Name: Tuan Pham
ID: 4652218

Math 8502: Differential Equations & Dynamical Systems

Homework # 1

① Consider the perturbed Jordan block with parameter $\mu \in \mathbb{R}$.

$$A(\mu) = \begin{pmatrix} b_0(\mu) & 1 & & & \\ b_1(\mu) & b_0(\mu) & 1 & & \\ b_2(\mu) & b_1(\mu) & b_0(\mu) & 1 & \\ \vdots & \ddots & \ddots & \ddots & 1 \\ b_{n-1}(\mu) & \cdots & b_2(\mu) & b_1(\mu) & b_0(\mu) \end{pmatrix}$$

Suppose b_j's are analytic functions, $b_j(0) = 0$ for all $0 \le j \le n-1$ and $b_{n-1}'(0) \neq 0$. We determine all eigenvalues to the leading order of μ near $\mu = 0$.

$\lambda \in \mathbb{C}$ is an eigenvalue of $A(\mu)$ if the equation $(A(\mu) - \lambda)u = 0$ has a nonzero solution $u = (u_1, u_2, \dots, u_n)$. In matrix form, the equation becomes

$$\begin{pmatrix} b_0(\mu)-\lambda & 1 & & & \\ b_1(\mu) & b_0(\mu)-\lambda & 1 & & \\ b_2(\mu) & b_1(\mu) & b_0(\mu)-\lambda & 1 & \\ \vdots & & \ddots & \ddots & 1 \\ b_{n-1}(\mu) & \cdots & \cdots & b_1(\mu) & b_0(\mu)-\lambda \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

We get

$$\begin{cases} u_2 = -(b_0 - \lambda) u_1, & (1) \\ u_3 = -b_1 u_1 - (b_0 - \lambda) u_2, \\ u_4 = -b_2 u_1 - b_2 u_2 - (b_0 - \lambda) u_3, \\ \dots \\ u_n = -b_{n-2} u_1 - b_{n-3} u_2 - \dots - b_1 u_{n-2} - (b_0 - \lambda) u_{n-1}, \\ b_{n-1} u_1 + b_{n-2} u_2 + \dots + b_1 u_{n-1} + (b_0 - \lambda) u_n = 0. & (2) \end{cases}$$

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We show by induction on $k \in \{2,3,\dots,n\}$ that

$$u_k = \left[\lambda^{k-1} + \sum_{j=0}^{k-2} P_{k,j}(b_0, b_1, \dots, b_{n-1}) \lambda^j \right] u_1 \qquad (3)$$

where each $P_{k,j}$ is a polynomial with the constant term equal to zero.

For $k=2$, (3) is true because of (1). Suppose that (3) is true for $k \in \{2,3,\dots,m\}$ where $m<n$. Then

$$u_{m+1} = \underbrace{-b_{m-1} u_1 - b_{m-2} u_2 - \cdots - b_1 u_{m-1}} - (b_0 - \lambda) u_m$$

$$= \sum_{j=0}^{m-2} Q_j(b_0, b_1, \dots, b_{n-1}) \lambda^j u_1 - (b_0 - \lambda) \left[\lambda^{m-1} + \sum \tilde{Q}_j(b_0, \dots, b_{n-1}) \lambda^j \right] u_1$$

(where Q_j and $\tilde{Q}_j$ are polynomials with the constant terms equal to zero)

$$= \left[\lambda^m + \sum_{j=0}^{m-1} P_{m+1,j}(b_0, b_1, \dots, b_{n-1}) \lambda^j \right] u_1$$

where $P_{m+1,j}$ is a polynomial with constant term equal to zero.

Therefore, (3) is true for $k=m+1$. Substituting $u_2, u_3, \dots, u_n$ obtained from (3) into (2), we get the equation

$$0 = b_{n-1} u_1 + b_{n-2} u_2 + \cdots + b_1 u_{n-1} + (b_0 - \lambda) u_n$$

$$= b_{n-1} u_1 + \sum_{j=0}^{n-1} P_j(b_0, b_1, \dots, b_{n-1}) \lambda^j u_1 + (b_0 - \lambda) \left[\lambda^{n-1} + \sum_{j=0}^{n-2} \tilde{P}_j(b_0, \dots, b_{n-1}) \lambda^j \right] u_1$$

