Name : Tuan Pham

ID: 4652218

Math 8502: Theory of ODE

Homework #2

① (i) Consider the ODE

$$x' = \mu_1 + \mu_2 x - x^3, \qquad (1)$$

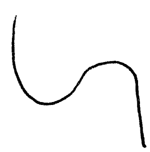
where $\mu_1, \mu_2 \in \mathbb{R}$ are parameters. We determine the region for (μ_1, μ_2) in $\mathbb{R}^2$ so that the ODE has two stable equilibria. Let

$$g(y) = \mu_1 + \mu_2 y - y^3 \quad \forall y \in \mathbb{R}.$$

Then (1) becomes

$$x' = g(x) \qquad (2)$$

An equilibrium of (2) is a real root of g. To get two stable equilibria, g must have at least two distinct real roots. Because $\lim_{y \to \infty} g(y) = -\infty$, its graph looks like



Thus, g has two distinct real roots if and only if g' has two distinct roots y_1 and y_2 and $g(y_1) g(y_2) \leq 0$.

$$g'(y) = \mu_2 - 3y^2.$$

We require $\mu_2 > 0$ so that g' has two distinct roots $y_1 = \frac{-\sqrt{\mu_2}}{\sqrt{3}}$ and $y_2 = \frac{\sqrt{\mu_2}}{\sqrt{3}}$.

$$g(y_1) = \mu_1 + \mu_2 y_1 - y_1^3 = \mu_1 - \frac{2\mu_2^{3/2}}{3^{3/2}},$$

$$g(y_2) = \mu_2 + \mu_2 y_2 - y_2^3 = \mu_1 + \frac{2\mu_2^{3/2}}{3^{3/2}}.$$

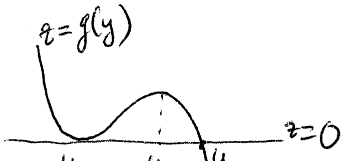
Then $\quad g(y_1) g(y_2) = \mu_1^2 - \frac{4\mu_2^3}{27}.$

$\boxed{2}$

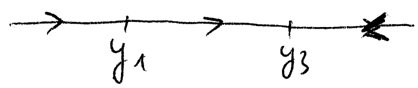
Then $g(y_1)g(y_2) \le 0 \iff |\mu_1| \le \frac{2}{3^{3/2}} \mu_2^{3/2}$.

Consider the following cases.

- $\underline{g(y_1) = 0}$.

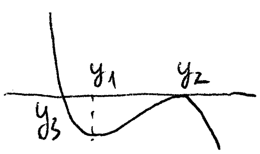


Then g has roots y_1 with multiplicity 2, and y_3. Based on the signs of g in the intervals $(-\infty, y_1)$, (y_1, y_3), (y_3, ∞), we obtain the phase diagram for (2).

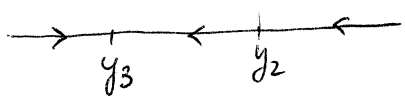


In this case, (2) has only one stable equilibrium.

- $\underline{g(y_2) = 0}$.



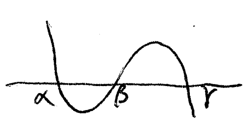
Then g has roots y_2 with multiplicity 2, and y_3. Based on the sign of g in the intervals $(-\infty, y_3)$, (y_3, y_2), (y_2, ∞), we get the phase diagram for (2).



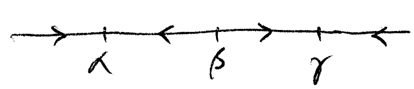
In this case, (2) has only one stable equilibrium.

- $\underline{g(y_1) \neq 0 \text{ and } g(y_2) \neq 0}$

Then g has three distinct roots α, β, γ.

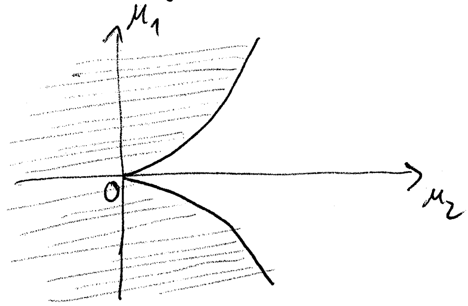


The phase diagram is



Then g has two stable equilibria α and γ. Therefore, the region for (μ_1, μ_2) we were to determine is
$$\begin{cases} \mu_2 > 0, \\ |\mu_1| < \frac{2}{3^{3/2}} \mu_2^{3/2}. \end{cases}$$

It is the unshaded region in the picture.



