Math 8501, Fall 2014

Theory of Ordinary Differential Equations

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— *Homework 1* —

(1) We want to show that there exists a unique solution $u : \mathbb{R} \to \mathbb{R}^n$ of $u' = f(u)$, $u(0) = u_0$ provided f is globally Lipschitz, $|f(u_1) - f(u_2)| \leq L|u_1 - u_2|$ for some $L > 0$ and all u_1, u_2. Therefore define Y_η as the set of continuous functions for which the norm in Y_η,

$$|u|_{Y_\eta} := \sup_{t\in\mathbb{R}} |u(t)\mathrm{e}^{-\eta|t|}|,$$

is bounded, and define the usual fixed point operator

$$[Tu](t) = u_0 + \int_0^t f(u(s))\mathrm{d}s.$$

(i) Show that Y_η is a Banach space.

(i) Show that T maps Y_η into itself.

(i) Show that T is a contraction for η large enough.

(2) Consider the system

$$\dot{x} = y^2$$
$$\dot{y} = -yx^3,$$

with initial conditions (x_0, y_0). Show that all solutions converge as $t \to \infty$, and find the limit $\lim_{t\to\infty} x(t)$ in terms of (x_0, y_0). For this, find an Euler multiplier so that the system becomes Hamiltonian and analyze the level sets of the Hamiltonian.

Homework is due on Friday, September 19, in class