where P_j is a polynomial whose each monomial has degree ≥ 2, and $\tilde{P}_j$ is a polynomial with constant term equal to zero. Thus,

$$0 = \left[b_{n-1} + \sum_{j=0}^{n-1} \tilde{\tilde{P}}_j(b_0, b_1, \dots, b_{n-1}) \lambda^j - \lambda^n \right] u_1 \qquad (4)$$

where $\tilde{\tilde{P}}_0$ is a polynomial whose each monomial has degree ≥ 2, and $\tilde{\tilde{P}}_1, \tilde{\tilde{P}}_2, \dots, \tilde{\tilde{P}}_{n-1}$ are polynomials with constant terms equal to zero.

Because $b_{n-1}(0)=0$ and $b'_{n-1}(0)\neq 0$, we have $b_{n-1}(\mu)=b'_{n-1}(0)\mu+O(\mu^2)$.

Because $b_j(0)=0$, $b_j(\mu)=O(\mu)$. Then

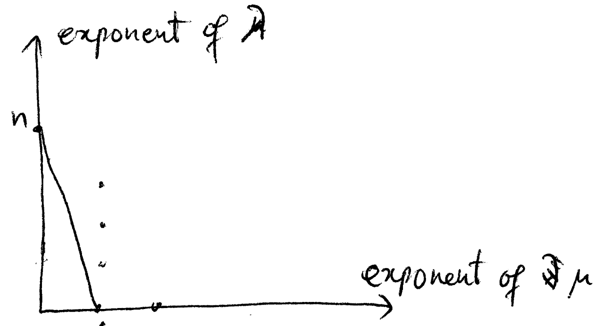
$$\tilde{P}_0(b_0(\mu),b_1(\mu),\dots,b_{n-1}(\mu)) = O(\mu^2),$$

$$\tilde{P}_j(b_0(\mu),b_1(\mu),\dots,b_{n-1}(\mu))\lambda^j = O(\mu)O(\lambda^j) \quad \forall 1\leq j\leq n-1.$$

For $u=(u_1,u_2,\dots,u_n)$ to be nonzero, it is necessary that $u_1\neq 0$. Then (4), becomes

$$b'_{n-1}(0)\mu-\lambda^n+O(\mu^2)+\sum_{j=1}^{n-1}O(\mu)O(\lambda^j)=0. \qquad (5)$$

The Newton polygon corresponding to this equation looks as follows.



There is only one edge not lying on either axis. That is the edge connecting $(1,0)$ to $(0,n)$. In order to get an appropriate balance of the exponents in (5), we need $\lambda^n=\alpha\mu+o(\mu)$. Then (5) becomes

$$(b'_{n-1}(0)-\alpha)\mu+o(\mu)+O(\mu^2)+\sum_{j=1}^{n-1}O(\mu)O(|\mu|^{\frac{j}{n}})=0.$$

The coefficient corresponding to the lowest order of μ must be zero. Thus, $\alpha=b'_{n-1}(0)$. Then $\lambda^n=b'_{n-1}(0)\mu+o(\mu)$, which gives us n complex values

$$\lambda\in\sqrt[n]{b'_{n-1}(0)\mu}+o(|\mu|^{\frac{1}{n}}).$$

(2) Consider a Hamiltonian system

$$\begin{cases}\dot{x}=\partial_y H(x,y),\\ \dot{y}=-\partial_x H(x,y),\end{cases}$$

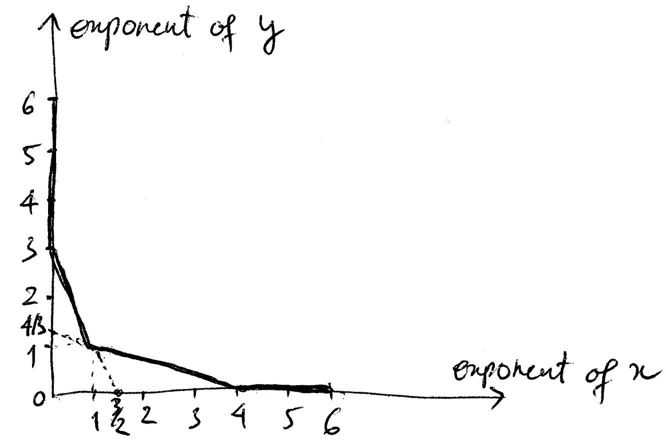
with analytic function H. Suppose $H(x,y) = x^4 + 5xy + y^3 + O(x^6+y^6)$ as $(x,y) \to (0,0)$. We have

$$\frac{dH}{dt}(x,y) = \partial_x H(x,y)\dot{x} + \partial_y H(x,y)\dot{y} = \partial_x H(x,y)\partial_y H(x,y) - \partial_y H(x,y)\partial_x H(x,y) = 0$$