(ii) Consider the ODE
$$x' = a_0(\mu) + a_1(\mu)x + a_2(\mu)x^2 + a_3(\mu)x^3, \qquad (3)$$
where $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$, a_0, a_1, a_2, a_3 are smooth functions with $a_0(0) = a_1(0) = a_2(0) = 0$ and $a_3(0) = -1$. Put $y = x + \frac{a_2(\mu)}{a_3(\mu)}$. Then (3) becomes
$$y' = a_0(\mu) + a_1(\mu)\left(y - \frac{a_2(\mu)}{a_3(\mu)}\right) + a_2(\mu)\left(y - \frac{a_2(\mu)}{a_3(\mu)}\right)^2 + a_3(\mu)\left(y - \frac{a_2(\mu)}{a_3(\mu)}\right)^3$$
$$\Leftrightarrow \quad y' = \underbrace{\left(\frac{2a_2(\mu)^2}{27a_3(\mu)^2} - \frac{a_1(\mu)a_2(\mu)}{3a_3(\mu)} + a_0(\mu)\right)}_{b_0(\mu)} + \underbrace{\left(a_1(\mu) - \frac{a_2(\mu)^2}{3a_3(\mu)}\right)}_{b_1(\mu)} y + \underbrace{a_3(\mu)}_{b_3(\mu)} y^3$$
$$\Leftrightarrow \quad y' = b_0(\mu) + b_1(\mu)y + b_3(\mu)y^3. \qquad (4)$$

Here b_0, b_1, b_3 are smooth functions in a neighborhood of 0 in $\mathbb{R}^2$. Since $a_0(0) = a_1(0) = a_2(0) = 0$, $b_0(0) = b_1(0) = 0$. Since $a_0(0) = -1$, $b_3(0) = -1$. Therefore, (4) is of the same form as (3) in which $a_2(\mu) \equiv 0$.

(iii) Assume $D_\mu(a_0, a_1)(0)$ is invertible. First, we show that there is a local diffeomorphism $\mu = (\mu_1, \mu_2) \mapsto \tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2)$ between neighborhoods of 0 in $\mathbb{R}^2$ such that the ODE $x' = a_0(\mu) + a_1(\mu)x + a_3(\mu)x^3$ becomes

$$x' = \tilde{\mu}_1 + \tilde{\mu}_2 x + \tilde{a}_3(\tilde{\mu})x^3, \qquad (5)$$

~~wh~~ with $\tilde{a}_3(0) = -1$.

Because $D_\mu(a_0, a_1)(0)$ is invertible, by the Implicit Function theorem, there exist neighborhoods U and V of 0 in $\mathbb{R}^2$ such that the map $(a_0, a_1)|_V : U \to V$ is a diffeomorphism. Write $\tilde{\mu} = (a_0, a_1)(\mu)$. In terms of components,

$$\tilde{\mu}_1 = a_0(\mu), \qquad \tilde{\mu}_2 = a_1(\mu).$$

Define $\tilde{a}_3(\tilde{\mu}) = a_3(\mu) = a_3((a_0, a_1)^{-1}(\tilde{\mu}))$. Then $\tilde{a}_3(0) = a_3((a_0, a_1)^{-1}(0)) = a_3(0) = -1$.

We have

$$\begin{aligned} x' &= a_0(\mu) + a_1(\mu)x + a_3(\mu)x^3 \\ &= \tilde{\mu}_1 + \tilde{\mu}_2 x + \tilde{a}_3(\tilde{\mu})x^3. \end{aligned}$$

Next, we describe the region of $\tilde{\mu} \in \mathbb{R}^2$ such that (5) has two stable equilibria by using Newton's polygon. Put

$$h(x) = \tilde{\mu}_1 + \tilde{\mu}_2 x + \tilde{a}_3(\tilde{\mu})x^3,$$

$$h'(x) = \tilde{\mu}_2 + 3\tilde{a}_3(\tilde{\mu})x^2.$$

Since $\tilde{\mu}$ is close to 0, $\tilde{a}_3(\tilde{\mu})$ is close to $\tilde{a}_3(0) = -1$. Then by the same arguments as in Part (i), we conclude that (5) has two stable equilibria if and only if

$$\begin{cases} \tilde{\mu}_2 > 0, \\ h(\tilde{y}_1)h(\tilde{y}_2) < 0, \end{cases}$$

where $\tilde{y}_1, \tilde{y}_2$ are the roots of h'.

$$\tilde{y}_1 = -\sqrt{-\frac{\tilde{\mu}_2}{3\tilde{a}_3(\tilde{\mu})}} \quad \text{and} \quad \tilde{y}_2 = \sqrt{-\frac{\tilde{\mu}_2}{3\tilde{a}_3(\tilde{\mu})}}.$$

We describe the curve $\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2)$ corresponding to $h(\tilde{y}_1) = 0$.

