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Math 8501: Differential Equations & Dynamical Systems
Homework #5

① Let $C^0(\mathbb{R}, \mathbb{R}^n)$ be the space of all continuous bounded functions from $\mathbb{R}$ to $\mathbb{R}^n$. Define

$$C^1(\mathbb{R}, \mathbb{R}^n) = \{ u : \mathbb{R} \to \mathbb{R}^n \mid u, u' \in C^0(\mathbb{R}, \mathbb{R}^n) \}.$$

Then $C^0(\mathbb{R}, \mathbb{R}^n)$ and $C^1(\mathbb{R}, \mathbb{R}^n)$ are Banach spaces with norms

$$\|u\|_{C^0} = \sup_{t \in \mathbb{R}} |u(t)|,$$

$$\|u\|_{C^1} = \sup_{t \in \mathbb{R}} \left(|u(t)| + |u'(t)| \right).$$

(i) Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear hyperbolic transformation. We show that the operator $L : C^1(\mathbb{R}, \mathbb{R}^n) \to C^0(\mathbb{R}, \mathbb{R}^n)$, $Lu = \frac{du}{dt} - Au$ has a continuous inverse.

First, we show that L is injective. Let $u \in \ker L$. Then $\frac{du}{dt} = Au$, which gives $u(t) = e^{At} u(0)$ for all $t \in \mathbb{R}$. Because A is hyperbolic, $\mathbb{R}^n = E_s \oplus E_u$ where E_s and E_u are the stable and unstable subspaces with respect to the flow e^{tA}. They are A-invariant subspaces of $\mathbb{R}^n$. If viewed as a flow on E_s, e^{tA} is a contraction. If viewed as a flow on E_u, e^{tA} is an expansion. By Theorem 13.3, Amann "Ordinary Differential Equations" 1990, page 174, there exist numbers

$C_1, C_2 > 0$ such that

$$|e^{tA}u_1| \leq C_1 e^{-\alpha t}|u_1| \qquad \forall t \geq 0, u_1 \in E_s, \qquad (1)$$

$$|e^{tA}u_1| \geq C_1 e^{-\alpha t}|u_1| \qquad \forall t \leq 0, u_1 \in E_s, \qquad (2)$$

$$|e^{tA}u_2| \geq C_2 e^{\alpha t}|u_2| \qquad \forall t \geq 0, u_2 \in E_u, \qquad (3)$$

$$|e^{tA}u_2| \leq C_2 e^{\alpha t}|u_2| \qquad \forall t \leq 0, u_2 \in E_u, \qquad (4)$$

where $\alpha = \frac{1}{2} \min_{\lambda \in \sigma(A)} |\lambda| > 0$.

Let $P_s : \mathbb{R}^n \to E_s$ and $P_u : \mathbb{R}^n \to E_u$ be the linear projection maps induced by the direct sum $\mathbb{R}^n = E_s \oplus E_u$. Take $u_1 = P_s(u(0)) \in E_s$ and $u_2 = P_u(u(0)) \in E_u$. Then

$$|u(t)| = |e^{tA}u(0)| = |e^{tA}u_1 + e^{tA}u_2| \geq |e^{tA}u_1| - |e^{tA}u_2|$$

$$\overset{(2),(4)}{\geq} C_1 e^{-\alpha t}|u_1| - C_2 e^{\alpha t}|u_2| \quad \forall t \leq 0$$

Because $u((-\infty, 0])$ is bounded, $u_1 = 0$. Similarly,

$$|u(t)| = |e^{tA}u_1 + e^{tA}u_2| \geq |e^{tA}u_2| - |e^{tA}u_1|$$

$$\overset{(1),(3)}{\geq} C_2 e^{\alpha t}|u_2| - C_1 e^{-\alpha t}|u_1| \qquad \forall t \geq 0.$$

Because $u([0,\infty))$ is bounded, $u_2 = 0$. Therefore, $u = u_1 + u_2 = 0$. We have showed that L is injective.

Next, we show that L is surjective. Let $\varphi \in C^0(\mathbb{R}, \mathbb{R}^n)$. We are to solve for $u \in C^1(\mathbb{R}, \mathbb{R}^n)$ the equation

$$\frac{du}{dt} - Au = \varphi(t) \quad \forall t \in \mathbb{R}. \qquad (5)$$

Because L is injective, the above equation has at most one solution.

It suffices to verify that

$$u(t) = \underbrace{\int_{-\infty}^{t} e^{A(t-\tau)} P_s(\varphi(\tau))\, d\tau}_{\{1\}} + \underbrace{\int_{\infty}^{t} e^{A(t-\tau)} P_u(\varphi(\tau))\, d\tau}_{\{2\}} \quad \forall t \in \mathbb{R}$$

✓

belongs to $C^1(\mathbb{R}, \mathbb{R}^n)$ and satisfies (5). For $\tau \in (-\infty, t]$, the estimation at (1) yields

$$\left| e^{A(t-\tau)} P_s(\varphi(\tau)) \right| \le C_1 e^{-\alpha(t-\tau)} |P_s(\varphi(\tau))| \le C_1 e^{-\alpha(t-\tau)} \|P_s\| \, |\varphi(\tau)|$$
$$\le C_1 e^{-\alpha(t-\tau)} \|P_s\| \, \|\varphi\|_{C^0}. \qquad (6)$$

Thus, $|\{1\}| \le \int_{-\infty}^{t} C_1 \|P_s\| e^{-\alpha(t-\tau)} \|\varphi\|_{C^0} d\tau = C_1 \|P_s\| \|\varphi\|_{C^0} \int_{-\infty}^{t} e^{-\alpha(t-\tau)} d\tau$

$$= \frac{1}{\alpha} C_1 \|P_s\| \, \|\varphi\|_{C^0}. \qquad (7)$$

For $\tau \in [t, \infty)$, the estimation at (4) yields

$$\left| e^{A(t-\tau)} P_u(\varphi(\tau)) \right| \le C_2 e^{\alpha(t-\tau)} |P_u(\varphi(\tau))| \le C_2 e^{\alpha(t-\tau)} \|P_u\| \, |\varphi(\tau)|$$
$$\le C_2 \|P_u\| e^{\alpha(t-\tau)} \|\varphi\|_{C^0}. \qquad (8)$$

Then $|\{2\}| \le \int_{t}^{\infty} C_2 \|P_u\| e^{\alpha(t-\tau)} \|\varphi\|_{C^0} d\tau = C_2 \|P_u\| \|\varphi\|_{C^0} \int_{t}^{\infty} e^{\alpha(t-\tau)} d\tau$

$$= \frac{1}{\alpha} C_2 \|P_u\| \|\varphi\|_{C^0}. \qquad (9)$$

By (7) and (9) we get

$$|u(t)| \leq |\{1\}| + |\{2\}| \leq \underbrace{\frac{1}{\alpha}\left(C_1 \|P_s\| + C_2 \|P_u\|\right)}_{C_3} \|\varphi\|_{C^0} \quad \forall t \in \mathbb{R}$$

Thus, $u \in C^0(\mathbb{R}, \mathbb{R}^n)$ and $\|u\|_{C^0} \leq C_3 \|\varphi\|_{C^0}$.

