Name: Tuan Pham
ID : 4652218

Math 8501: Differential Equations & Dynamical Systems

Homework #6

① Consider the system of ODE for $A, B : \mathbb{R} \to \mathbb{C}$.

$$\begin{cases} A' = B, \\ B' = -A + A^2\bar{A}. \end{cases} \qquad (\text{I})$$

This system has a family of periodic solutions parametrized by $k \in (-1,1)$.

$$A_*(t) = R e^{ikt}, \quad B_*(t) = ikRe^{ikt} \quad \text{where } R = \sqrt{1-k^2}.$$

Indeed,

$$A_*'(t) = R(ik)e^{ikt} = B_*(t), \qquad (1)$$

$$B_*'(t) = (ik)^2 R e^{ikt} = (-k^2)A_*(t) = (-1+R^2)A_*(t) = \left(-1 + A_*(t)\bar{A}_*(t)\right)A_*(t)$$
$$= -A_*(t) + A_*(t)^2\bar{A}_*(t). \quad (2)$$

(i) If we view (I) as a system of ODE of only A and B, the right hand side is not a C^1-function because the complex conjugate map $z \in \mathbb{C} \mapsto \bar{z} \in \mathbb{C}$ is not C^1. The lack of smoothness prevents us from linearizing (I). We now view (I) as a system of ODE of four complex-valued functions $A, B, \bar{A}, \bar{B}$.

$$\begin{cases} A' = B, \\ B' = -A + A^2\bar{A}, \\ \bar{A}' = \bar{B}, \\ \bar{B}' = -\bar{A} + \bar{A}^2 A. \end{cases} \qquad (\text{II})$$

System (II) is of the form $(A', B', \bar{A}', \bar{B}') = f(A, B, \bar{A}, \bar{B})$ where $f : \mathbb{C}^4 \to \mathbb{C}^4$,

$$f(z_1, z_2, z_3, z_4) = (z_2, -z_1 + z_1^2 z_3, z_4, -z_3 + z_3^2 z_1).$$

Because f is a smooth function, system (II) has a linearization about $(A_*, B_*, \bar{A}_*, \bar{B}_*)$.

$$(A', B', \bar{A}', \bar{B}') = f(A_*, B_*, \bar{A}_*, \bar{B}_*) + Df(A_*, B_*, \bar{A}_*, \bar{B}_*) \cdot (A - A_*, B - B_*, \bar{A} - \bar{A}_*, \bar{B} - \bar{B}_*). \quad (3)$$

Because $(A_*, B_*, \bar{A}_*, \bar{B}_*)$ solves system (II),

$$f(A_*, B_*, \bar{A}_*, \bar{B}_*) = (A_*', B_*', \bar{A}_*', \bar{B}_*'). \quad (4)$$

We have

$$Df(z_1, z_2, z_3, z_4) = \begin{pmatrix} \frac{\partial}{\partial z_1}(z_2) & \frac{\partial}{\partial z_2}(z_2) & \frac{\partial}{\partial z_3}(z_2) & \frac{\partial}{\partial z_4}(z_2) \\ \frac{\partial}{\partial z_1}(-z_1 + z_1^2 z_3) & \frac{\partial}{\partial z_2}(-z_1 + z_1^2 z_3) & \frac{\partial}{\partial z_3}(-z_1 + z_1^2 z_3) & \frac{\partial}{\partial z_4}(-z_1 + z_1^2 z_3) \\ \frac{\partial}{\partial z_1}(z_4) & \frac{\partial}{\partial z_2}(z_4) & \frac{\partial}{\partial z_3}(z_4) & \frac{\partial}{\partial z_4}(z_4) \\ \frac{\partial}{\partial z_1}(-z_3 + z_3^2 z_1) & \frac{\partial}{\partial z_2}(-z_3 + z_3^2 z_1) & \frac{\partial}{\partial z_3}(-z_3 + z_3^2 z_1) & \frac{\partial}{\partial z_4}(-z_3 + z_3^2 z_1) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 + 2z_1 z_3 & 0 & z_1^2 & 0 \\ 0 & 0 & 0 & 1 \\ z_3^2 & 0 & -1 + 2z_3 z_1 & 0 \end{pmatrix}.$$