$$0 = h(\tilde{y}_1) = \tilde{\mu}_1 + \tilde{\mu}_2 \tilde{y}_1 + \tilde{a}_3(\tilde{\mu}) \tilde{y}_1^3 = \tilde{\mu}_1 - \tilde{\mu}_2^{3/2} \frac{-3\tilde{a}_3(\tilde{\mu}) - 1}{(-3\tilde{a}_3(\tilde{\mu}))^{3/2}}. \qquad (6)$$

Since $\tilde{a}_3(\tilde{\mu}) = -1 + O(|\tilde{\mu}|)$,

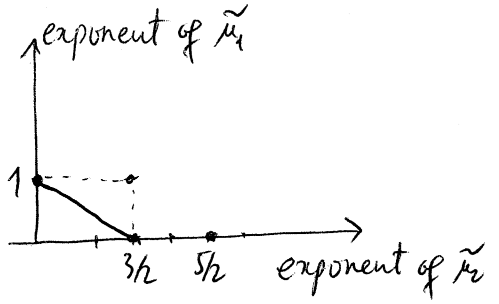
$$\frac{1}{-\tilde{a}_3(\tilde{\mu})} = \frac{1}{1 - (\tilde{a}_3(\tilde{\mu}) + 1)} = 1 + (\tilde{a}_3(\tilde{\mu}) + 1) + (\tilde{a}_3(\tilde{\mu}) + 1)^2 + \dots = 1 + O(|\tilde{\mu}|).$$

Thus, $$\left(\frac{1}{-\tilde{a}_3(\tilde{\mu})}\right)^{3/2} = \left(1 + O(|\tilde{\mu}|)\right)^{3/2} = 1 + O(|\tilde{\mu}|)$$

because the map $t \mapsto (1+t)^{3/2}$ is differentiable at $t = 0$. Replacing this result into (6), we get

$$\begin{aligned} 0 &= \tilde{\mu}_1 - \tilde{\mu}_2^{3/2} \frac{1}{3^{3/2}} \left(-3\tilde{a}_3(\tilde{\mu}) - 1\right)\left(1 + O(|\tilde{\mu}|)\right) \\ &= \tilde{\mu}_1 - \left(\frac{\tilde{\mu}_2}{3}\right)^{3/2} \left(2 + O(|\tilde{\mu}|)\right)\left(1 + O(|\tilde{\mu}|)\right) \\ &= \tilde{\mu}_1 - 2\left(\frac{\tilde{\mu}_2}{3}\right)^{3/2} + \left(\frac{\tilde{\mu}_2}{3}\right)^{3/2} O(|\tilde{\mu}|). \qquad (7) \end{aligned}$$

The Newton's polygon for this equation looks like the following.



The only edge facing toward the origin is the one connecting $(0,1)$ with $(\frac{3}{2}, 0)$. It suggests $\tilde{\mu}_1 = \alpha \tilde{\mu}_2^{3/2} + O(\tilde{\mu}_2^{5/2})$. Then (7) becomes

$$0 = \alpha \tilde{\mu}_2^{3/2} - 2\left(\frac{\tilde{\mu}_2}{3}\right)^{3/2} + O(\tilde{\mu}_2^{5/2}).$$

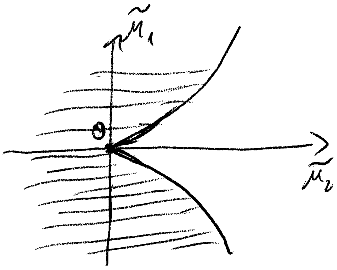
Dividing both sides by $\tilde{\mu}_2^{3/2}$, we get $\alpha = \frac{2}{3^{3/2}}$. Therefore, the curve $\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2)$ describing the equation $h(\tilde{y}_1) = 0$ is

$$\tilde{\mu}_1 = \frac{2}{3^{3/2}} \tilde{\mu}_2^{3/2} + O(\tilde{\mu}_2^{5/2}).$$

Similarly, the curve describing the equation $h(\tilde{y}_2) = 0$ is

$$\tilde{\mu}_1 = \frac{-2}{3^{3/2}} \tilde{\mu}_2^{3/2} + O(\tilde{\mu}_2^{5/2}).$$

These two curves determines a cusp shape in the plane $\tilde{\mu} \in \mathbb{R}^2$. Since $\tilde{\mu}_2 > 0$, the region to be determined is the unshaded region in the picture.



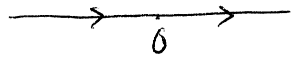
(2) We determine the equilibria, their stability, and the heteroclinic orbits for the following ODEs.

(i) $x' = \mu x + x^2$.

The equilibrium equation is $\mu x + x^2 = 0$, which gives two ~~slo~~ solutions $x = 0$ and $x = -\mu$. The stability (of these equilibria) and the heteroclinic orbits are determined by our being able to specify the signs of $f(x) = \mu x + x^2$ and draw the phase portrait. For this reason, we need to compare 0 with $-\mu$.