Thus, $H(x(t),y(t)) \equiv C$ (constant). A trajectory $(x(t),y(t))$ that converges to $(0,0)$ as $t \to \infty$ or $t \to -\infty$ corresponds to the choice $C = H(0,0) = 0$. We get

$$x^4 + 5xy + y^3 + O(x^6+y^6) = 0. \qquad (1)$$

We now find $x = \varphi(y)$ by giving the first order of φ near $y = 0$. The Newton polygon corresponding to Equation (1) is as follows.



There are two edges not lying on either axis. One connects $(1,1)$ to $(4,0)$, the extension of which meets the y-axis at $4/3$, one connects $(1,1)$ to $(0,3)$, the extension of which meets the x-axis at $3/2$. This means there are two acceptable ways to balance the exponents in Equation (1).

- $\underline{y^{4/3} \sim x^4}$

Write $x = \alpha\sqrt[3]{y} + o(y^{1/3})$ where $\alpha \neq 0$ is to be determined. Then (1) becomes

$$\alpha^4 y^{4/3} + 5\alpha y^{4/3} + y^3 + o(y^{4/3}) = 0$$

$$\Longleftrightarrow \alpha y^{4/3}(\alpha^3+5) + o(y^{4/3}) = 0.$$

Thus, $\alpha^3+5=0$, which gives $\alpha = -\sqrt[3]{5}$. We get $x=\varphi(y) = -\sqrt[3]{5y} + o(y^{1/3})$.

$\underline{y^3 \sim x^{3/2}}$

Write $x = \beta y^2 + o(y^3)$ where $\beta \neq 0$ is to be determined. Then (1) becomes

$$\beta^4 y^8 + 5\beta y^3 + y^3 + O(y^6) = 0$$

$$\Longleftrightarrow (5\beta+1)y^3 + O(y^6) = 0.$$

Thus, $5\beta+1=0$, which gives $\beta = -1/5$. We get $x = \varphi(y) = -\frac{1}{5}y^2 + o(y^2)$.

In conclusion, we get two trajectories $x=\varphi(y)$ distinguished by their rates of convergence to 0 as $y \to 0$.

(i) $x = \varphi(y) = -\sqrt[3]{5y} + o(y^{1/3})$.

(ii) $x = \varphi(y) = -\frac{1}{5}y^2 + o(y^2)$.

③ Consider a ring of 3 cells with anti-diffusive coupling.

$$\begin{cases} \dot{u}_0 = f(u_0) + d(u_1 - 2u_0 + u_2) \\ \dot{u}_1 = f(u_1) + d(u_2 - 2u_1 + u_0) \\ \dot{u}_2 = f(u_2) + d(u_0 - 2u_2 + u_1) \end{cases} \qquad (I)$$

where $f(0)=0$, $f'(0)=-1$, $f''(0)\neq 0$. First, we determine $d \in (-1,0)$ such that the equilibrium $(u_0,u_1,u_2) \equiv (0,0,0)$ is not hyperbolic. Put $u = (u_0,u_1,u_2)$. Then

$$\begin{pmatrix} f(u_0) \\ f(u_1) \\ f(u_2) \end{pmatrix} = -u + g(u), \text{ where } g(u) = \begin{pmatrix} f(u_0)+u_0 \\ f(u_1)+u_1 \\ f(u_2)+u_2 \end{pmatrix} = \mathcal{O}(|u|^2) \text{ as } u \to 0.$$

Thus, system (I) is of the form

$$\dot{u} = -u + g(u) + d\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} u.$$

Put

$$A = A(d) = \begin{pmatrix} -2d-1 & d & d \\ d & -2d-1 & d \\ d & d & -2d-1 \end{pmatrix}.$$

We get $\dot{u} = Au + g(u)$. The equilibrium $u \equiv 0$ is not hyperbolic if and only if A has a purely imaginary eigenvalue. Since A is a real symmetric matrix, all of its eigenvalues are real. Thus, the equilibrium $u \equiv 0$ is not hyperbolic if and only if 0 is an eigenvalue of A. This is equivalent to $\det A = 0$.

$$\det A = -(3d+1)^2.$$

Thus, $\det A = 0$ if and only if $d = -1/3$.