Thus, $Df(A_*, B_*, \bar{A}_*, \bar{B}_*) \cdot (A - A_*, B - B_*, \bar{A} - \bar{A}_*, \bar{B} - \bar{B}_*)$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 + 2A_* \bar{A}_* & 0 & A_*^2 & 0 \\ 0 & 0 & 0 & 1 \\ \bar{A}_*^2 & 0 & -1 + 2\bar{A}_* A_* & 0 \end{pmatrix} \begin{pmatrix} A - A_* \\ B - B_* \\ \bar{A} - \bar{A}_* \\ \bar{B} - \bar{B}_* \end{pmatrix}$$

$$= \begin{pmatrix} B - B_* \\ (-1+2A_*\bar{A}_*)(A-A_*) + A_*^2(\bar{A}-\bar{A}_*) \\ \bar{B} - \bar{B}_* \\ (-1+2\bar{A}_*A_*)(\bar{A}-\bar{A}_*) + \bar{A}_*^2(A-A_*) \end{pmatrix} \qquad (5)$$

Substituting (4) and (5) into (3), we get

$$\begin{pmatrix} X' \\ Y' \\ \bar{X}' \\ \bar{Y}' \end{pmatrix} = \begin{pmatrix} Y \\ (-1+2A_*\bar{A}_*)X + A_*^2\bar{X} \\ \bar{Y} \\ (-1+2A_*\bar{A}_*)\bar{X} + \bar{A}_*^2 X \end{pmatrix} \qquad (6)$$

where $X = A - A_*$ and $Y = B - B_*$. We have

$$-1 + 2A_*\bar{A}_* = -1 + 2|A_*|^2 = -1 + 2R^2,$$
$$A_*^2 = R^2 e^{2ikt}.$$

Then (6) is equivalent to

$$\begin{pmatrix} X' \\ Y' \\ \bar{X}' \\ \bar{Y}' \end{pmatrix} = \begin{pmatrix} Y \\ (-1+2R^2)X + R^2 e^{2ikt}\bar{X} \\ \bar{Y} \\ (-1+2R^2)\bar{X} + R^2 e^{-2ikt} X \end{pmatrix} \qquad (7)$$

System (7) is the linearization of system (II).

Next, we verify that $(A_*', B_*', \bar{A}_*', \bar{B}_*')$ satisfies (7). It is sufficient to verify $A_*''(t) = B_*'(t)$ and $B_*''(t) = (-1+2R^2)A_*'(t) + R^2 e^{2ikt}\bar{A}_*'(t)$.

By (1), $A_*''(t) = B_*'(t)$. By (2),

$$B_*'(t) = (-1+R^2)A_*'(t) = (-1+2R^2)A_*'(t) - R^2 \underbrace{A_*'(t)}_{= Rike^{ikt}}$$

$$= (-1+2R^2)A_x'(t) + R^2 e^{2ikt}\underbrace{(-ik\, e^{-ikt})}_{=\bar{A}_x'(t)}.$$

(ii) Put $(a(t), b(t), \bar{a}(t), \bar{b}(t)) = (e^{-ikt}X(t), e^{-ikt}Y(t), e^{ikt}\bar{X}(t), e^{ikt}\bar{Y}(t))$. We show that $(a, b, \bar{a}, \bar{b})$ satisfies a linear equation with constant coefficients. The inverse relation is

$$(X, Y, \bar{X}, \bar{Y}) = (e^{ikt}a, e^{ikt}b, e^{-ikt}\bar{a}, e^{-ikt}\bar{b}).$$

Taking the derivative with respect to t of both sides, we get

$$LHS(7) = \begin{pmatrix} X' \\ Y' \\ \bar{X}' \\ \bar{Y}' \end{pmatrix} = \begin{pmatrix} (ika + a')e^{ikt} \\ (ikb + b')e^{ikt} \\ (-ik\bar{a} + \bar{a}')e^{-ikt} \\ (-ik\bar{b} + \bar{b}')e^{-ikt} \end{pmatrix}. \qquad (8)$$

$$RHS(7) = \begin{pmatrix} e^{ikt}b \\ (-1+2R^2)e^{ikt}a + R^2 e^{2ikt}(e^{-ikt}\bar{a}) \\ e^{-ikt}\bar{b} \\ (-1+2R^2)e^{-ikt}\bar{a} + R^2 e^{-2ikt}(e^{ikt}a) \end{pmatrix} = \begin{pmatrix} e^{ikt}b \\ (-1+2R^2)e^{ikt}a + R^2 e^{ikt}\bar{a} \\ e^{-ikt}\bar{b} \\ (-1+2R^2)e^{-ikt}\bar{a} + R^2 e^{-ikt}a \end{pmatrix}. \qquad (9)$$

