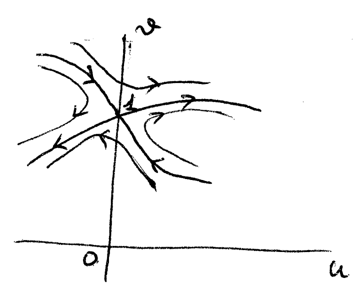
If $b \neq 2$ then $\lambda_1 \neq \lambda_2$. Then $A$ is conjugate to the matrix $\tilde{A} = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$.

The phase portrait of $\begin{pmatrix}u\\v\end{pmatrix}' = A\begin{pmatrix}u\\v\end{pmatrix}$ is thus a linear deformation of the phase portrait of $\begin{pmatrix}u\\v\end{pmatrix}' = \tilde{A}\begin{pmatrix}u\\v\end{pmatrix}$.

If $0<b<1$ then $\lambda_2 < 0 < \lambda_1$.

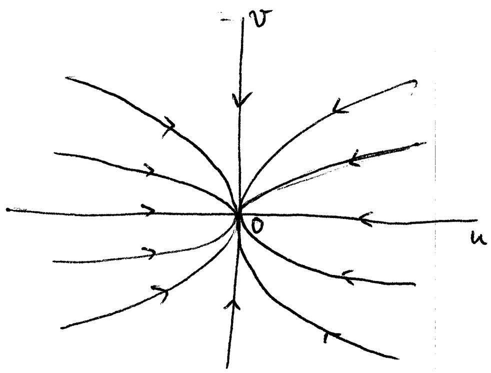



Phase portrait around $(0,0)$ of $\begin{pmatrix}u\\v\end{pmatrix}' = \tilde{A}\begin{pmatrix}u\\v\end{pmatrix}$



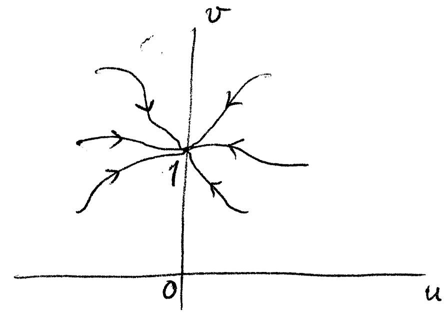

Phase portrait around $(0,1)$ of $(u,v)' = f(u,v)$ looks like this.

If $1<b<2$ or $b>2$ then $\lambda_1, \lambda_2 < 0$.




Phase portrait around $(0,0)$ of

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \tilde{A} \begin{pmatrix} u \\ v \end{pmatrix}$$

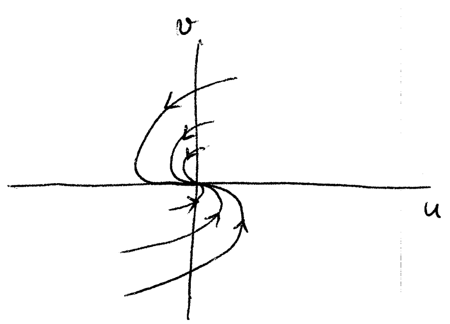



Phase portrait around $(0,1)$ of $(u,v)' = f(u,v)$ looks like this.

Consider the case $b=2$. Then $\lambda_1 = \lambda_2 = -1$, and $A = \begin{pmatrix} -1 & 0 \\ -a & -1 \end{pmatrix}$. We find the geometric multiplicity of $-1$.

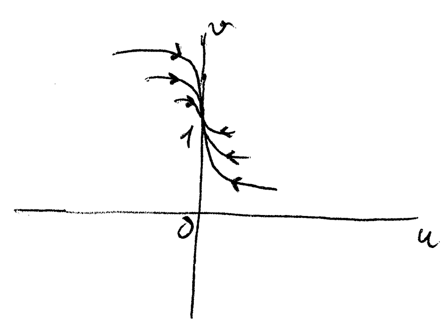
$$A + I_2 = \begin{pmatrix} 0 & 0 \\ -a & 0 \end{pmatrix} \text{ has rank } 1 \text{ because } a \neq 0.$$

Thus, the geometric multiplicity is $1$.




Phase portrait around $(0,0)$ of

$$\begin{pmatrix} u \\ v \end{pmatrix}' = A \begin{pmatrix} u \\ v \end{pmatrix}$$



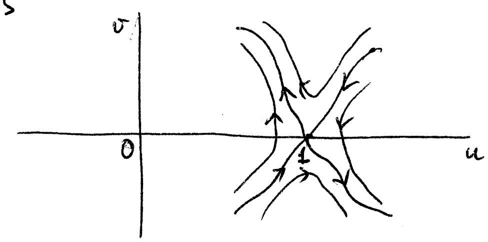

Phase portrait around $(0,1)$ of $(u,v)' = f(u,v)$ looks like this.