Now we show that $u' \in C^0(\mathbb{R}, \mathbb{R}^n)$. By the chain rule of differentiation,

$$\frac{d\{1\}}{dt} = P_s(\varphi(t)) + \underbrace{\int_{-\infty}^{t} e^{A(t-\tau)} A P_s(\varphi(\tau))\, d\tau}_{\{3\}} \qquad (10)$$

$$\frac{d\{2\}}{dt} = P_u(\varphi(t)) + \underbrace{\int_{\infty}^{t} e^{A(t-\tau)} A P_u(\varphi(\tau))\, d\tau}_{\{4\}} \qquad (11)$$

For $\tau \in (-\infty, t]$, the estimation at (6) gives

$$\left| e^{A(t-\tau)} A P_s(\varphi(\tau)) \right| \leq C_1 \|A\| \|P_s\| e^{-\alpha(t-\tau)} \|\varphi\|_{C^0}.$$

Thus,

$$|\{3\}| \leq \int_{-\infty}^{t} C_1 \|A\| \|P_s\| e^{-\alpha(t-\tau)} \|\varphi\|_{C^0}\, d\tau = \frac{1}{\alpha} C_1 \|A\| \|P_s\| \|\varphi\|_{C^0}. \qquad (12)$$

For $\tau \in [t, \infty)$, the estimation at (8) gives

$$\left| e^{A(t-\tau)} A P_u(\varphi(\tau)) \right| \leq C_2 \|A\| \|P_u\| e^{\alpha(t-\tau)} \|\varphi\|_{C^0}.$$

Thus,

$$|\{4\}| \leq \int_{t}^{\infty} C_2 \|A\| \|P_u\| e^{\alpha(t-\tau)} \|\varphi\|_{C^0}\, d\tau = \frac{1}{\alpha} C_2 \|A\| \|P_u\| \|\varphi\|_{C^0}. \qquad (13)$$

Replacing (12) into (10), we get

$$\left|\frac{d\{1\}}{dt}\right| \leq |P_s(\varphi(t))| + |\{3\}| \leq \|P_s\| \|\varphi\|_{C^0} + \frac{1}{\alpha} C_1 \|A\| \|P_s\| \|\varphi\|_{C^0}.$$

Replacing (13) into (11), we get

$$\left|\frac{d\{2\}}{dt}\right| \leq |P_u(\varphi(t))| + |\{4\}| \leq \|P_u\| \|\varphi\|_{C^0} + \frac{1}{\alpha} C_2 \|A\| \|P_u\| \|\varphi\|_{C^0}.$$

Thus,

$$\left|\frac{du}{dt}\right| \leq \left|\frac{d\{1\}}{dt}\right| + \left|\frac{d\{2\}}{dt}\right| \leq \Big(\underbrace{\|P_s\| + \|P_u\| + \frac{1}{\alpha} C_1 \|A\| \|P_s\| + \frac{1}{\alpha} C_2 \|A\| \|P_u\|}_{C_4}\Big) \|\varphi\|_{C^0} \quad \forall t \in \mathbb{R}.$$

Hence $u' \in C^0(\mathbb{R}, \mathbb{R}^n)$ and $\|u'\|_{C^0} \leq C_4 \|\varphi\|_{C^0}$. We have showed that $u \in C^1(\mathbb{R}, \mathbb{R}^n)$.

Moreover,

$$\|u\|_{C^1} = \|u + u'\|_{C^0} \leq \|u\|_{C^0} + \|u'\|_{C^0} \leq (C_3 + C_4) \|\varphi\|_{C^0}. \qquad (14)$$

Summing (10) and (11), we get

$$\frac{d\{1\}}{dt} + \frac{d\{2\}}{dt} = P_s(\varphi(t)) + P_u(\varphi(t)) + A\left(\int_{-\infty}^{t} e^{A(t-\tau)} P_s(\varphi(\tau))\,d\tau + \int_{\infty}^{t} e^{A(t-\tau)} P_u(\varphi(\tau))\,d\tau\right)$$

$$= \varphi(t) + Au(t).$$

This means $\frac{du}{dt} - Au = \varphi(t)$.

We have showed that L is surjective. Thus, L is bijective and

$$L^{-1}\varphi(t) = \int_{-\infty}^{t} e^{A(t-\tau)} P_s(\varphi(\tau))\,d\tau + \int_{\infty}^{t} e^{A(t-\tau)} P_u(\varphi(\tau))\,d\tau \quad \forall t \in \mathbb{R}.$$

The estimation at (14) says that L^{-1} is a continuous map from $C^0(\mathbb{R}, \mathbb{R}^n)$ to $C^1(\mathbb{R}, \mathbb{R}^n)$.

(ii) Let $f: \mathbb{R}^n \to \mathbb{R}^n$ and $g: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable maps. Suppose $f(0) = 0$ and $A = Df(0)$ is a hyperbolic transformation on $\mathbb{R}^n$.

We are asked to show that there exists a small neighborhood $\mathcal{U}(0)$ in $\mathbb{R}^n$ such that for all $\varepsilon > 0$ sufficiently small, there exists a unique solution $u(t)$ to the problem

$$\frac{du}{dt} = f(u) + \varepsilon g(t,u) \quad \forall t \in \mathbb{R} \qquad (15)$$

such that $u(t) \in \mathcal{U}(0)$ for all $t \in \mathbb{R}$. However, this is true only if certain boundedness condition of function g with respect to t is satisfied. Consider the following example: $n = 1$, $f(x) = x$, $g(t,x) = t$. Equation (15) becomes

$$\frac{du}{dt} = u + \varepsilon t \quad \forall t \in \mathbb{R},$$

which gives us solutions $u(t) = Ce^t - \varepsilon t - \varepsilon$. For every choice of constant $C \in \mathbb{R}$, u is an unbounded function.

We add, for instance, the following conditions on g.

(a) $\displaystyle\sup_{\substack{x \in \mathbb{R}^n, |x| \le 1 \\ t \in \mathbb{R}}} |g(t,x)| < \infty$,

✓

(b) $|D_x g(t,x_1) - D_x g(t,x_2)| \le \omega(|x_1 - x_2|) \quad \forall t \in \mathbb{R}, \forall x_1, x_2 \in \mathbb{R}^n, |x_1|, |x_2| \le 2$,

where $\lim_{h \to 0} \omega(h) = 0$.

Put $\displaystyle M = \sup_{x \in \mathbb{R}^n, |x| \le 1} |f(x)| + \sup_{\substack{x \in \mathbb{R}^n, |x| \le 1 \\ t \in \mathbb{R}}} |g(t,x)| < \infty$.

Denote $B_1 = \{u \in C^1(\mathbb{R}, \mathbb{R}^n) : \|u\|_{C^1} < 1\}$. Define a map $F: B_1 \times (-1,1) \to C^0(\mathbb{R}, \mathbb{R}^n)$

$$F(u,\varepsilon) = \frac{du}{dt} - f(u) - \varepsilon g(t,u).$$

First, we show that F is well-defined. That is to show $F(u,\varepsilon) \in C^0(\mathbb{R},\mathbb{R}^n)$ for all $u \in B_1$, $\varepsilon \in (-1,1)$. Since $u \in B_1$, $\left\| \frac{du}{dt} \right\| \leq 1$ and ✓

$$|f(u(t))| \leq M \qquad \forall t \in \mathbb{R},$$
$$|g(t,u(t))| \leq M \qquad \forall t \in \mathbb{R}.$$

Thus, $\|F(u,\varepsilon)\|_{C^0} \leq \left\| \frac{du}{dt} \right\|_{C^0} + \|f(u)\|_{C^0} + |\varepsilon| \, \|g(\cdot,u(\cdot))\|_{C^0} \leq 1 + M + M < \infty$.