Equating (8) with (9), we get

$$\begin{pmatrix} ika + a' \\ ikb + b' \\ -ik\bar{a} + \bar{a}' \\ -ik\bar{b} + \bar{b}' \end{pmatrix} = \begin{pmatrix} b \\ (-1+2R^2)a + R^2\bar{a} \\ \bar{b} \\ (-1+2R^2)\bar{a} + R^2 a \end{pmatrix}$$

Equivalently,

$$\begin{pmatrix} a' \\ b' \\ \bar{a}' \\ \bar{b}' \end{pmatrix} = \begin{pmatrix} -ika + b \\ (-1+2R^2)a - ikb + R^2\bar{a} \\ ik\bar{a} + \bar{b} \\ (-1+2R^2)\bar{a} + ik\bar{b} + R^2 a \end{pmatrix} = \mathcal{B}\begin{pmatrix} a \\ b \\ \bar{a} \\ \bar{b} \end{pmatrix}, \qquad (10)$$

where

$$\mathcal{B} = \begin{pmatrix} -ik & 1 & 0 & 0 \\ -1+2R^2 & -ik & R^2 & 0 \\ 0 & 0 & ik & 1 \\ R^2 & 0 & -1+2R^2 & ik \end{pmatrix}. \tag{11}$$

By the theory of linear ODEs with constant coefficients, we know that the solution of (10) is

$$\begin{pmatrix} a(t) \\ b(t) \\ \bar{a}(t) \\ \bar{b}(t) \end{pmatrix} = e^{\mathcal{B}t} \begin{pmatrix} a(0) \\ b(0) \\ \bar{a}(0) \\ \bar{b}(0) \end{pmatrix}. \tag{12}$$

We have showed that $(a, b, \bar{a}, \bar{b})$ satisfies a linear ODE with constant coefficients. Define a map $\mathcal{P} : \mathbb{R} \to GL(4, \mathbb{C})$,

$$\mathcal{P}(t) = \begin{pmatrix} e^{ikt} & 0 & 0 & 0 \\ 0 & e^{ikt} & 0 & 0 \\ 0 & 0 & e^{ikt} & 0 \\ 0 & 0 & 0 & e^{ikt} \end{pmatrix}. \tag{13}$$

Then

$$\begin{pmatrix} X \\ Y \\ \bar{X} \\ \bar{Y} \end{pmatrix} = \mathcal{P}(t) \begin{pmatrix} a \\ b \\ \bar{a} \\ \bar{b} \end{pmatrix} \overset{(12)}{=} \mathcal{P}(t)\, e^{\mathcal{B}t} \begin{pmatrix} a(0) \\ b(0) \\ \bar{a}(0) \\ \bar{b}(0) \end{pmatrix}$$

$$= \mathcal{P}(t)\, e^{\mathcal{B}t} \underbrace{\mathcal{P}(0)^{-1}}_{= I_4} \begin{pmatrix} X(0) \\ Y(0) \\ \bar{X}(0) \\ \bar{Y}(0) \end{pmatrix}.$$

Thus,
$$\begin{pmatrix} X \\ Y \\ \bar{X} \\ \bar{Y} \end{pmatrix} = \mathcal{P}(t)\, e^{\mathcal{B}t} \begin{pmatrix} X(0) \\ Y(0) \\ \bar{X}(0) \\ \bar{Y}(0) \end{pmatrix} \qquad (14)$$

Next, we relate the representation (14) to Floquet theory. We can rewrite (7) as
$$\begin{pmatrix} X' \\ Y' \\ \bar{X}' \\ \bar{Y}' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1+2R^2 & 0 & R^2 e^{2ikt} & 0 \\ 0 & 0 & 0 & 1 \\ R^2 e^{-2ikt} & 0 & -1+2R^2 & 0 \end{pmatrix}}_{\mathcal{A}(t)} \begin{pmatrix} X \\ Y \\ \bar{X} \\ \bar{Y} \end{pmatrix} \qquad (15)$$

We see that $\mathcal{A}(t)$ is periodic with period $T = \pi/k$. Put $M = \{(z_1, z_2, \bar{z}_1, \bar{z}_2) : z_1, z_2 \in \mathbb{C}\}$