Now we calculate all equilibria of the form $u = (u_0, u_1, u_1)$ in terms of d when d is close to $-1/3$. Put $\mu = d + \frac{1}{3}$. System (I) becomes

$$\begin{cases} f(u_0) + 2(\mu - \frac{1}{3})(u_1 - u_0) = 0 & (1) \\ f(u_1) + (\mu - \frac{1}{3})(u_0 - u_1) = 0. & (2) \end{cases}$$

Equation (1) implies

$$u_1 = u_0 - \frac{3}{2}\frac{f(u_0)}{3\mu - 1}. \qquad (3)$$

Substituting this expression of u_1 into (2), we get

$$f\left(u_0 - \frac{3}{2}\frac{f(u_0)}{3\mu-1}\right) + \frac{1}{2}f(u_0) = 0. \qquad (4)$$

We have

$$u_0 - \frac{3}{2}\frac{f(u_0)}{3\mu-1} = u_0 - \frac{3}{2}\left(-u_0 + \frac{f''(0)}{2}u_0^2 + O(u_0^3)\right)\left(1+3\mu+9\mu^2+O(\mu^3)\right)$$

$$= -\frac{u_0}{2} - \frac{9}{2}u_0\mu + \frac{3}{4}f''(0)u_0^2 - \frac{27}{2}u_0\mu^2 + O(u_0^2)O(\mu^2) + O(\mu^3)u_0 + O(u_0^3). \quad (5)$$

Thus,

$$f\left(u_0 - \frac{3}{2}\frac{f(u_0)}{3\mu-1}\right) = -\left(u_0 - \frac{3}{2}\frac{f(u_0)}{3\mu-1}\right) + \frac{f''(0)}{2}\left(u_0 - \frac{3}{2}\frac{f(u_0)}{3\mu-1}\right)^2 + O\left(\left(u_0 - \frac{3}{2}\frac{f(u_0)}{3\mu-1}\right)^3\right)$$

$$\overset{(5)}{=\!=} \left(\frac{u_0}{2} + \frac{9}{2}u_0\mu - \frac{3}{4}f''(0)u_0^2 + \frac{27}{\cancel{4}\,2}u_0\mu^2\right) + \frac{f''(0)}{2}\left(\frac{u_0^2}{4} + \frac{9}{2}u_0^2\mu\right)$$

$$+ O(u_0^2)O(\mu^2) + O(\mu^3)u_0 + O(u_0^3). \quad (6)$$

Also, $\frac{1}{2}f(u_0) = -\frac{u_0}{2} + \frac{f''(0)}{4}u_0^2 + O(u_0^3)$. (7)

Substituting (6), (7) into (4), we get

$$\frac{9}{2}u_0\mu - \frac{3}{8}f''(0)u_0^2 + \frac{27}{2}f''(0)u_0\mu^2 + \frac{9}{2}f''(0)u_0^2\mu + O(u_0^2)O(\mu^2) +$$

$$+ O(\mu^3)u_0 + O(u_0^3) = 0. \quad (8)$$

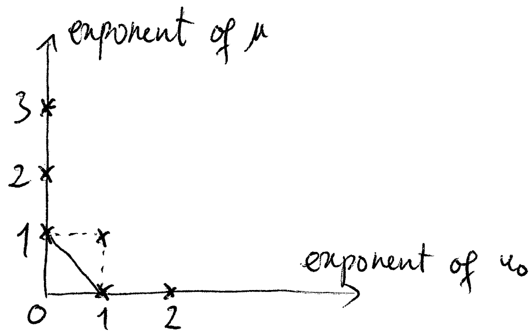
If $u_0 = 0$ then $u_1 = 0$. Thus we get the trivial equilibrium $u \equiv 0$.