Next, we show that F is continuously differentiable. For $u, v \in B_1$ and $\varepsilon, \delta \in (-1,1)$ such that $u+v \in B_1$, $\varepsilon + \delta \in (-1,1)$, we have

$$\begin{aligned} F(u+v,\varepsilon+\delta) &= \frac{du}{dt} + \frac{dv}{dt} - f(u+v) - (\varepsilon+\delta) g(t,u+v) \\ &= \frac{du}{dt} + \frac{dv}{dt} - \left[f(u) + \int_0^1 Df(u+sv).v \, ds \right] \\ &\qquad - (\varepsilon+\delta)\left[g(t,u) + \int_0^1 D_x g(t,u+sv).v \, ds \right] \\ &= F(u,\varepsilon) + \frac{dv}{dt} - \int_0^1 Df(u+sv).v \, ds - \varepsilon \int_0^1 D_x g(t,u+sv).v \, ds \\ &\qquad - \delta g(t,u) - \delta \int_0^1 D_x g(t,u+sv).v \, ds. \end{aligned}$$

Put $\tilde{L}_{u,\varepsilon}(v,\delta) = \frac{dv}{dt} - Df(u).v - \varepsilon D_x g(t,u).v - \delta g(t,u)$. (16)

This is a linear continuous map from $C^1(\mathbb{R},\mathbb{R}^n) \times \mathbb{R}$ to $C^0(\mathbb{R},\mathbb{R}^n)$. Then

$$F(u+v,\varepsilon+\delta) = F(u,\varepsilon) + \tilde{L}_{u,\varepsilon}(v,\delta) - \underbrace{\int_0^1 [Df(u+sv) - Df(u)].v \, ds}_{\{1\}} -$$

$$- \underbrace{\varepsilon \int_0^1 [D_x g(t,u+sv) - D_x g(t,u)].v \, ds}_{\{2\}} - \underbrace{\delta \int_0^1 [D_x g(t,u+sv) - D_x g(t,u)].v \, ds}_{\{3\}} \qquad (17)$$

We need to show $\dfrac{\|\{1\}\|_{C^0}}{\|v\|_{C^1}+|\delta|}$, $\dfrac{\|\{2\}\|_{C^0}}{\|v\|_{C^1}+|\delta|}$, $\dfrac{\|\{3\}\|_{C^0}}{\|v\|_{C^1}+|\delta|} \longrightarrow 0$ as $\|v\|_{C^1}\to 0$

and $|\delta| \to 0$.

$$|\{1\}| \le \|v\|_{C^0} \int_0^1 |Df(u+sv) - Df(u)| \, ds \le \|v\|_{C^1} \sup_{\substack{|x|\le 1 \\ |y| \le \|v\|_{C^1}}} |Df(x+y) - Df(x)| .$$

Thus, $$\frac{\|\{1\}\|_{C^0}}{\|v\|_{C^1}+|\delta|} \le \frac{\|\{1\}\|_{C^0}}{\|v\|_{C^1}} \le \sup_{\substack{|x|\le 1 \\ |y| \le \|v\|_{C^1}}} |Df(x+y) - Df(x)| .$$

The quantity on the right hand side goes to 0 as $\|v\|_{C^1} \to 0$ because Df is uniformly continuous on every compact subset of $\mathbb{R}^n$.

$$|\{2\}| \le \|v\|_{C^0} \int_0^1 |D_x g(t,u+sv) - D_x g(t,u)| \, ds$$

$$\le \|v\|_{C^1} \sup_{\substack{|x|\le 1 \\ |y| \le \|v\|_{C^1}}} |D_x g(t,x+y) - D_x g(t,x)| \le \|v\|_{C^1} \sup_{|y| \le \|v\|_{C^1}} \omega(y).$$

Then $$\frac{\|\{2\}\|_{C^0}}{\|v\|_{C^1}+|\delta|} \le \frac{\|\{2\}\|_{C^0}}{\|v\|_{C^1}} \le \sup_{|y|\le \|v\|_{C^1}} \omega(y) \longrightarrow 0 \quad \text{as } \|v\|_{C^1} \to 0.$$

We have

$$|\{3\}| \le |\delta| \, \|v\|_{C^0} \int_0^1 |D_x g(t, u+sv) - D_x g(t,u)| \, ds$$

$$\le |\delta| \, \|v\|_{C^1} \sup_{\substack{|x|\le 1 \\ |y| \le \|v\|_{C^1}}} |D_x g(t,x+y) - D_x g(t,x)| \le |\delta| \, \|v\|_{C^1} \sup_{|y| \le \|v\|_{C^1}} \omega(y).$$

Thus,

$$\frac{\|\{3\}\|_{C^0}}{\|v\|_{C^1}+|\delta|} \leq \frac{|\delta|\,\|v\|_{C^1}}{\|v\|_{C^1}+|\delta|} \sup_{|y|\leq\|v\|_{C^1}} \omega(y) \leq \sup_{|y|\leq\|v\|_{C^1}} \omega(y) \longrightarrow 0 \text{ as } \|v\|_{C^1}\to 0.$$

Therefore, F is differentiable and $DF(u,\varepsilon) = \tilde{L}_{u,\varepsilon}$. Because f and g are continuously differentiable maps, the map $(u,\varepsilon)\mapsto \tilde{L}_{u,\varepsilon}$ given by (16) is continuous. Thus, F is continuously differentiable.

Now we compute $D_u F(u,\varepsilon)$. By (17), we have

$$F(u+v,\varepsilon) = F(u,\varepsilon) + \tilde{L}_{u,\varepsilon}(v,0) - \{1\} - \{2\}.$$

Hence, $D_u F(u,\varepsilon)v = \tilde{L}_{u,\varepsilon}(v,0) = \frac{dv}{dt} - Df(u).v - \varepsilon D_x g(t,u).v.$

Then $D_u F(0,0)v = \frac{dv}{dt} - Df(0)v = \frac{dv}{dt} - Av = Lv.$

Thus, $D_u F(0,0)$ is an invertible element of $\mathcal{L}(C^1(\mathbb{R},\mathbb{R}^n), C^0(\mathbb{R},\mathbb{R}^n))$. By the Implicit Function Theorem, there exist a neighborhood $V(0)$ in $\mathbb{R}$, a neighborhood $W(0)$ in $C^1(\mathbb{R},\mathbb{R}^n)$, and a map $G: V(0)\to W(0)$ such that the solution of $F(u,\varepsilon)=0$ for $\varepsilon\in V(0)$ is precisely the graph of G. Because $W(0)$ is a neighborhood in $C^1(\mathbb{R},\mathbb{R}^n)$, there exists $\delta\in(0,1)$ such that $\{u\in C^1(\mathbb{R},\mathbb{R}^n): \|u\|_{C^1}<\delta\}\subset W(0)$. Because f is continuous and $f(0)=0$, there exists $\delta'\in(0,\frac{\delta}{4})$ such that $|f(x)|\leq\frac{\delta}{4}$ for all $x\in\mathbb{R}^n$, $|x|<\delta'$. Suppose $u\in C^1(\mathbb{R},\mathbb{R}^n)$, $\|u\|_{C^0}<\delta'$ satisfies $\dot{u}=f(u)+\varepsilon g(t,u)$. Then

$$|\dot{u}(t)| \le |f(u)| + |\varepsilon|\,|g(t,u)| \le \frac{\delta}{4} + |\varepsilon| M .$$

Because $V(0)$ is a neighborhood in $\mathbb{R}$, there exists $\varepsilon_0 \in \left(0, \frac{\delta}{4M}\right)$ such that $(-\varepsilon_0, \varepsilon_0) \subset V(0)$. For $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $|\dot{u}(t)| < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}$. Thus,

$$\|u\|_{C^1} = \|u + u'\|_{C^0} \le \|u\|_{C^0} + \|u'\|_{C^0} < \delta' + \frac{\delta}{2} < \delta,$$

which implies $u \in W(0)$. Take $\mathcal{U}(0) = \{x \in \mathbb{R}^n : |x| < \delta\}$.