Let $(X, Y, \bar{X}, \bar{Y})$ be the solution of (15). For $t, s \in \mathbb{R}$, we define a map $\phi_{t,s} : M \to M$
$$\phi_{t,s} \begin{pmatrix} X(s) \\ Y(s) \\ \bar{X}(s) \\ \bar{Y}(s) \end{pmatrix} = \begin{pmatrix} X(t) \\ Y(t) \\ \bar{X}(t) \\ \bar{Y}(t) \end{pmatrix}$$

Note that (15) is a ~~non~~ linear non-autonomous equation. By Theorem 7.9, Amann "Ordinary Differential Equations" 1990, page 103, (15) together with an initial condition has a unique solution. This uniqueness makes $\phi_{t,s}$ well-defined. Floquet theory (Theorem 20.9, Amann, page 282) says that there exist a matrix $\tilde{\mathcal{B}} \in M_4(\mathbb{C})$ and a map $\tilde{\mathcal{P}} \in C^1(\mathbb{R}, GL(4, \mathbb{C}))$ periodic with period $T = \pi/k$ and satisfying $\tilde{\mathcal{P}}(0) = id_{\mathbb{C}^4}$ such that $\phi_{t,0} = \tilde{\mathcal{P}}(t) \circ e^{\tilde{\mathcal{B}}t}$. Then $e^{\tilde{\mathcal{B}}t} = \tilde{\mathcal{P}}(t)^{-1} \circ \phi_{t,0}$.
This implies there exists a coordinate change (depending on t) so that (15) becomes a linear autonomous equation.

$$\begin{pmatrix} X(t) \\ Y(t) \\ \bar{X}(t) \\ \bar{Y}(t) \end{pmatrix} = \phi_{t,0} \begin{pmatrix} X(0) \\ Y(0) \\ \bar{X}(0) \\ \bar{Y}(0) \end{pmatrix} = \tilde{\mathcal{P}}(t) e^{\tilde{\mathcal{B}} t} \begin{pmatrix} X(0) \\ Y(0) \\ \bar{X}(0) \\ \bar{Y}(0) \end{pmatrix}.$$

Comparing this equation with (14), we see that we can take $\tilde{\mathcal{P}} = \mathcal{P}$ and $\tilde{\mathcal{B}} = \mathcal{B}$.

(iii) We compute the eigenvalues of $\mathcal{B}$ and their algebraic multiplicities.

$$\det(\lambda I_4 - \mathcal{B}) = \begin{vmatrix} -ik-\lambda & 1 & 0 & 0 \\ -1+2R^2 & -ik-\lambda & R^2 & 0 \\ 0 & 0 & ik-\lambda & 1 \\ R^2 & 0 & -1+2R^2 & ik-\lambda \end{vmatrix} \underset{\text{row 1}}{=} (-ik-\lambda) \begin{vmatrix} -ik-\lambda & R^2 & 0 \\ 0 & ik-\lambda & 1 \\ 0 & -1+2R^2 & ik-\lambda \end{vmatrix}$$

$$- \begin{vmatrix} -1+2R^2 & R^2 & 0 \\ 0 & ik-\lambda & 1 \\ R^2 & -1+2R^2 & ik-\lambda \end{vmatrix}$$

$$= (-ik-\lambda)(-ik-\lambda) \begin{vmatrix} ik-\lambda & 1 \\ -1+2R^2 & ik-\lambda \end{vmatrix} - \left\{ (-1+2R^2) \begin{vmatrix} ik-\lambda & 1 \\ -1+2R^2 & ik-\lambda \end{vmatrix} - R^2 \begin{vmatrix} 0 & 1 \\ R^2 & ik-\lambda \end{vmatrix} \right\}$$

$$= \left\{ (-ik-\lambda)^2 + 1 - 2R^2 \right\} \begin{vmatrix} ik-\lambda & 1 \\ -1+2R^2 & ik-\lambda \end{vmatrix} - R^4$$

$$= (k^2 - 1 + 2ik\lambda + \lambda^2)(k^2 - 1 - 2ik\lambda + \lambda^2) - (1-k^2)^2 \qquad (\text{recall } R^2 = 1-k^2)$$

$$= (k^2 - 1 + \lambda^2)^2 + (2k\lambda)^2 - (k^2-1)^2$$

$$= \lambda^2(\lambda^2 + 6k^2 - 2).$$

Thus, the characteristic polynomial of $\mathcal{B}$ is $f_{\mathcal{B}}(\lambda) = \lambda^2(\lambda^2 + 6k^2 - 2)$.