Consider $u_0 \neq 0$. Dividing both sides of (8) by u_0, we get

$$\frac{9}{2}\mu - \frac{3}{8}f''(0)u_0 + \frac{27}{2}f''(0)\mu^2 + \frac{9}{2}f''(0)u_0\mu + O(u_0)O(\mu^2) + O(\mu^3) +$$

$$+ O(u_0^2) = 0. \quad (9)$$

The corresponding Newton polygon looks as follows.



There is only one edge not lying on either axis. It connects $(1,0)$ with $(0,1)$. This means $u_0 \sim \mu$ as $\mu \to 0$. Write $u_0 = \alpha\mu + o(\mu)$. Then (9) becomes

$$\left(\frac{9}{2} - \frac{3}{8} f''(0)\alpha\right)\mu + o(\mu) = 0,$$

which gives $$\alpha = \frac{12}{f''(0)}.$$

Thus, $u_0 = \dfrac{12}{f''(0)}\mu + o(\mu)$. By (5),

$$u_1 = u_0 - \frac{3}{2}\frac{f(u_0)}{3\mu - 1} = -\frac{u_0}{2} + o(\mu) = -\frac{6}{f''(0)}\mu + o(\mu).$$

Therefore,
$$\begin{cases} u_0 = \dfrac{12}{f''(0)}\left(d+\dfrac{1}{3}\right) + o\left(d+\dfrac{1}{3}\right), \\[2ex] u_1 = u_2 = -\dfrac{6}{f''(0)}\left(d+\dfrac{1}{3}\right) + o\left(d+\dfrac{1}{3}\right). \end{cases}$$

4 Let $\{A(\mu_1,\mu_2)\}_{\mu_1,\mu_2 \in [0,1]}$ be a family of $n \times n$ matrices with real coefficients. Suppose each coefficient is a C^r-function in (μ_1,μ_2) for some $r > \max\{0, 2-n^2\}$. We show that there exists an $n \times n$ matrix B, with norm less than a given number $\varepsilon > 0$, such that $\ker(A(\mu_1,\mu_2)+B)$ is at most one-dimensional for all $\mu_1,\mu_2 \in [0,1]$.

Put $M = [0,1]\times[0,1]$ and $N = \mathbb{R}^{n\times n}$. Define a map $F: M \times N \to N$,

$$F(M_1, M_2, B) = A(M_1, M_2) + B \quad \forall (M_1, M_2) \in M, \forall B \in N.$$

Then F is a C^1-function. For each $j \in \{0, 1, \dots, n-1\}$, put

$$W_j = \{X \in \mathbb{R}^{n \times n} : \operatorname{rank}(X) = j\}.$$

We explain why W_j is a smooth manifold of dimension $j(2n-j)$. This is true for $j = 0$ because $W_0 = \{0\}$. Consider $1 \le j \le n-1$. For $1 \le i_1 < i_2 < \dots < i_j \le n$, put

$$U_{i_1, \dots, i_j} = \{X \in \mathbb{R}^{n \times n} : \text{the columns } i_1, i_2, \dots, i_j \text{ are linearly independent, other columns are linear combinations of these columns}\}.$$

~~Define a map~~ $\cancel{\phi_{i_1, \dots, i_j} : \mathbb{R}^{j(n-j)} \times \mathbb{R}^{nj} \to U_{i_1, \dots, i_j}}$

For each $X \in U_{i_1, \dots, i_j}$, we write $X = (v_1 \ v_2 \ \cdots \ v_n)$ where each v_i is a column vector. For $l \notin \{i_1, \dots, i_j\}$, v_l is a linear combination of $\{v_{i_1}, \dots, v_{i_j}\}$. Thus, v_l corresponds to a ~~ty~~ tuple of j real numbers $\alpha_1^{(l)}, \alpha_2^{(l)}, \dots, \alpha_j^{(l)}$. Then the columns $(v_l)_{l \notin \{i_1, \dots, i_j\}}$ corresponds to a tuple of $j(n-j)$ real numbers $(\alpha_k^{(l)})_{\substack{l \notin \{i_1, \dots, i_j\} \\ 1 \le k \le j}}$. We need jn real numbers to constitute the vectors $v_{i_1}, v_{i_2}, \dots, v_{i_j}$ from the standard basis of $\mathbb{R}^n$. Totally, we need $j(n-j) + jn = j(2n-j)$ real numbers to constitute an element $X \in U_{i_1, \dots, i_j}$. This correspondance gives us a bijective map $\phi_{i_1, \dots, i_j} : \mathbb{R}^{j(2n-j)} \to U_{i_1, \dots, i_j}$. It is actually a homeomorphism because all dependences are linear. The transition map $\phi_{i_1, \dots, i_j} \circ \phi^{-1}_{k_1, \dots, k_j}$ involves only addition, multiplication and division by nonzero numbers. Thus, it is a smooth map. Because