For each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the equation $\dot{u} = f(u) + \varepsilon g(t,u)$ for all $t \in \mathbb{R}$ has a unique solution in $W(0)$, which is $u = G(\varepsilon)$. This solution satisfies $u(t) \in \mathcal{U}(0)$ for all $t \in \mathbb{R}$. Any other solution $\tilde{u}$ which satisfies $\tilde{u}(t) \in \mathcal{U}(0)$ for all $t \in \mathbb{R}$ must lie in $W(0)$ according to the above arguments, and thus coincides u. ✓

(iii) Assume the same setting as in Part (ii). Suppose $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is 1-periodic in the first argument, i.e.

$$g(t+1, x) = g(t, x) \quad \forall x \in \mathbb{R}^n, \forall t \in \mathbb{R}.$$

We show that when $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the unique solution $u \in C^1(\mathbb{R}, \mathbb{R}^n)$ to the problem

$$\dot{u} = f(u) + \varepsilon g(t,u) \quad \forall t \in \mathbb{R} \qquad (18)$$

is 1-periodic. That is to show $u(t+1) = u(t)$ for all $t \in \mathbb{R}$. Define a function $\tilde{u} : \mathbb{R} \to \mathbb{R}^n$, $\tilde{u}(t) = u(t+1)$. Then $\tilde{u} \in C^1(\mathbb{R}, \mathbb{R}^n)$ and

$$\frac{d\tilde{u}}{dt} = \frac{du}{dt}(t+1) = f(u(t+1)) + \varepsilon g(t+1, u(t+1))$$
$$= f(\tilde{u}(t)) + \varepsilon g(t+1, \tilde{u}(t))$$

$$= f(\tilde{u}(t)) + \varepsilon g(t, \tilde{u}(t)) \quad \forall t \in \mathbb{R}.$$

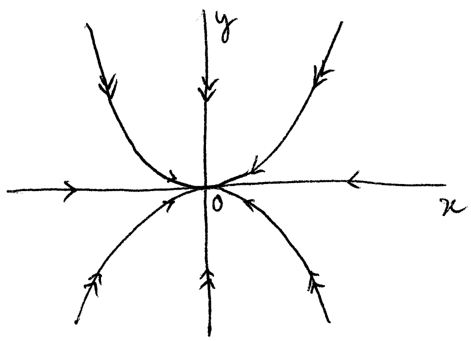
This implies $\tilde{u}$ is also a solution to Problem (18). The uniqueness of solutions suggests $\tilde{u} = u$. Therefore, u is 1-periodic. ✓

(2) Consider the system of ODE

$$\begin{cases} x' = -x + ay^2 \\ y' = -2y + bx^2 \end{cases} \qquad (I)$$

(i) The linearization of (I) about the origin in $\mathbb{R}^2$ is

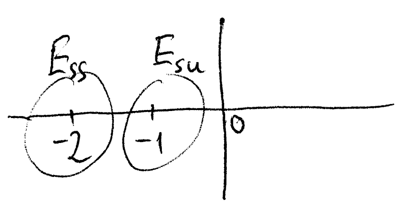
$$\begin{cases} x' = -x \\ y' = -2y \end{cases} \qquad (II)$$



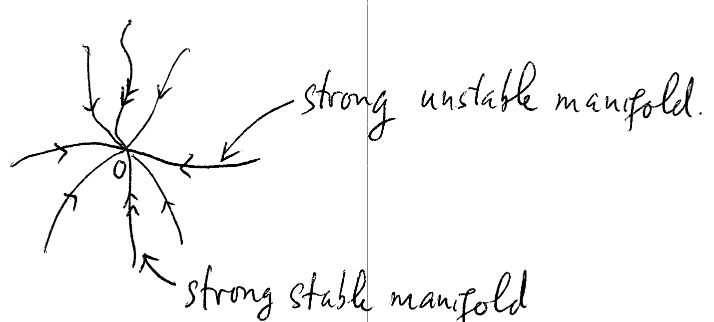
The equation $x' = -x$ gives $x(t) = C_1 e^{-t}$. The equation $y' = -2y$ gives $y(t) = C_2 e^{-2t}$. Thus, all trajectories $(x(t), y(t))$ are parabolae $y = Cx^2$, except for the vertical line $x = 0$.

(ii) The system (II) can be written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$



The strong stable manifold E_{ss} is the eigenspace of the eigenvalue -2. Thus, E_{ss} is the y-axis. This manifold can be viewed as the graph of $x = 0$. This gives us a hint that the strong stable manifold for nonlinear system (I) is also described as the graph of $x = h(y)$.



Suppose the regularity of h is at least C^2 so that we have a Taylor expansion $x = h(y) = a_0 + a_1 y + a_2 y^2 + O(y^3)$. Assume x and y satisfy

$$x' = -x + ay^2; \quad (1)$$

$$y' = -2y + bx^2. \quad (2)$$

We determine a_0, a_1, a_2. Because $(x,y) = (0,0)$ is an equilibrium and we want to describe the strong stable manifold around this point, we take $a_0 = 0$. Then

$$x' = (h(y))' = y' h'(y) \overset{(2)}{=\!=} (-2y + bx^2)(a_1 + 2a_2 y + O(y^2)).$$

Substituting x' from (1) into the above equation, we get

$$-x + ay^2 = (-2y + bx^2)(a_1 + 2a_2 y + O(y^2)).$$

Substituting $x = a_1 y + a_2 y^2 + O(y^3)$ into the above equation, we get

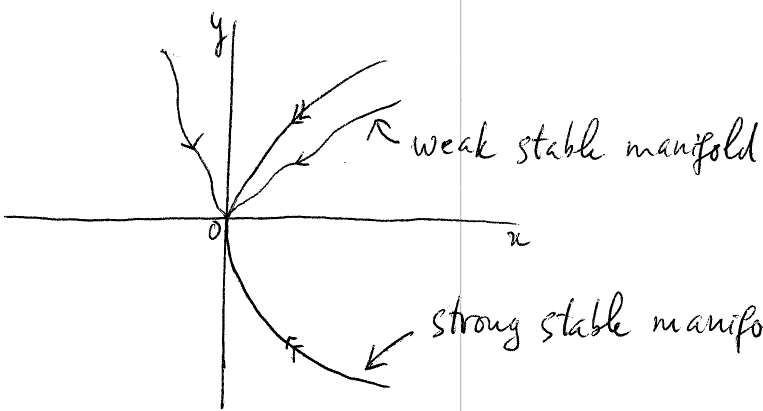
$$-a_1 y - a_2 y^2 + ay^2 = [-2y + b(a_1 y + a_2 y^2)^2](a_1 + 2a_2 y) + O(y^3)$$

Equating the coefficients of y^1, we get $-a_1 = -2a_1$. Thus, $a_1 = 0$.

Equating the coefficients of y^2, we get $-a_2 + a = -4a_2$. Thus, $a_2 = -\frac{a}{3}$.

We get $x = h(y) = -\frac{a}{3} y^2 + O(y^3)$. ✓

(iii) We try to find a "weak stable" manifold as a graph over the x-axis, i.e. as a graph $y=h(x)$. Note that the notation h here does not refer to the function h in Part (ii).