$\underline{\mu = 0}$

There is only one equilibrium: $x=0$. Since $f(x)\geq 0$, the phase ~~diag~~ portrait is



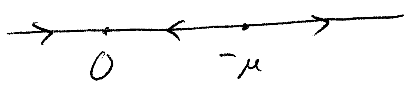
The equilibrium is unstable. There is no heteroclinic orbit.

- $\mu < 0$

There are two equilibria: $x=0$ and $x=-\mu$.

x		0		$-\mu$	
$f(x)$	$+$	0	$-$	0	$+$

The phase portrait is



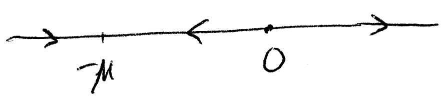
We see that 0 is stable, $-\mu$ is unstable, and the only heteroclinic orbit is the segment $(0,-\mu)$ with direction from $-\mu$ to 0.

- $\mu > 0$

There are two equilibria: $x=0$ and $x=-\mu$.

x		$-\mu$		0	
$f(x)$	$+$	0	$-$	0	$+$

The phase portrait is

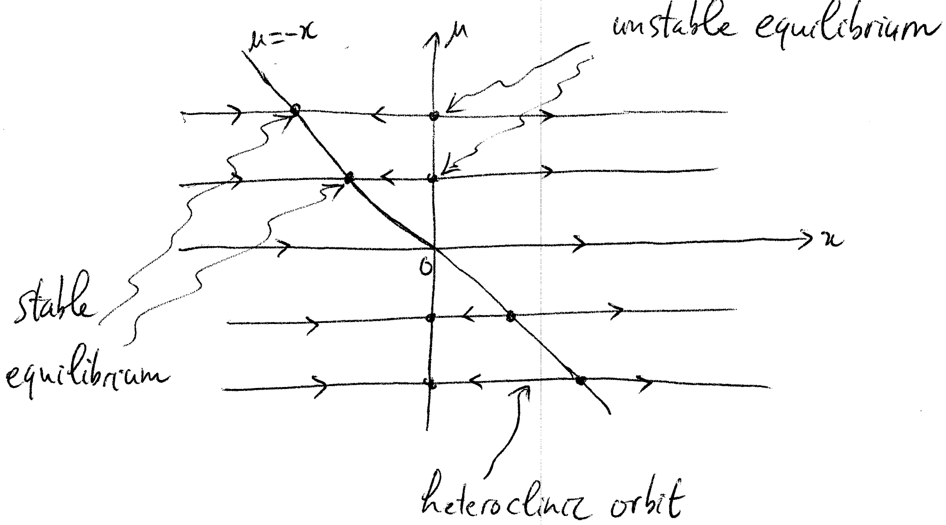


We see that 0 is unstable, $-\mu$ is stable, and the only heteroclinic orbit is the segment $(-\mu,0)$ with direction from 0 to $-\mu$.

All information that we have found is indicated in the following

8

bifurcation diagram.

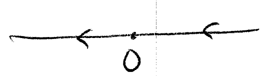


(i') $x' = \mu x - x^2$.

Put $f(x) = \mu x - x^2$. It has two zeros $x=0$ and $x=\mu$.

- $\underline{\mu = 0}$

There is only one equilibrium: $x=0$. Since $f(x) \leq 0$, the phase portrait is



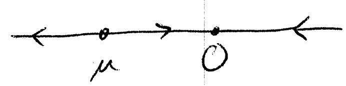
The equilibrium is unstable. There is no heteroclinic orbit.

- $\underline{\mu < 0}$

There are two equilibria: $x=0$ and $x=\mu$.

x		μ		0	
$f(x)$	$-$	0	$+$	0	$-$

The phase portrait is



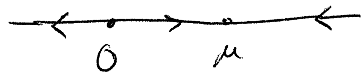
We see that 0 is stable, μ is unstable, and the only heteroclinic orbit is the segment $(\mu, 0)$ with direction from μ to 0.

- $\underline{\mu > 0}$

There are two equilibria : $x=0$ and $x=\mu$.