- $\underline{(u_0, v_0) = (1,0)}$:

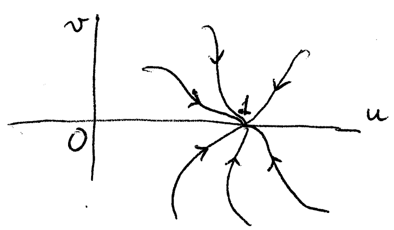
Then $A = \begin{pmatrix} -1 & -b \\ 0 & 1-a \end{pmatrix}$. The method to study this case is the same as the one used in the case $(u_0, v_0) = (0,1)$, except that $b$ is replaced by $a$.

$A$ has two eigenvalues $\lambda_1 = 1-a$ and $\lambda_2 = -1$. If $a=1$ then $A$ is not hyperbolic. This is out of our consideration. If $a \neq 2$ then $\lambda_1 \neq \lambda_2$. Then $A$ is conjugate to the matrix $\tilde{A} = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$. The phase portrait of $\begin{pmatrix} u \\ v \end{pmatrix}' = A \begin{pmatrix} u \\ v \end{pmatrix}$ is thus a linear deformation of that of $\begin{pmatrix} u \\ v \end{pmatrix}' = \tilde{A} \begin{pmatrix} u \\ v \end{pmatrix}$.

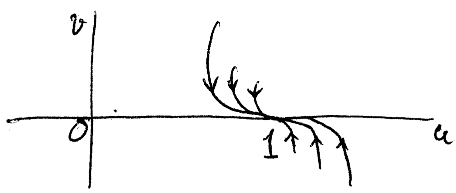
If $0 < a < 1$ then $\lambda_2 < 0 < \lambda_1$. The phase portrait around $(1,0)$ of $(u,v)' = f(u,v)$ looks like this




If $1 < a < 2$ or $a > 2$ then $\lambda_1, \lambda_2 < 0$. The phase portrait around $(1,0)$ of $(u,v)' = f(u,v)$ looks like this




If $a = 2$ then $\lambda_1 = \lambda_2 = -1$ with geometric multiplicity 1.




(iii) Consider the equilibrium $(u_0, v_0) = (1,0)$. We compute the spectral projection on the unstable and on the stable subspace. The linear flow $\begin{pmatrix} u \\ v \end{pmatrix}' = A \begin{pmatrix} u \\ v \end{pmatrix}$ resides in $E = \mathbb{R}^2$. We have $A = \nabla f(1,0) = \begin{pmatrix} -1 & -b \\ 0 & 1-a \end{pmatrix}$.

For $a \neq 1$, $A$ is a hyperbolic matrix. It has two eigenvalues $\lambda_1 = 1-a$, $\lambda_2 = -1$. To determine the stable subspace $E_s$ and the unstable subspace $E_u$ of $E$, we need to decide the sign of $\lambda_1$.

- $\underline{0<a<1}$: $\lambda_2 < 0 < \lambda_1$.

Then $E_s = \ker(A - \lambda_2 I_2)$ and $E_u = \ker(A - \lambda_1 I_2)$.

$$A - \lambda_2 I_2 = A + I_2 = \begin{pmatrix} 0 & -b \\ 0 & 2-a \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus, $E_s = \{(\alpha, 0) : \alpha \in \mathbb{R}\}$.

$$A - \lambda_1 I_2 = A - (1-a) I_2 = \begin{pmatrix} a-2 & -b \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{-b}{a-2} \\ 0 & 0 \end{pmatrix}.$$

Thus, $E_u = \left\{ \left( \frac{b\beta}{a-2}, \beta \right) : \beta \in \mathbb{R} \right\}$.

For each $(x, y) \in \mathbb{R}^2$, we find $\alpha, \beta \in \mathbb{R}$ such that

$$(x, y) = (\alpha, 0) + \left( \frac{b\beta}{a-2}, \beta \right).$$

This system of linear equations gives $\alpha = x - \dfrac{by}{a-2}$ and $\beta = y$.

Thus, the stable projection $P_s : E \to E_s$ is $P_s(x,y) = (\alpha, 0) = \left( x - \frac{by}{a-2}, 0 \right)$.

The unstable projection $P_u : E \to E_u$ is $P_u(x,y) = \left( \frac{b\beta}{a-2}, \beta \right) = \left( \frac{by}{a-2}, y \right)$.