Suppose the regularity of h is at least C^2 so that we have a Taylor expansion $y=h(x)=b_0+b_1x+b_2x^2+O(x^3)$. Assume x and y satisfy

$$x' = -x + ay^2, \qquad (3)$$

$$y' = -2y + bx^2. \qquad (4)$$

We determine b_0, b_1, b_2. Because $(x,y)=(0,0)$ is an equilibrium and that we want to ~~describl~~ describe a manifold around this point, we take $b_0=0$.

$$y' = (h(x))' = x'h'(x) \overset{(3)}{=\!=} (-x+ay^2)(b_1+2b_2x+O(x^2)).$$

Substituting y' from (4) into the equation, we get

$$-2y + bx^2 = (-x+ay^2)(b_1+2b_2x+O(x^2)).$$

Substituting $y=b_1x+b_2x^2+O(x^3)$ into the above equation, we get

$$-2b_1x - 2b_2x^2 = [-x+a(b_1x+b_2x^2)^2](b_1+2b_2x) + O(x^3).$$

Equating the coefficients of x^1, we get $-2b_1=-b_1$. Thus, $b_1=0$.

$(+b$

Equating the coefficients of x^2, we get $-2b_2 = -2b_2$! We cannot determine b_2 from this equation! One implication could be that the condition $h \in C^2$ cannot be ~~sast~~ satisfied. when $b \neq 0$

(iv) For $a = 0$, the system (I) becomes

$$\begin{cases} x' = -x, & (5) \\ y' = -2y + bx^2. & (6) \end{cases}$$

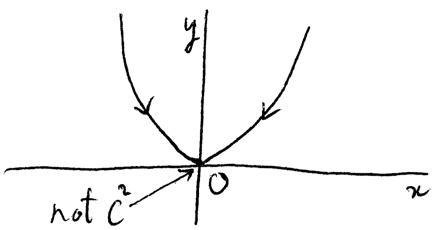
Equation (5) gives us $x(t) = C_1 e^{-t}$. Then (6) becomes $y' + 2y = C_1^2 b e^{-2t}$.
The homogeneous equation has solutions $y_1(t) = C_2 e^{-2t}$. A particular solution is $y_2(t) = C_1^2 b t e^{-2t}$. Thus,

$$y(t) = y_1(t) + y_2(t) = C_2 e^{-2t} + C_1^2 b t e^{-2t}.$$

For $C_1 \neq 0$, $\quad y(t) = \frac{C_2}{C_1^2}(C_1 e^{-t})^2 + bt(C_1 e^{-t})^2 = \frac{C_2}{C_1^2} x^2 + btx^2.$

Assume $C_1 > 0$. Then $t = \log C_1 - \log x = C_3 - \log x$. Then

$$y = \frac{C_2}{C_1^2} x^2 + b(C_3 - \log x)x^2 = C_4 x^2 - bx^2 \log x.$$



As the parameter C_4 varies, we get many solutions $y = h(x)$. We show that for $b \neq 0$, none of these curves is C^2 at $x = 0$. It suffices to show that the map $\varphi : [0, \infty) \to \mathbb{R}$, $\varphi(x) = \begin{cases} x^2 \log x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$

cannot be extended beyond $x = 0$ to a C^2 function.

✓

$$\varphi'(x) = 2x\log x + x^2\frac{1}{x} = x(2\log x + 1) \qquad \forall x > 0,$$

$$\varphi''(x) = (2\log x + 1) + x\frac{2}{x} = 2\log x + 3 \qquad \forall x > 0.$$

Because $\lim_{x\to 0^+} \varphi''(x) = -\infty$, φ cannot extend to a C^2 function on $(-\varepsilon, \infty)$ for some $\varepsilon > 0$.

(3) Consider the linear ODE on $\mathbb{R}^n$

$$\dot{x} = Ax \qquad (1)$$

where $A = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_n$.

(i) We want to write the equation for the projectived flow.

The solution of (1) is $x(t) = e^{At}x(0)$ for all $t \in \mathbb{R}$. If $x(0) = 0$ then $x(t) = 0$ for all $t \in \mathbb{R}$. In this case, there is no projectivized flow associate with $x(t)$. Consider the case $x(0) \neq 0$. Then $x(t) \neq 0$ for all $t \in \mathbb{R}$.

Let $(e_1, e_2, \ldots, e_n)$ be the standard basis of $\mathbb{R}^n$. It provides an inner product in $\mathbb{R}^n$, namely

$$\left\langle \sum_{k=1}^n a_k e_k, \sum_{j=1}^n b_j e_j \right\rangle = \sum_{k=1}^n a_k b_k.$$

Because $|x|^2 = x.x > 0$ is continuously differentiable in $t \in \mathbb{R}$, $|x|$ is also continuously differentiable as a function of t. Put $u = \frac{x}{|x|} \in S^{n-1}$. Then u is continuously differentiable and $|u| = 1$.

$$\dot{u} = \frac{d}{dt}\left(\frac{x}{|x|}\right) = \frac{\dot{x}|x| - \frac{d|x|}{dt}x}{|x|^2} = \frac{\dot{x}}{|x|} - \frac{x}{|x|}\frac{1}{|x|}\frac{d|x|}{dt} = \frac{Ax}{|x|} - \frac{u}{|x|}\frac{d|x|}{dt}.$$

Thus, $\dot{u} = Au - u\frac{1}{|x|}\frac{d|x|}{dt}$. (2)

Taking the inner product with u on both sides, we get

$$\frac{1}{2}\frac{d}{dt}(\underbrace{|u|^2}_{=1}) = \langle Au, u\rangle - \underbrace{|u|^2}_{=1}\frac{1}{|x|}\frac{d|x|}{dt}.$$

Thus, $\frac{1}{|x|}\frac{d|x|}{dt} = \langle Au, u\rangle$. Replacing this term into (2), we obtain

$$\dot{u} = Au - \langle Au, u\rangle u. \checkmark \quad (3).$$

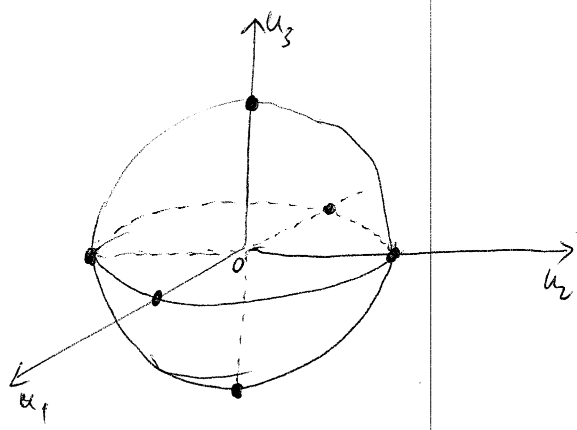
Since the ODE $\dot{x} = Ax$ gives the flow $(t, x_0) \mapsto e^{At}x_0$, the ODE $\dot{u} = Au - \langle Au, u\rangle u$ gives a flow $\phi: \mathbb{R}\times S^{n-1} \to S^{n-1}$, $\phi(t, u_0) = \frac{e^{At}u_0}{|e^{At}u_0|}$.

This is the projectivized flow associate with x.

Next, we find all equilibria of the flow ϕ on S^{n-1}. An equilibrium u is a point on S^{n-1} that satisfies $Au - \langle Au, u\rangle u = 0$. In other words, $u \in S^{n-1}$ is an equilibrium if and only if $Au = \lambda u$ for some $\lambda \in \mathbb{R}$. Write $u = (u_1, u_2, \dots, u_n)^T$. Then the equation $Au = \lambda u$ becomes

$$\lambda_k u_k = \lambda u_k \quad \forall\, 1 \le k \le n.$$

Because $u \ne 0$, there is $1 \le j \le n$ such that $u_j \ne 0$. Then $\lambda = \lambda_j$. Because $\lambda_k \ne \lambda_j$ for all $k \ne j$, $u_k = 0$ for all $k \ne j$. Then $1 = |u| = |u_j|$. Thus, $u_j = \pm 1$, which implies $u = \pm e_j$. Therefore, there are $2n$ equilibria $\pm e_1, \pm e_2, \dots, \pm e_n$. They are the intersection points of S^{n-1} and the coordinate axes in $\mathbb{R}^n$.