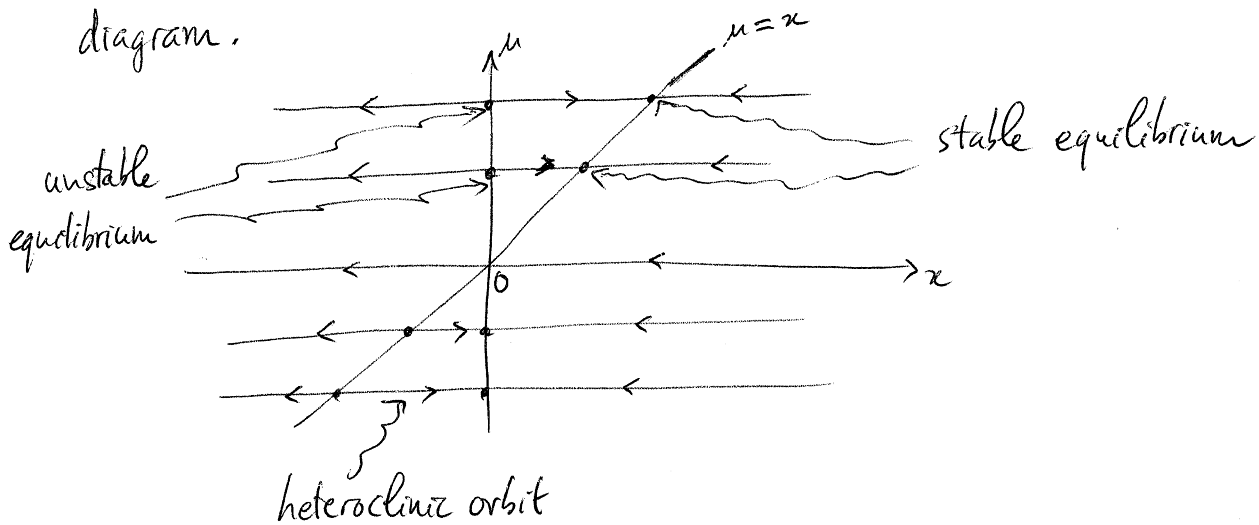
x		0		μ	
$f(x)$	$-$	0	$+$	0	$-$

The phase portrait is



We see that 0 is unstable, μ is stable, and the only heteroclinic orbit is the segment $(0,\mu)$ with direction from 0 to μ.

All information we have found is indicated in the following bifurcation diagram.



(ii) $x' = \mu_1 x + \mu_2 x^3 - x^5$.

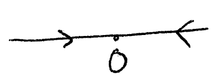
Put $f(x) = \mu_1 x + \mu_2 x^3 - x^4 = x(\mu_1 + \mu_2 x^2 - x^4) = x\,g(x^2)$, where $g(t) = \mu_1 + \mu_2 t - t^2$.

The discriminant of $g(t)$ is $\Delta = \mu_2^2 + 4\mu_1$. Consider the following cases

- $\mu_2^2 + 4\mu_1 < 0$

Then $g(t) < 0$ for all $t \in \mathbb{R}$. There is only one equilibrium : $x=0$.

x		0	
$f(x)$	$+$	0	$-$

The phase portrait is .

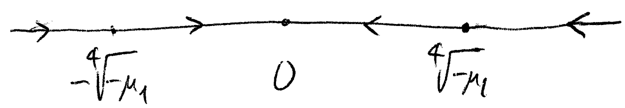
We see that 0 is stable and there is no heteroclinic orbit.

- $\mu_2^2+4\mu_1=0,\ \mu_1<0.$

Then $g(x^2)=-(x^2-\sqrt{-\mu_1})^2$. There are three equilibria: $x=0$, $x=\pm\sqrt[4]{-\mu_1}$.

x		$-\sqrt[4]{-\mu_1}$		0		$\sqrt[4]{-\mu_1}$	
$f(x)$	$+$	0	$+$	0	$-$	0	$-$

The phase portrait is



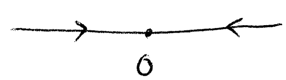
We see that 0 is stable, $\pm\sqrt[4]{-\mu_1}$ are unstable. There are two heteroclinic orbits: $(-\sqrt[4]{-\mu_1}, 0)$ and $(0, \sqrt[4]{-\mu_1})$ with direction shown in the above picture.

- $\mu_2^2+4\mu_1=0,\ \mu_1=0.$

Then $g(x^2)=-x^4$. There is only one equilibrium: $x=0$.

x		0	
$f(x)$	$+$	0	$-$

The phase portrait is

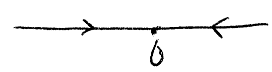


We see that 0 is stable. There is no heteroclinic orbit.

- $\mu_2^2+4\mu_1>0,\ \mu_1<0,\ \mu_2<0.$

The polynomial $g(t)$ has two negative roots. Thus, $g(x^2)<0$. There is only one equilibrium: $x=0$.

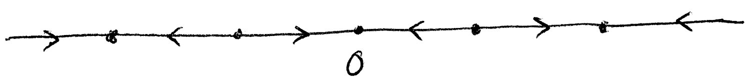
x		0	
$f(x)$	$+$	0	$-$

The phase ~~d~~ portrait is 

We see that 0 is stable. There is no~~t~~ heteroclinic orbit.

- $\underline{\mu_2^2+4\mu_1>0,\ \mu_1<0,\ \mu_2>0.}$

The polynomial $g(t)$ has two distinct positive roots. Thus, there are five equilibria: $x=0$, $x=\pm\sqrt{\dfrac{\mu_2\pm\sqrt{\mu_2^2+4\mu_1}}{2}}$.