- $\underline{1<a<2 \text{ or } a>2}$: $\lambda_1, \lambda_2 < 0$, $\lambda_1 \neq \lambda_2$

Then $E_s = \ker(A - \lambda_1 I_2) \oplus \ker(A - \lambda_2 I_2) = E$ and $E_u = \{0\}$.

The stable projection is $P_s = id_E$. The unstable projection is $P_u \equiv 0$.

- $\underline{a=2}$: $\lambda_1 = \lambda_2 = -1 < 0$

The algebraic multiplicity of $-1$ is equal to 2. Thus, $E_s = \ker[(A-\lambda_1 I_2)^2] = E$. Then $E_u = \{0\}$. The stable projection is $P_s = id_E$. The unstable projection is $P_u \equiv 0$.

(iv) We draw the phase portraits near the equilibria $(u_0, v_0)$ with $u_0, v_0 > 0$. According to Part (i), such equilibria exist in 3 following cases: $0 < a \leq b < 1$, $1 < a \leq b$ and $a = b = 1$.

- $\underline{0 < a \leq b < 1}$

Then $(u_0, v_0) = \left(\frac{1-b}{1-ab}, \frac{1-a}{1-ab}\right)$. Recall that $(u_0, v_0)$ satisfies $u_0 + bv_0 = 1$ and $au_0 + v_0 = 1$. We have

$$A = \nabla f(u_0, v_0) = \begin{pmatrix} 1-2u_0-bv_0 & -bu_0 \\ -av_0 & 1-2v_0-au_0 \end{pmatrix} = \begin{pmatrix} -u_0 & -bu_0 \\ -av_0 & -v_0 \end{pmatrix}.$$

The eigenvalues of $A$ solve the equation

$$(\lambda + u_0)(\lambda + v_0) - ab\, u_0 v_0 = 0$$

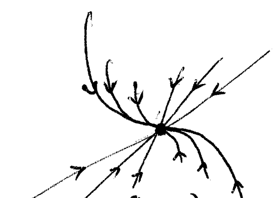
$$\Leftrightarrow \lambda^2 + (u_0 + v_0)\lambda + (1-ab) u_0 v_0 = 0 \qquad (1)$$

Because $u_0 + v_0 > 0$ and $(1-ab) u_0 v_0 > 0$, the two roots of (1) have negative real parts. Thus, $A$ is hyperbolic. Moreover, the origin of $E = \mathbb{R}^2$ is a sink of the flow $\binom{u}{v}' = A\binom{u}{v}$. Grobman-Hartman's theorem states that the nonlinear flow $(u,v)' = f(u,v)$ in a neighborhood of $(u_0, v_0)$ is homeomorphic to the linear flow $\binom{u}{v}' = A\binom{u}{v}$ in a neighborhood of the origin $(0,0)$. Thus,

$(u_0, v_0)$ is also a sink of the nonlinear flow. We can draw the phase portrait around $(u_0, v_0)$ by solving (forward in time) the ODEs

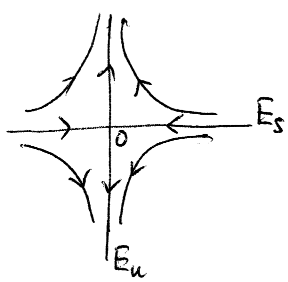
$$(u,v)' = f(u,v), \quad (u(0), v(0)) = (u_0 \pm \varepsilon_1, v_0 \pm \varepsilon_2).$$

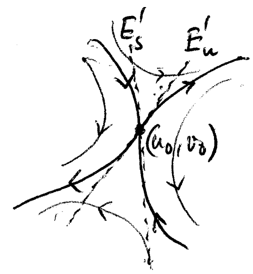
We use the command ode45 in Matlab to solve this family of ODEs with various small values of $\varepsilon_1, \varepsilon_2 > 0$ over a short period of time.




- $1 < a \leq b$

The expressions for $(u_0, v_0)$ and $A$ are the same as those in the previous case. However, Equation (1) ~~ha~~ now has two roots $\lambda_1 < 0 < \lambda_2$. The matrix $A$ is still hyperbolic, but the origin of $E$ is no longer a sink.