(ii) Define a map $V: \mathbb{R}^n \to \mathbb{R}$, $V(x) = -\frac{1}{2}\langle Ax, x\rangle$, which is called the Rayleigh quotient. We show that V is a strict Lyapunov function for the projectivized flow ϕ.

Taking the inner product with Au on both sides of (3), we get

$$\langle Au, \dot{u}\rangle = \langle Au, Au\rangle - |\langle Au, u\rangle|^2 \qquad (4)$$

Because A is a symmetric matrix, it is self-adjoint. Then

$$\langle Au, \dot{u}\rangle = \langle u, A\dot{u}\rangle = \frac{\langle Au, \dot{u}\rangle + \langle A\dot{u}, u\rangle}{2} = \frac{d}{dt}\frac{1}{2}\langle Au, u\rangle = -\frac{d}{dt}V(u).$$

Then (4) becomes

$$\frac{d}{dt}V(u) = |\langle Au, u\rangle|^2 - |Au|^2. \qquad (5)$$

We have $|\langle Au, u\rangle| \le |Au||u| = |Au|$. Then (5) implies $\frac{d}{dt}V(u) \le 0$. ✓

If u is not an equilibrium then $Au(t)$ and $u(t)$ are never parallel because otherwise u would pass through an equilibrium. In this case, $|\langle Au, u\rangle| < |Au||u|$, and thus $\frac{d}{dt}V(u) < 0$ for all $t \in \mathbb{R}$. Therefore, V is a strict Lyapunov function.

✓

Next, we show that only $\pm e_1$ are stable equilibria of the flow ϕ. Other equilibria are unstable. View S^{n-1} as a metric space with metric induced from $\mathbb{R}^n$. Suppose $v \in \{\pm e_1, \pm e_2, \dots, \pm e_n\}$ is stable with respect to ϕ.

First, we show that v is a local minimum of V on S^{n-1}. We have

$$V(v) = -\frac{1}{2}\langle Av, v\rangle \in \left\{-\frac{1}{2}\langle Ae_1, e_1\rangle, \dots, -\frac{1}{2}\langle Ae_n, e_n\rangle\right\}.$$

Thus,
$$V(v) \in \left\{-\frac{\lambda_1}{2}, -\frac{\lambda_2}{2}, \dots, -\frac{\lambda_n}{2}\right\}.$$

Assume $v = e_i$ or $-e_i$. Let
$$\varepsilon = \min\left\{\frac{1}{2}\min_{j\neq i}\left|-\frac{\lambda_i}{2}+\frac{\lambda_j}{2}\right|, \frac{1}{2}\right\}.$$

Because v is stable, there exists a neighborhood $\mathcal{U}(v)$ in S^{n-1} such that if $v_0 \in \mathcal{U}(v)$ then $|\phi_t(v_0) - v| < \varepsilon$ for all $t \geq 0$. By shrinking $\mathcal{U}(v)$ if necessary, we can assume $|V(v_0) - V(v)| < \varepsilon$ for all $v_0 \in \mathcal{U}(v)$. Take any $v_0 \in \mathcal{U}(v) \setminus \{v\}$. Because $\phi_t(v_0) \in S^{n-1}$, which is a compact set, $\omega(v_0) \neq \emptyset$. Because V is a strict Lyapunov function, $\omega(v_0)$ is contained in the set of equilibria $\{\pm e_1, \pm e_2, \dots, \pm e_n\}$. This is because of La Salle's invariance principle. Each $v' \in \omega(v_0)$ belongs to $\overline{\{\phi_t(v_0) : t \geq 0\}}$. Because $|\phi_t(v_0) - v| < \varepsilon \leq \frac{1}{2}$ for all $t \geq 0$, no equilibrium other than v could belong to $\overline{\{\phi_t(v_0) : t \geq 0\}}$. Thus, $\omega(v_0) = \{v\}$. Then there is an increasing sequence $t_k \to \infty$ such that $v = \lim_{k\to\infty} \phi_{t_k}(v_0)$. Since V is continuous,

$$V(v) = \lim_{k\to\infty} V(\phi_{t_k}(v_0)).$$

Because V is a strict Lyapunov function, $\{V(\phi_{t_k}(v_0))\}$ is a strictly decreasing sequence. Thus, $V(v) < V(\phi_0(v_0)) = V(v_0)$. We have showed that v is a local minimum of V on S^{n-1}.

As a consequence, v is a solution to the problem $V(x) \to \min$ under the constraint $|x| = 1$. Thus, v is a solution to the problem $W(x) := V(x) + \frac{\lambda}{2}|x|^2 \to \min$, where λ is a Lagrange multiplier. Then v has to satisfy two requirements.

$$\begin{cases} \nabla W(v) = 0 \\ \nabla^2 W(v) \text{ is positive semi-definite} \end{cases}$$

We have $\nabla W(v) = \nabla V(v) + \lambda v = -(\lambda_1 v_1, \lambda_2 v_2, \dots, \lambda_n v_n) + \lambda(v_1, \dots, v_n)$

$$= ((\lambda - \lambda_1) v_1, \dots, (\lambda - \lambda_n) v_n).$$

Because $v = e_i$ or $-e_i$, $\nabla W(v) = 0$ if and only if $\lambda = \lambda_i$. ~~the~~ The Hessian matrix of W at v is

$$\nabla^2 W(v) = \left(\frac{\partial^2 W}{\partial x_k \partial x_j}(v) \right)_{1 \le k,j \le n} = \begin{pmatrix} \lambda - \lambda_1 & & & \\ & \lambda - \lambda_2 & & \\ & & \ddots & \\ & & & \lambda - \lambda_n \end{pmatrix}.$$

Since $\nabla^2 W(v)$ is positively semi-definite, $\lambda - \lambda_j \geq 0$ for all $1 \le j \le n$. Thus, $\lambda_i = \lambda \geq \lambda_j$ for all $1 \le j \le n$. This is possible only if $\lambda_i = \lambda_1$, i.e. $i=1$.

✓ Therefore, $\pm e_1$ are the only possible stable equilibria of ϕ.

Next, we show that $\pm e_1$ are indeed stable equilibria of ϕ.

$$V(u) = -\frac{1}{2}\langle Au, u\rangle = -\frac{1}{2}\left(\lambda_1 u_1^2 + \lambda_2 u_2^2 + \cdots + \lambda_n u_n^2\right)$$

$$= -\frac{1}{2}\lambda_1 \underbrace{(u_1^2 + \cdots + u_n^2)}_{=1} + \frac{1}{2}(\lambda_1 - \lambda_2)u_2^2 + \cdots + \frac{1}{2}(\lambda_1 - \lambda_n)u_n^2$$

$$= V(\pm e_1) + \frac{1}{2}(\lambda_1 - \lambda_2)u_2^2 + \cdots + \frac{1}{2}(\lambda_1 - \lambda_n)u_n^2.$$

Thus, $$V(u) - V(\pm e_1) = \frac{1}{2}(\lambda_1 - \lambda_2)u_2^2 + \cdots + \frac{1}{2}(\lambda_1 - \lambda_n)u_n^2. \quad (6)$$

We will give the proof for e_1. The proof for $-e_1$ is achieved by the same way.