The stable equilibria are $x=0$, $x=\pm\sqrt{\dfrac{\mu_2+\sqrt{\mu_2^2+4\mu_1}}{2}}$.

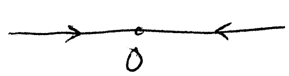
The unstable equilibria are $x=\pm\sqrt{\dfrac{\mu_2-\sqrt{\mu_2^2+4\mu_1}}{2}}$.

There are four heteroclinic orbits, indicated as four finite segments in the above picture.

- $\underline{\mu_2^2+4\mu_1>0,\ \mu_1=0,\ \mu_2<0}$

$f(x)=x^3(\mu_2-x^2)$. There is only one equilibrium: $x=0$

x		0	
$f(x)$	$+$	0	$-$

The phase portrait is 

We see that 0 is stable. There is no heteroclinic orbit.

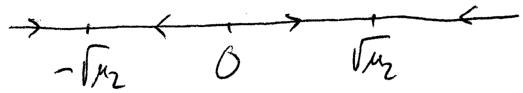
- $\underline{\mu_2^2+4\mu_1>0,\ \mu_1=0,\ \mu_2>0}$

$f(x)=x^3(\mu_2-x^2)$. There are three equilibria: $x=0$, $x=\pm\sqrt{\mu_2}$.

12

x		$-\sqrt{\mu_2}$		0		$\sqrt{\mu_2}$	
$f(x)$	$+$	0	$-$	0	$+$	0	$-$

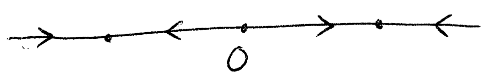
The phase portrait is



The stable equilibria are $\pm\sqrt{\mu_2}$. The unstable equilibrium is 0. There are two heteroclinic orbits: $(-\sqrt{\mu_2}, 0)$ and $(0, \sqrt{\mu_2})$ with direction shown in the above picture.

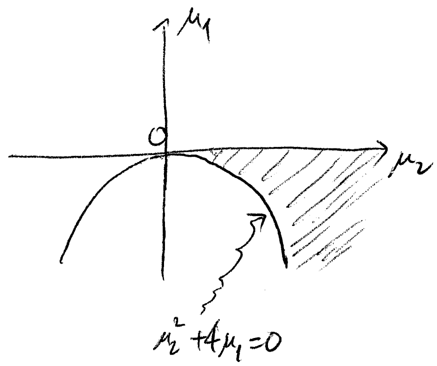
- $\underline{\mu_2^2 + 4\mu_1 > 0,\ \mu_1 > 0}$

The polynomial $g(t)$ has one positive root and one negative root. There are three equilibria: $x = 0$, $x = \pm\sqrt{\dfrac{\mu_2 + \sqrt{\mu_2^2 + 4\mu_1}}{2}}$.



The stable equilibria are $x = \pm\sqrt{\dfrac{\mu_2 + \sqrt{\mu_2^2 + 4\mu_1}}{2}}$. The unstable equilibrium is $x = 0$. There are two heteroclinic orbits indicated as two finite segments in the above picture.

The region for (μ_1, μ_2) where there are five equilibria is $\begin{cases} \mu_2^2 + 4\mu_1 > 0, \\ \mu_1 < 0, \mu_2 > 0. \end{cases}$



(3) Consider a smooth function $f(x,\mu): \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$. Define a map

$$F: \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \times [0,1] \longrightarrow \mathbb{R}^n \times \mathbb{R}^n,$$

$$F(x,v,\mu) = \left(f(x,\mu), \partial_x f(x,\mu)\cdot v\right).$$

(i) We show that the zeros of F are exactly the bifurcation points of f, that is points where the Implicit Function Theorem is not applicable.

Suppose $F(x_0, v_0, \mu_0) = 0$. Then $f(x_0,\mu_0) = 0$ and $\partial_x f(x_0,\mu_0)\cdot v_0 = 0$. Because $v_0 \neq 0$, the linear operator $\partial_x f(x_0,\mu_0): \mathbb{R}^n \to \mathbb{R}^n$ is not invertible. Thus, (x_0,μ_0) is a bifurcation point of f.

Conversely, suppose $f(x_0,\mu_0) = 0$ and $\partial_x f(x_0,\mu_0)$ is not invertible. Then there exists $v_0 \in \mathbb{R}^n \setminus \{0\}$ such that $\partial_x f(x_0,\mu_0)\cdot v_0 = 0$. Thus, $F(x_0, v_0, \mu_0) = 0$.