$$W_j = \bigcup_{1 \le i_1 < \dots < i_j \le n} U_{i_1, i_2, \dots, i_j},$$

W_j becomes a smooth manifold with $\binom{n}{j}$ coordinate charts $(\phi_{i_1, \dots, i_j})$. Moreover, W_j has topological dimension $j(2n-j)$.

Next, we show that F is transverse to every W_j, $0 \le j \le n-1$. Because $F(\mu_1, \mu_2, B) = A(\mu_1, \mu_2) + B$,

$$DF(\mu_1, \mu_2, B) = \underbrace{\left(\begin{array}{c|c|c} D_{\mu_1}A & D_{\mu_2}A & I_{n^2} \end{array} \right)}_{n^2+2} \Big\} n^2$$

Since $DF(\mu_1, \mu_2, B)$ has full rank, it is surjective as a map from $\mathbb{R}^{n^2+2}$ to $\mathbb{R}^{n^2}$. Thus,

$$\mathrm{Range}(DF(\mu_1, \mu_2, B)) + T_{(\mu_1, \mu_2, B)} W_j = \mathbb{R}^{n^2},$$

at any intersection point (μ_1, μ_2, B) of F and W_j. Thus, F is transverse to W_j.

For each $0 \le j \le n-1$, we put $\mathcal{A}_j = \{B \in \mathbb{R}^{n \times n} : F(\cdot, B) \text{ is transverse to } W_j\}$. By Theorem 10.14, Chow-Hale "Methods of Bifurcation Theory" 1982, page 65, $\mathcal{A}_j$ is residual in $\mathbb{R}^{n^2}$, i.e. equal to the countable intersection of open dense sets. Put $\mathcal{A} = \bigcap_{j=0}^{n-1} \mathcal{A}_j$. Then $\mathcal{A}$ is equal to the countable intersection of open dense subsets of $\mathbb{R}^{n^2}$. By Baire category theorem, $\mathcal{A}$ is dense in $\mathbb{R}^{n^2}$. For each $B \in \mathcal{A}$, $F(\cdot, B)$ is transverse to $W_0, W_1, \dots, W_{n-1}$.

Suppose by contradiction that $F(\mu_1^*, \mu_2^*, B) \in W_j$ for some $(\mu_1^*, \mu_2^*) \in M$, $0 \le j < n-1$. Then

$$D_{(\mu_1, \mu_2)} F(\mu_1^*, \mu_2^*, B) + T_{(\mu_1^*, \mu_2^*)} W_j = \mathbb{R}^{n^2} \qquad (1)$$

We have

$$\dim D_{(\mu_1,\mu_2)} F(\mu_1^*, \mu_2^*, B) \le 2,$$

$$\dim T_{(\mu_1^*, \mu_2^*)} W_j = \dim W_j = j(2n-j).$$

Then the equality (1) implies $n^2 \le 2 + j(2n-j)$, or in other words $(n-j)^2 \le 2$. This is a contradiction because $n-j \geq 2$. Therefore, $F(\mu_1, \mu_2, B) \notin W_j$ for all $(\mu_1, \mu_2) \in M$, $B \in \mathcal{A}$, $0 \le j < n-1$

We have showed that $F(\mu_1, \mu_2, B)$ has at least rank $n-1$ for all $(\mu_1, \mu_2) \in M$, $B \in \mathcal{A}$. In other words, $A(\mu_1, \mu_2) + B$ has kernel with dimension $\le n-(n-1) = 1$. Because $\mathcal{A}$ is dense in $\mathbb{R}^{n^2}$, B can be chosen arbitrarily close to 0 such that the above property still holds. Therefore, the matrices $A(\mu_1, \mu_2)$, $\mu_1, \mu_2 \in [0,1]$, can be pertubed simultaneously by adding a small matrix so that their kernels have dimension ≤ 1.