Let $\varepsilon > 0$. We determine $\delta > 0$ such that if $v_0 = (v_1, \ldots, v_n) \in S^{n-1}$ satisfies $|v_0 - e_1| < \delta$ then $|\phi_t(v_0) - e_1| < \varepsilon$ for all $t \geq 0$. Write

$$\phi_t(v_0) = w(t) = (w_1(t), w_2(t), \ldots, w_n(t)).$$

We have
$$V(\phi_t(v_0)) - V(e_1) < V(v_0) - V(e_1).$$

Then by the identity (6),

$$\frac{1}{2}(\lambda_1 - \lambda_2)w_2^2 + \cdots + \frac{1}{2}(\lambda_1 - \lambda_n)w_n^2 < \frac{1}{2}(\lambda_1 - \lambda_2)v_2^2 + \cdots + \frac{1}{2}(\lambda_1 - \lambda_n)v_n^2 \quad (7)$$

$$LHS(7) \geq \frac{1}{2}(\lambda_1 - \lambda_2)(w_2^2 + \cdots + w_n^2) = \frac{1}{2}(\lambda_1 - \lambda_2)(1 - w_1^2)$$

$$RHS(7) \leq \frac{1}{2}(\lambda_1 - \lambda_n)(v_2^2 + \cdots + v_n^2) \leq \frac{1}{2}(\lambda_1 - \lambda_n)|v_0 - e_1|^2 < \frac{\delta^2}{2}(\lambda_1 - \lambda_n).$$

Then (7) implies $\frac{1}{2}(\lambda_1-\lambda_2)(1-w_1^2) < \frac{\delta^2}{2}(\lambda_1-\lambda_n)$,

which yields
$$w_1^2 > 1-\delta^2\frac{\lambda_1-\lambda_n}{\lambda_1-\lambda_2}. \qquad (8)$$

For δ sufficiently small, this inequality implies
$$w_1(t) < -\sqrt{1-\delta^2\frac{\lambda_1-\lambda_n}{\lambda_1-\lambda_2}} \quad \text{or} \quad w_1(t) > \sqrt{1-\delta^2\frac{\lambda_1-\lambda_n}{\lambda_1-\lambda_2}} \quad \forall t \geq 0.$$

Because w_1 is a continuous function, one of these inequalities must hold for all $t \geq 0$. Because $|v_0-e_1|<\delta$, $1-v_1<\delta$. Thus, $w_1(0)=v_1>1-\delta$. This implies
$$w_1(t) > \sqrt{1-\delta^2\frac{\lambda_1-\lambda_n}{\lambda_1-\lambda_2}} \quad \forall t \geq 0 \qquad (9)$$

(8) implies
$$w_2^2(t)+\dots+w_n^2(t) < \delta^2\frac{\lambda_1-\lambda_n}{\lambda_1-\lambda_2}. \qquad (10)$$

Thus,
$$|\phi_t(v_0)-e_1|^2 = |w(t)-e_1|^2 = (1-w_1(t))^2 + w_2^2(t)+\dots+w_n^2(t)$$
$$\overset{(7),(8)}{\leq} \delta^2\frac{\lambda_1-\lambda_n}{\lambda_1-\lambda_2} + \delta^2\frac{\lambda_1-\lambda_n}{\lambda_1-\lambda_2}$$
$$= 2\delta^2\frac{\lambda_1-\lambda_n}{\lambda_1-\lambda_n} \quad \forall t \geq 0.$$

alternative:
stereographic projection coordinates near e_1
$y_j = x_j/x_1$: $S^{n-1} \to \mathbb{R}^{n-1}$
$\hookrightarrow$ linear system for y_j, $\dot y_j = (\lambda_j-\lambda_1)y_j$ ....

Then with δ sufficiently small, $|\phi_t(v_0)-e_1|^2 < \varepsilon^2$ for all $t \geq 0$. ✓

(iii) We show that all trajectories are either heteroclinic or the equilibria.

Take $u_0 \in S^{n-1} \setminus \{\pm e_1, \pm e_2, \dots, \pm e_n\}$. We show that $u(t) = \phi_t(u_0)$ is heteroclinic.

$$|V(u)| = \frac{1}{2}|\langle Au, u\rangle| = \frac{1}{2}|\lambda_1 u_1^2+\dots+\lambda_n u_n^2| \leq \frac{1}{2}\max_{1\leq k\leq n}|\lambda_k| \quad \forall t \in \mathbb{R}$$

Thus, $\{V(u(t)) : t \in \mathbb{R}\}$ is a bounded set. Because $V(u)$ is an decreasing function of t, there exist $a = \lim_{t\to\infty} V(u(t))$ and $b = \lim_{t\to-\infty} V(u(t))$.

$$0 \geq \frac{d}{dt} V(u(t)) \geq V(u(t)) - V(u(t-1)).$$

Thus,

$$0 \geq \lim_{t\to\infty} \frac{d}{dt} V(u(t)) \geq a - a = 0$$

$$0 \geq \lim_{t\to-\infty} \frac{d}{dt} V(u(t)) \geq b - b = 0.$$

We get $\lim_{t\to\infty} \frac{d}{dt} V(u(t)) = \lim_{t\to-\infty} \frac{d}{dt} V(u(t)) = 0$.

Because S^{n-1} is compact, there exists a sequence $t_k \to \infty$ such that $u(t_k)$ converges to some $a_0 \in S^{n-1}$. Then $a = \lim_{k\to\infty} V(u(t_k)) = V(a_0)$. By (5), we have

$$\frac{d}{dt} V(u(t_k)) = |\langle Au(t_k), u(t_k)\rangle|^2 - |Au(t_k)|^2.$$

Letting $k \to \infty$, we get $0 = |\langle Aa_0, a_0\rangle|^2 - |Aa_0|^2$. Thus a_0 is an equilibrium of φ. Thus, every point in the ω-limit set of u_0 is an equilibrium. Let $\mathcal{U}^+(e_1), \mathcal{U}^-(-e_1), \ldots, \mathcal{U}^+(e_n), \mathcal{U}^-(-e_n)$ be neighborhoods in S^{n-1} that are pairwise disjoint. Because S^{n-1} is compact, for every sequence $t_k \to \infty$, $u(t_k)$ has a convergent subsequence and the limit must be in $\{\pm e_1, \ldots, \pm e_n\}$. If the limit $\lim_{t\to\infty} u(t)$ does not exist, there exists a sequence $t_k \to \infty$, such that $u(t_k)$ lies in the complement of $\mathcal{U}^+(e_1) \cup \mathcal{U}^-(-e_1) \cup \cdots \cup \mathcal{U}^+(e_n) \cup \mathcal{U}^-(-e_n)$.

This sequence has no subsequence that converges to a point in $\{\pm e_1, \dots, \pm e_n\}$. This is a contradiction. Thus, $\lim_{t\to\infty} u(t)$ exists and belongs to $\{\pm e_1, \dots, \pm e_n\}$.