We consider three following cases of $k \in (-1, 1)$.

- $k^2 < 1/3$

Then B has eigenvalues 0 with multiplicity 2, $\sqrt{2-6k^2}$ with multiplicity 1, and $-\sqrt{2-6k^2}$ with multiplicity 1. The Floquet multipliers are the eigenvalues of $e^{BT} = e^{\frac{\pi}{k}B}$. They are 1 with multiplicity 2, $e^{+\frac{\pi}{k}\sqrt{2-6k^2}}$ with multiplicity 1, and $e^{-\frac{\pi}{k}\sqrt{2-6k^2}}$ with multiplicity 1.

- $k^2 = 1/3$

Then $f_B(\lambda) = \lambda^4$. In this case, B has eigenvalue 0 with multiplicity 4. The Floquet multiplier is 1 with multiplicity 4.

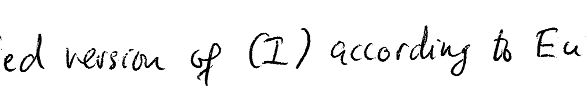
- $k^2 > 1/3$

Then B has eigenvalue 0 with multiplicity 2, $i\sqrt{6k^2-2}$ with multiplicity 1, and $-i\sqrt{6k^2-2}$ with multiplicity 1. The Floquet multiplier are 1 with multiplicity 2, $e^{i\frac{\pi}{k}\sqrt{6k^2-2}}$ with multiplicity 1, and $e^{-i\frac{\pi}{k}\sqrt{6k^2-2}}$ with multiplicity 1.

(2) Consider the system of ODEs which model the van der Pol oscilator.

$$\begin{cases} x' = \frac{1}{\varepsilon}(x - x^3 - y), \\ y' = x - y. \end{cases} \qquad (I)$$

(i) We solve (I) numerically by Euler's method. Put $x_0 = x(0)$ and $y_0 = y(0)$. Let dt be the time step and T be the ending time.



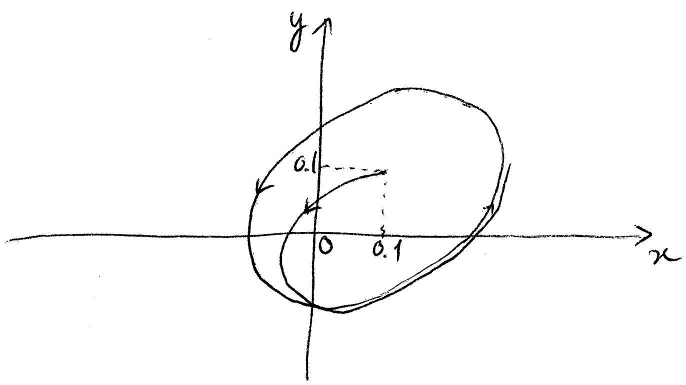
The approximated version of (I) according to Euler's method is

$$\begin{cases} \dfrac{x((k+1)dt)-x(kdt)}{dt} = \dfrac{1}{\varepsilon}\left(x(kdt)-x(kdt)^3-y(kdt)\right), \\ \dfrac{y((k+1)dt)-y(kdt)}{dt} = x(kdt)-y(kdt), \end{cases} \quad \forall\, 0 \le k \le N-1 \qquad (\mathrm{II})$$

where $N = \lfloor T/dt \rfloor$, the greatest integer not exceeding T/dt. In (II), we denote $x_k = x(kdt)$ and $y_k = y(kdt)$. Then (II) becomes

$$\begin{cases} x_{k+1} = x_k + \dfrac{dt}{\varepsilon}(x_k - x_k^3 - y_k) \\ y_{k+1} = y_k + dt\,(x_k - y_k). \end{cases} \quad \forall\, 0 \le k \le N-1 \qquad (\mathrm{III})$$

We have got the Euler scheme for (I). The implementation with $x_0 = y_0 = 0.1$, $\varepsilon = 0.3$, $dt = 0.001$, $T = 100$ yields the following picture.



The orbit (x, y) looks periodic after $T = 10$.