We have $f(u,v) \approx f(u_0, v_0) + \nabla f(u_0, v_0) \cdot (u - u_0, v - v_0)$

$$= A \begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix}.$$

Let $E_s$ and $E_u$ be the stable and unstable subspaces of $E$ with respect to the linear flow $\begin{pmatrix} u \\ v \end{pmatrix}' = A \begin{pmatrix} u \\ v \end{pmatrix}$. Then the "stable" and "unstable" subspaces of $E$ with respect to the nonlinear flow $(u,v)' = f(u,v)$ are

$$E_s' = \{(u,v) : (u-u_0, v-v_0) \in E_s\} = (u_0, v_0) + E_s,$$

$$E_u' = (u_0, v_0) + E_u.$$

We have

$$\lambda_1 = \frac{-u_0 - v_0 - \sqrt{(u_0+v_0)^2 - 4(1-ab)u_0 v_0}}{2},$$

$$\lambda_2 = \frac{-u_0 - v_0 + \sqrt{(u_0+v_0)^2 - 4(1-ab)u_0 v_0}}{2}.$$

$$A - \lambda_1 I_2 = \begin{pmatrix} -u_0 - \lambda_1 & -bu_0 \\ -av_0 & -v_0 - \lambda_1 \end{pmatrix} \sim \begin{pmatrix} -u_0 - \lambda_1 & -bu_0 \\ 0 & 0 \end{pmatrix}$$

$$A - \lambda_2 I_2 = \begin{pmatrix} -u_0 - \lambda_2 & -bu_0 \\ -av_0 & -v_0 - \lambda_2 \end{pmatrix} \sim \begin{pmatrix} -u_0 - \lambda_2 & -bu_0 \\ 0 & 0 \end{pmatrix}.$$

Thus, $E_s = \ker(A - \lambda_1 I_2) = \text{linear span}\,\{(bu_0, -u_0 - \lambda_1)\}$.

$E_u = \ker(A - \lambda_2 I_2) = \text{linear span}\,\{(bu_0, -u_0 - \lambda_2)\}$.

Hence, $E_s'$ is the line passing through $(u_0, v_0)$ and parallel to the vector $\gamma_s = (bu_0, -u_0 - \lambda_1)$. $E_u'$ is the line passing through $(u_0, v_0)$ and parallel to the vector $\gamma_u = (bu_0, -u_0 - \lambda_2)$.

On the nonlinear flow that passes through $(u_0, v_0)$ and is tangent to $E_s'$, $(u_0, v_0)$ is a sink. Thus, we can find this curve in a neighborhood of $(u_0, v_0)$ by solving (forward in time) the ODEs

$$(u,v)' = f(u,v), \quad (u(0), v(0)) = (u_0, v_0) \pm \varepsilon \gamma_s$$

for various small values $\varepsilon > 0$.

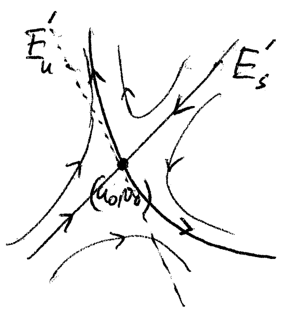
On the nonlinear flow that passes through $(u_0, v_0)$ and is tangent to $E'_u$, $(u_0, v_0)$ is a source. Thus, we can find this curve in a neighborhood of $(u_0, v_0)$ by solving backward in time the ODEs

$$(u,v)' = f(u,v), \quad (u(0), v(0)) = (u_0, v_0) \pm \varepsilon \gamma_u$$

for various small values $\varepsilon > 0$. Note that by reparametrizing the time $t \mapsto -t$, it is the same as solving forward in time the ODEs

$$(u,v)' = -f(u,v), \quad (u(0), v(0)) = (u_0, v_0) \pm \varepsilon \gamma_u.$$

Using Matlab, we see that the flow tangent to $E'_s$ at $(u_0, v_0)$ looks like a straight line.




- $\underline{a = b = 1}$:

Then $(u_0, v_0)$ can be any point such that $0 < u_0 < 1$, $v_0 = 1 - u_0$.

$$A = \begin{pmatrix} -u_0 & -u_0 \\ -v_0 & -v_0 \end{pmatrix}$$

The eigenvalues of $A$ solve the equation

$$(\lambda + u_0)(\lambda + v_0) - u_0 v_0 = 0$$

$$\Leftrightarrow \lambda^2 + (u_0 + v_0)\lambda = 0$$

$$\Leftrightarrow \lambda(\lambda + 1) = 0.$$

They are $\lambda_1 = -1$ and $\lambda_2 = 0$. Then $A$ is not a hyperbolic matrix. We stop because Grobman–Hartman's theorem is not applicable in this case. In other words, we don't know how to describe the nonlinear flow $(u,v)' = f(u,v)$ via the linear flow $\begin{pmatrix} u \\ v \end{pmatrix}' = A\begin{pmatrix} u \\ v \end{pmatrix}$.