By a similar proof, $\lim_{t\to-\infty} u(t)$ exist and belongs to $\{\pm e_1, \dots, \pm e_n\}$. To conclude that u is a heteroclinic orbit, we need to show

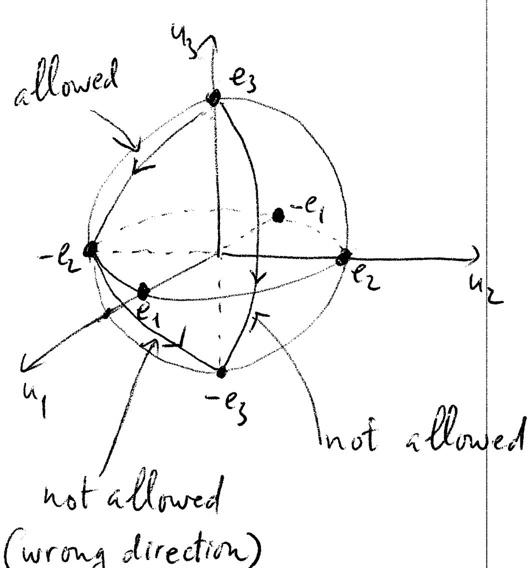
$$a_0 = \lim_{t\to\infty} u(t) \neq \lim_{t\to-\infty} u(t) = b_0.$$

Because $V(u(t))$ decreases in t,

$$V(a_0) \le V(u(t)) \le V(b_0) \quad \forall t \in \mathbb{R}.$$

If $V(a_0) = V(b_0)$ then $V(u(t)) \equiv \text{const}$. In that case, $u(t)$ is an equilibrium because V is a strict Lyapunov function. This is a contradiction because $u_0 \notin \{\pm e_1, \pm e_2, \dots, \pm e_n\}$. Hence $V(a_0) \neq V(b_0)$. This implies $a_0 \neq \pm b_0$ ✓

Next, we describe the heteroclinic orbits of ϕ. by the information we have obtained. A heteroclinic orbit $u(t) = \phi_t(u_0)$ satisfies



$$\lim_{t\to\infty} u(t) = \pm e_i,$$

$$\lim_{t\to-\infty} u(t) = \pm e_j,$$

for $1 \le i < j \le n$. The reason for $i<j$ is

$$-\frac{\lambda_i}{2} = V(\pm e_i) < V(\pm e_j) = -\frac{\lambda_j}{2}.$$

do all those exist? yes, restrict to $\mathrm{Span}\langle e_i, e_j \rangle$ .....

(iv) Consider $n=2$ and $A = \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix}$. The projectivized flow of $\dot{x} = Ax$ is derived by the same method as in Part (i).

$$\dot{u} = Au - \langle Au, u\rangle u \qquad (10)$$

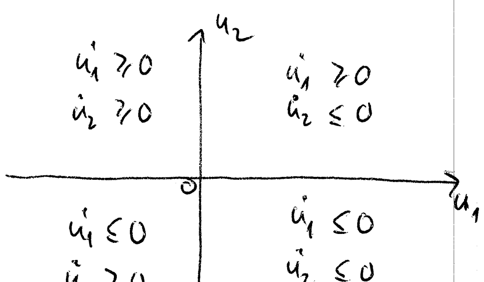
This equation gives a flow ϕ on the circle $S^1 \subset \mathbb{R}^2$. As in Part (i), the equilibria of ϕ are points $u \in S^1$ such that Au is parallel to u. In other words, the equilibria of ϕ are the unit eigenvectors of A.

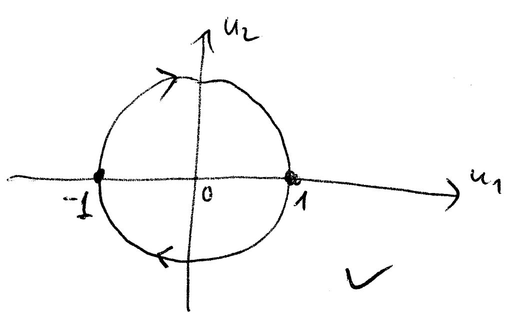
$$\det(\lambda I_2 - A) = \det\begin{pmatrix} \lambda & -1 \\ -\mu & \lambda \end{pmatrix} = \lambda^2 - \mu .$$

- If $\mu < 0$ then ϕ has no equilibrium. Thus, it has no heteroclinic orbit.
- Consider $\mu = 0$. Then A has only eigenvalue $\lambda = 0$. The matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has unit eigenvectors $u = (u_1, u_2) = (\pm 1, 0)$. The equilibria of ϕ are $\{(1,0), (-1,0)\}$. To describe the heteroclinic orbit, we draw the phase portrait of ϕ. Write $u = (u_1, u_2)$. Then (10) becomes

$$\begin{cases} \dot{u}_1 = u_2(1 - u_1^2), \\ \dot{u}_2 = -u_1 u_2^2 . \end{cases}$$

Because $|u_1| \leq 1$, $1 - u_1^2 \geq 0$.





There are two heteroclinic orbits.

- Consider $\mu > 0$. Then A has two distinct eigenvalues $\lambda_1 = -\sqrt{\mu}$ and $\lambda_2 = \sqrt{\mu}$.

$$\lambda_2 I_2 - A = \begin{pmatrix} \sqrt{\mu} & -1 \\ -\mu & \sqrt{\mu} \end{pmatrix} \sim \begin{pmatrix} \sqrt{\mu} & -1 \\ 0 & 0 \end{pmatrix}.$$

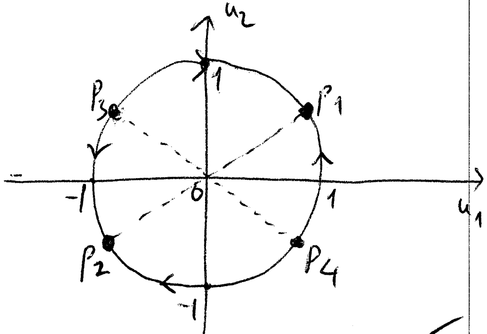
The corresponding eigenspace is $\{c(1,\sqrt{\mu}) : c \in \mathbb{R}\}$. It has two unit vectors $P_{1,2} = \pm\left(\frac{1}{\sqrt{1+\mu}}, \frac{\sqrt{\mu}}{\sqrt{1+\mu}}\right)$.

$$\lambda_1 I_2 - A = \begin{pmatrix} -\sqrt{\mu} & -1 \\ -\mu & -\sqrt{\mu} \end{pmatrix} \sim \begin{pmatrix} \sqrt{\mu} & 1 \\ 0 & 0 \end{pmatrix}.$$

The corresponding eigenspace is $\{c(-1,\sqrt{\mu}) : c \in \mathbb{R}\}$. It has two unit vectors

$$P_{3,4} = \pm\left(\frac{-1}{\sqrt{1+\mu}}, \frac{\sqrt{\mu}}{\sqrt{1+\mu}}\right).$$

Therefore, ϕ has four equilibria $\{P_1, P_2, P_3, P_4\}$.



P_1 lies in the first quadrant, P_2 in the third, P_3 in the second and P_4 in the fourth. None of them lie on the axes. The flow ϕ has four heteroclinic orbits. To determine the arrow on each of those orbits, we only need to determine the signs of $\dot{u}_1$ and $\dot{u}_2$ at four points $(\pm 1, 0)$, $(0, \pm 1)$.

We have $$Au = \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ \mu u_1 \end{pmatrix}.$$

$$\langle Au, u \rangle = \begin{pmatrix} u_2 \\ \mu u_1 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (1+\mu) u_1 u_2.$$

Thus (10) becomes

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ \mu u_1 \end{pmatrix} - (1+\mu) u_1 u_2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad (11)$$

At point $(u_1, u_2) = (1,0)$, (11) becomes $\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \mu \end{pmatrix}$. This gives us the arrow on the orbit between P_4 and P_1.

At point $(u_1, u_2) = (-1, 0)$, (11) becomes $\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\mu \end{pmatrix}$. This gives us the arrow on the orbit between p_2 and p_3.

At point $(u_1, u_2) = (0, 1)$, (11) becomes $\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This gives us the arrow on the orbit between p_3 and p_1.

At point $(u_1, u_2) = (0, -1)$, (11) becomes $\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. This gives us the arrow on the orbit between p_2 and p_4.

(or use previous part after diagonalizing)

good