Now we show that for any sufficiently small initial data $(x_0, y_0) \neq (0,0)$, ~~the~~ and $\varepsilon \in (0, \frac{4}{13})$, the trajectory of the solution of (I) eventually looks periodic. That is to show the set $\omega(x_0, y_0)$ is equal to a periodic orbit.

First, we show that (I) has a unique solution for $t \in [0, \infty)$ for any initial data (x_0, y_0). Put $z = x - y$. Then

$$z' = x' - y' = \frac{1}{\varepsilon}(z - x^3) - z = \frac{1}{\varepsilon}\left(z - (z+y)^3\right) - z$$

$$= -\frac{z}{\varepsilon}\left(z^2 + 3zy + 3y^2 + \varepsilon - 1\right) - \frac{y^3}{\varepsilon}.$$

Then (I) is equivalent to

$$\begin{cases} y' = z, \\ z' = -\frac{z}{\varepsilon}(z^2 + 3zy + 3y^2 + \varepsilon - 1) - \frac{y^3}{\varepsilon}. \end{cases}$$

Define a map $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(y,z) = \left(z, -\frac{z}{\varepsilon}(z^2+3zy+3y^2+\varepsilon-1) - \frac{y^3}{\varepsilon}\right)$.

Then the above system can be written as

$$(y', z') = f(y, z). \qquad (1)$$

Because $f \in C^1(\mathbb{R}^2; \mathbb{R}^2)$, (1) has a unique solution on a maximal interval of existence. To show that (1) has a solution for $t \in [0, \infty)$ is to show that its local solution does not blow up after finite times. Put

$$F(y, z) = \frac{z^2}{2} + \frac{y^4}{4\varepsilon}.$$

Then $\frac{d}{dt} F(y,z) = \nabla F(y,z) \cdot (y', z') = \left(\frac{y^3}{\varepsilon}, z\right) \cdot \left(z, \frac{-z}{\varepsilon}(z^2+3zy+3y^2+\varepsilon-1) - \frac{y^3}{\varepsilon}\right)$.

$$= \frac{y^3 z}{\varepsilon} - \frac{z^2}{\varepsilon}(z^2+3zy+3y^2+\varepsilon-1) - \frac{zy^3}{\varepsilon}$$

$$= -\frac{z^2}{\varepsilon}(z^2+3zy+3y^2+\varepsilon-1).$$

The initial conditions of (1) are $y(0) = y_0$ and $z(0) = z_0 = x_0 - y_0$. We want to choose a constant $C_1 > 0$ such that $F(y,z) \le C_1$ for all $t \ge 0$. It suffices to choose $C_1 > 0$ such that $F(y_0, z_0) < C_1$ and if $F(y(t_1), z(t_1)) = C_1$ for some $t_1 \in [0, \infty)$ then $\left.\frac{d}{dt}\right|_{t=t_1} F(y,z) \le 0$. Put $y_1 = y(t_1)$ and $z_1 = z(t_1)$.

$$C_1 = F(y_1, z_1) = \frac{z_1^2}{2} + \frac{y_1^4}{4\varepsilon}. \qquad (2)$$

As a consequence, $y_1^4 \le 4\varepsilon C_1$. Thus, $y_1^2 \le 2\sqrt{\varepsilon C_1}$. Then $y_1^4 \le 2\sqrt{\varepsilon C_1}\, y_1^2$.

Then (2) implies

$$C_1 \le \frac{z_1^2}{2} + \frac{2\sqrt{\varepsilon C_1}\, y_1^2}{4\varepsilon} = \frac{z_1^2}{2} + \frac{\sqrt{C_1}}{2\sqrt{\varepsilon}}\, y_1^2$$

$$\le \left(\frac{1}{2} + \frac{\sqrt{C_1}}{2\sqrt{\varepsilon}}\right) \max\{z_1^2, y_1^2\}.$$

Thus, $\max\{z_1^2, y_1^2\} \geqslant \frac{1}{C_1}\left(\frac{1}{2} + \frac{\sqrt{C_1}}{2\sqrt{\varepsilon}}\right)$. (3)

On the other hand,

$$z_1^2 + 3z_1 y_1 + 3y_1^2 = 3\left(\frac{z_1}{2} + y_1\right)^2 + \frac{z_1^2}{4} = \left(z_1 + \frac{3}{2}y_1\right)^2 + \frac{3y_1^2}{4}$$

$$\geqslant \max\left\{\frac{z_1^2}{4}, \frac{3y_1^2}{4}\right\} \geqslant \frac{1}{4}\max\{z_1^2, y_1^2\} \overset{(3)}{\geqslant} \frac{1}{4C_1}\left(\frac{1}{2} + \frac{\sqrt{C_1}}{2\sqrt{\varepsilon}}\right).$$