(ii) Assume 0 is not a critical value of F. Let (x_0,μ_0) be a bifurcation point of f, i.e. $f(x_0,\mu_0) = 0$ and $\partial_x f(x_0,\mu_0)$ is not invertible. We show that (x_0,μ_0) is a regular fold point, which means

- $\dim \ker \partial_x f(x_0,\mu_0) = 1$,
- $(e_0^*, \partial_\mu f(x_0,\mu_0)) \neq 0$,
- $(e_0^*, \partial_{xx} f[e_0, e_0]) \neq 0$,

where $e_0 \in \mathbb{R}^n$ is a vector generating $\ker \partial_x f(x_0,\mu_0)$, $e_0^* \in \mathbb{R}^n$ is a vector generating $\ker \underbrace{(\partial_x f(x_0,\mu_0))^*}_{\text{transpose of } \partial_x f(x_0,\mu_0)}$, and $(\cdot,\cdot)$ denotes the usual dot product in $\mathbb{R}^n$.

By Part (i), we know that there exists $v_0 \in \mathbb{R}^n \setminus \{0\}$ such that $F(x_0, v_0, \mu_0) = 0$.

Since 0 is not a critical value of F, $DF(x_0, v_0, \mu_0)$ is a surjective linear map from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n}$. Put $A = DF(x_0, v_0, \mu_0)$. We can view A as a $(2n)\times(2n+1)$ matrix. Denote by $x_1, x_2, \dots, x_n$ the coordinates in $\mathbb{R}^n$. Then

$$A = \begin{array}{c} \left(\begin{array}{ccc|ccc|c} \partial_{x_1} f & \cdots & \partial_{x_n} f & 0 & \cdots & 0 & \partial_\mu f \\ (\partial_{x_1 x} f)\cdot v_0 & \cdots & (\partial_{x_n x} f)\cdot v_0 & \partial_{x_1} f & \cdots & \partial_{x_n} f & (\partial_{x\mu} f)\cdot v_0 \end{array}\right) \begin{array}{l} \}\, n \\ \}\, n \end{array} \\ \begin{array}{ccc} \underbrace{\qquad\qquad\qquad\qquad}_{n} & \underbrace{\qquad\qquad\qquad}_{n} & \underbrace{\qquad\quad}_{1} \end{array} \end{array}$$

Let A^* be the dual map of A.

$$A^* = \left(\begin{array}{c|c} \partial_{x_1} f & (\partial_{x_1 x} f)\cdot v_0 \\ \vdots & \vdots \\ \partial_{x_n} f & (\partial_{x_n x} f)\cdot v_0 \\ \hline 0 & \partial_{x_1} f \\ \vdots & \vdots \\ 0 & \partial_{x_n} f \\ \hline \partial_\mu f & (\partial_{x\mu} f)\cdot v_0 \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c} a\\ a\\ a \end{array}}\right\}\, n \\ \left.\vphantom{\begin{array}{c} a\\ a\\ a \end{array}}\right\}\, n \\ \}\, 1 \end{array} \qquad (1)$$

$$\qquad\quad \underbrace{\qquad}_{n}\ \underbrace{\qquad}_{n}$$

We have $\operatorname{rank}(A^*) = \operatorname{rank}(A) = 2n$. Suppose by contradiction that $\operatorname{rank}(\partial_x f(x_0, \mu_0)) \leq n-2$. Then after row reduction among the first n rows of A^*, these rows become

$$\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \begin{array}{l} \}\, n-2 \\ \}\, 2 \end{array}$$

$$\underbrace{\quad}_{n}\ \underbrace{\quad}_{n}$$

Then A^* is row equivalent to

$$\begin{array}{c}\left(\begin{array}{c|c} * & * \\ \hline 0 & * \\ 0 & \partial_{x_1} f \\ \vdots & \\ 0 & \partial_{x_n} f \\ \hline \partial_\mu f & (\partial_{x\mu} f)\cdot v_0 \end{array}\right) \begin{array}{l} \}\, n-2 \\ \\ \}\, n+2 \\ \\ \\ \}\, 1 \end{array} \\ \underbrace{\qquad}_{n}\ \underbrace{\qquad}_{n} \end{array}$$

Among $(n+2)$ rows in the middle, there are at most n linearly independent rows. Thus, among $(2n+1)$ rows, there are at most $n+(n-2)+1 = 2n-1$ linearly independent rows. This is a contradiction because $\operatorname{rank} A^* = 2n$. Therefore, $\operatorname{rank}(\partial_x f(x_0,\mu_0)) \in \{n-1, n\}$. Because $\partial_x f(x_0,\mu_0)$ is not invertible, $\operatorname{rank}(\partial_x f(x_0,\mu_0)) = n-1$. Thus, $\dim \ker \partial_x f(x_0,\mu_0) = 1$. We have

$$\ker\left(\partial_x f(x_0,\mu_0)\right)^* = \left(\operatorname{range} \partial_x f(x_0,\mu_0)\right)^\perp.$$

Hence, $\dim \ker (\partial_x f(x_0,\mu_0))^* = n - \dim \operatorname{range} \partial_x f(x_0,\mu_0) = n - \operatorname{rank} \partial_x f(x_0,\mu_0) = n-(n-1) = 1$. Let $e_0 \in \mathbb{R}^n \setminus \{0\}$ be a vector generating $\ker \partial_x f(x_0,\mu_0)$, and e_0^* be a vector generating $\ker(\partial_x f(x_0,\mu_0))^*$.

