Math 8501, Fall 2014

Theory of Ordinary Differential Equations

Arnd Scheel, VinH 509, phone 625-4065, scheel@umn.edu

— *Homework 5* —

(1) Consider $\dot{u} = f(u) + \varepsilon g(t,u)$ with $f, g \in C^1$, $f(0) = 0$, and $A = Df(0)$ hyperbolic, for ε small.

(i) Show that the linearization $\frac{\mathrm{d}}{\mathrm{d}t} - A$ is invertible as an operator from $C^1(\mathbb{R}, \mathbb{R}^n)$ into $C^0(\mathbb{R}, \mathbb{R}^n)$.

(ii) Show that there exists a small neighborhood of the origin $\mathcal{U}(0)$ so that for all ε sufficiently small, there exists a unique solution $u(t)$ such that $u(t) \in \mathcal{U}(0)$ for all $t \in \mathbb{R}$.

(iii) Show that $u(t)$ is 1-periodic if $g(t,u)$ is 1-periodic in t.

(2) Consider

$$x' = -x + ay^2$$
$$y' = -2y + bx^2.$$

(i) Draw the phase portrait of the linearization and express trajectories as graphs $x = h(y)$ or $y = h(x)$.

(ii) Find the Taylor jet of the (smooth) strong stable manifold, $x = h(y)$ for the nonlinear system up to order two.

(iii) Try to find a "weak stable" manifold tangent to $\{y = 0\}$, by calculating a quadratic Taylor jet of $y = h(x)$ — what goes wrong?

(iv) Set $a = 0$ and compute the solutions explicitly. Express y as a function of x. Show that there are many invariant manifolds $y = h(x)$ but none is C^2 because of terms of the form $x^2 \log x$.

(3) Consider the linear equation $\dot{x} = Ax$, $A = \mathrm{diag}\,(\lambda_j)$, $\lambda_1 > \lambda_2 > \ldots > \lambda_n$.

(i) Derive an equation for the projectivized flow, that is, write $x = u \cdot |x|$ and derive an equation for $u \in S^{n-1}$. Find all equilibria of this flow on the sphere.

(ii) Show that the Rayleigh quotient $V(u) = -\frac{1}{2}\langle Au, u\rangle$ is a strict Lyapunov function, that is, strictly decreasing for non-equilibrium stolutions. Which equilibria are stable?

(iii) Conclude that all trajectories are heteroclinic and describe heteroclinic orbits.

(iv) Describe equilibria and heteroclinic orbits for the (non-self-adjoint) $A = \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix}$ for all $\mu \in \mathbb{R}$?

(v) *Alternative to the above:* Create a phase portrait on S^2 numerically when $\lambda_j = -j$.

(4) Implement classical Runge-Kutta for the competing species problem studied in class and demonstrate numerically that the method is of order 4.

(5) Study error propagation numerically in the Lorenz model,

$$\begin{aligned} x' &= \sigma(y - x) \\ y' &= x(\rho - z) - y \\ z' &= xy - \beta z, \end{aligned}$$

$\sigma = 10$, $\beta = 8/3$,$\rho = 28$. Therefore, start with $x = y = z = 0.1$ and integrate using Euler and/or Matlab's RK solver. Compare the solutions at various time intervals and for various step sizes: when do solutions differ qualitatively? Now study the difference between the solutions with initial conditions $x = y = 0.1$ and $x = y = z = 0.1 + \varepsilon$, ε small. Demonstrate numerically that the difference grows exponentially for a certain time.

(6) Find (complex) stability regions for

$$\begin{aligned} u_{n+1} &= u_n + hf(u_n + \frac{h}{2}f(u_n)), \\ u_{n+1} &= u_n + \frac{h}{2}(f(u_n) + f(u_{n+1})). \end{aligned}$$

Alternative/Optional: Find the (real) stability boundaries numerically.

Homework is due on Wednesday, November 26, in class. Choose three, or more for extra credit.