Then $\left.\frac{d}{dt}\right|_{t=t_1} F(y,z) = \frac{-z_1^2}{\varepsilon}\left(z_1^2 + 3z_1 y_1 + 3y_1^2 + \varepsilon - 1\right) \le \frac{-z_1^2}{\varepsilon}\left[\underbrace{\frac{1}{4C_1}\left(\frac{1}{2} + \frac{\sqrt{C_1}}{2\sqrt{\varepsilon}}\right) + \varepsilon - 1}_{\{1\}}\right]$

Since $\lim_{C_1 \to 0^+} \{1\} = \infty$, there exists $C_1 > 0$ such that $\{1\} \geqslant 0$.

Then $\left.\frac{d}{dt}\right|_{t=t_1} F(y,z) \le 0$. In the particular case $y_0 = 0.1$, $z_0 = x_0 - y_0 = 0$, $\varepsilon = 0.3$, we can choose $C_1 = F(y_0, z_0) = \frac{0.1^4}{4 \times 0.3} \approx 10^{-4}$.

Next, we show that if $\varepsilon \in (0, \frac{4}{13})$ and $(y_0, z_0) \neq (0,0)$ then $(0,0) \notin \omega(y_0, z_0)$. It suffices to find a constant $C_2 > 0$ such that $F(y_0, z_0) \geqslant C_2$ and if $F(y(t_2), z(t_2)) = C_2$ for some $t_2 \in [0, \infty)$ then $\left.\frac{d}{dt}\right|_{t=t_2} F(y,z) \geqslant 0$.

Put $y_2 = y(t_2)$ and $z_2 = z(t_2)$.

$$C_2 = F(y_2, z_2) = \frac{z_2^2}{2} + \frac{y_2^4}{4\varepsilon} \geqslant \frac{z_2^2}{2} + \frac{2\varepsilon y_2^2 - \varepsilon^2}{4\varepsilon} \geqslant \frac{z_2^2}{2} + \frac{y_2^2}{2} - \frac{\varepsilon}{4}.$$

Thus, $$\frac{z_2^2+y_2^2}{2} \leq C_2+\frac{\varepsilon}{4}. \qquad (4)$$

On the other hand,

$$z_2^2+3z_2y_2+3y_2^2 \leq z_2^2+\frac{3}{2}(z_2^2+y_2^2)+3y_2^2 \leq \frac{9}{2}(z_2^2+y_2^2) \overset{(4)}{\leq} 9\left(C_2+\frac{\varepsilon}{4}\right).$$

Thus, $$\left.\frac{d}{dt}\right|_{t=t_2} F(y,z) = \frac{-z_2^2}{\varepsilon}\left(z_2^2+3z_2y_2+3y_2^2+\varepsilon-1\right)$$
$$\geq -\frac{z_2^2}{\varepsilon}\Big(\underbrace{9\left(C_2+\frac{\varepsilon}{4}\right)+\varepsilon-1}_{\{2\}}\Big)$$

For any choice $0<C_2\leq\frac{1}{9}\left(1-\frac{13\varepsilon}{4}\right)$, we have $\{2\}\leq 0$. Then $\left.\frac{d}{dt}\right|_{t=t_2} F(y,z)\geq 0$.

In the particular case $y_0=0.1$, $z_0=0$, $\varepsilon=0.3$, we can choose $C_2=F(y_0,z_0)$ $=\frac{0.1^4}{4\times 0.3}\approx 10^{-4}$. We have showed that $(0,0)\notin\omega(y_0,z_0)$.

The flow given by (1) is a planar flow. Because $F(y,z)\leq C_1$ for all $t\geq 0$, the forward orbit $\gamma^+(y_0,z_0)$ is bounded. By Poincaré-Bendixson theorem (Theorem 24.6, Amann "Ordinary Differential Equations" 1990, page 333), $\omega(y_0,z_0)$ either contains an equilibrium or equals a periodic orbit. We have $f(z,y)=0$ if and only if $(z,y)=(0,0)$. Thus, $(0,0)$ is the only equilibrium of the flow given by (1). This means $\omega(x_0,y_0)$ is equal to a periodic orbit.