Math 8501, Fall 2014

Dynamical Systems and Differential Equations I

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— Homework 6 —

(1) Consider

$$A' = B, \quad B' = -A + A^2\bar{A}, \qquad A, B \in \mathbb{C},$$

which possesses a family of periodic orbits parameterized by $k \in (-1, 1)$,

$$A_*(t) = R\mathrm{e}^{\mathrm{i}kt}, \quad B_*(t) = \mathrm{i}kR\mathrm{e}^{\mathrm{i}kt}, \qquad R = \sqrt{1-k^2}.$$

Our goal is to compute the Floquet multipliers for the linearization at $(A_*, B_*)(t)$.

(i) Convince yourself that the linearization can be written in the form

$$\begin{aligned} A' &= B, \qquad & B' &= -A + 2R^2 A + R^2 \mathrm{e}^{2\mathrm{i}kt}\bar{A}, \\ \bar{A}' &= \bar{B}, \qquad & \bar{B}' &= -\bar{A} + 2R^2\bar{A} + R^2\mathrm{e}^{-2\mathrm{i}kt}A. \end{aligned}$$

Verify that $(A_*', B_*', \bar{A}_*', \bar{B}_*')$ are solutions to this linear equation.

(ii) Show that the change of coordinates $(A, B, \bar{A}, \bar{B}) = (\mathrm{e}^{\mathrm{i}kt}a, \mathrm{e}^{\mathrm{i}kt}b, \mathrm{e}^{-\mathrm{i}kt}\bar{a}, \mathrm{e}^{-\mathrm{i}kt}\bar{b})$ transforms the system into a constant-coefficient equation. Interpret this change of coordinates in terms of Floquet theory, that is, find $\mathcal{Q}(t)$ and $\mathcal{B}$.

(iii) Find the eigenvalues of $\mathcal{B}$ and their muiltiplicities for $k^2 < 1/3$, $k^2 > 1/3$, and $k^2 = 1/3$.

(2) Compute the Floquet exponents of the periodic orbit in the van der Pol oscillator numerically using Euler's method. Therefore, consider

$$x' = \frac{1}{\varepsilon}(x - x^3 - y), \quad y' = x - y$$

For the following, $\varepsilon = 0.3$ and $dt = 0.001$ are reasonable choices.

(i) Write a simple Euler scheme starting with $x = 0.1, y = 0.1$, say, and iterate for a long time ($T = 100$, say); verify that the trajectory "looks" periodic, eventually.

(ii) With initial condition taken as the final value from the previous step, integrate until $y = 0$ and $x > 0$ (for instance use a while loop to check for sign change of y during iteration).

(iii) Compute one further loop to compute the periodic orbit for precisely one more period; test the dependence of the period on ε and dt.

(iv) To this last loop over one period, add the Euler time stepping for the linear variational equation with initial conditions $(1,0)^T$ and $(0,1)^T$ (this is two separate Euler steppings!). The final values put together as a matrix give $\Phi_{T,0}$ (why?), the eigenvalues are the Floquet multipliers. Also compute the Floquet exponents (for checking: $\lambda_1 \sim 0, \lambda_2 \sim -1.7$).

Due December 10 in class. Choose one of the two exercieses or two from the last homework (that you did not already complete).