Now we show that $(e_0^*, \partial_\mu f(x_0,\mu_0)) \neq 0$. Suppose by contradiction that $(e_0^*, \partial_\mu f(x_0,\mu_0)) = 0$. Then $\partial_\mu f(x_0,\mu_0) \perp \ker(\partial_x f(x_0,\mu_0))$. Then $\partial_\mu f(x_0,\mu_0) \in \operatorname{range} \partial_x f(x_0,\mu_0)$. There exists $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ such that

$$\partial_\mu f(x_0,\mu_0) = \partial_x f(x_0,\mu_0)\cdot a = \sum_{k=1}^{n} \partial_{x_k} f(x_0,\mu_0)\, a_k .$$

In (1), ~~substituting~~ subtracting a_1 times the first row, a_2 times the second row, ..., a_n times the n'th row from the last row of A^*, we get a row equivalent form of A^*.

$$A^* \sim \left(\begin{array}{c|c} \partial_{x_1} f & (\partial_{x_1 x} f)\cdot v_0 \\ \vdots & \vdots \\ \partial_{x_n} f & (\partial_{x_n x} f)\cdot v_0 \\ \hline 0 & \partial_{x_1} f \\ \vdots & \vdots \\ 0 & \partial_{x_n} f \\ \hline 0 & * \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c} a\\ a\\ a \end{array}}\right\} n \\ \left.\vphantom{\begin{array}{c} a\\ a\\ a \end{array}}\right\} n \\ \left.\vphantom{a}\right\} 1 \end{array}$$

$$\underbrace{\qquad}_{n}\ \underbrace{\qquad}_{n}$$

Because $\operatorname{rank}(\partial_x f(x_0,\mu_0)) = n-1$, after suitable row reduction among the first n rows of A^*, we get

$$A^* \sim \left(\begin{array}{c|c} * & * \\ \hline 0 & * \\ 0 & \partial_{x_1} f \\ \vdots & \vdots \\ 0 & \partial_{x_n} f \\ 0 & * \end{array}\right) \begin{array}{l} \left.\vphantom{a}\right\} n-1 \\ \left.\vphantom{\begin{array}{c} a\\ a\\ a\\ a\\ a \end{array}}\right\} n+2 \end{array}$$

$$\underbrace{\qquad}_{n}\ \underbrace{\qquad}_{n}$$

There are at most n linearly independent rows among the last $(n+2)$ rows.

Thus, among the $(2n+1)$ rows, there are at most $n+(n-1) = 2n-1$ linearly independent rows. This contradicts the fact that $\operatorname{rank} A^* = 2n$.

Finally, we show that $(e_0^*, \partial_{xx} f[e_0, e_0]) \neq 0$. Recall that $\partial_{xx} f[e_0, e_0]$ is the short notation for vector $(e_0^t \partial_{xx} f_1(x_0, \mu_0) e_0, \dots, e_0^t \partial_{xx} f_n(x_0, \mu_0) e_0)$, where $f_1, f_2, \dots, f_n$ are the components of f. Suppose by contradiction that $(e_0^*, \partial_{xx} f[e_0, e_0]) = 0$.

Because $\partial_x f(x_0, \mu_0) \cdot v_0 = 0$, $v_0 \in \ker \partial_x f(x_0, \mu_0)$. Thus, $v_0 = \alpha e_0$ for some $\alpha \in \mathbb{R} \setminus \{0\}$. Then $(e_0^*, \partial_{xx} f[v_0, v_0]) = \alpha^2 (e_0^*, \partial_{xx} f[e_0, e_0]) = 0$. This implies

$$\partial_{xx} f[v_0, v_0] \perp \ker (\partial_x f(x_0, \mu_0))^*.$$

Hence, $\partial_{xx} f[v_0, v_0] \in \operatorname{range} \partial_x f(x_0, \mu_0)$. There exists vector $a = (a_1, \dots, a_n) \in \mathbb{R}^n \setminus \{0\}$ such that

$$\partial_{xx} f[v_0, v_0] = \partial_x f(x_0, \mu_0) \cdot a = \sum_{k=1}^{n} \partial_{x_k} f(x_0, \mu_0)\, a_k .$$

Write $v_0 = (v_1, v_2, \dots, v_n)$. Then

$$(\partial_{x_k x} f) \cdot v_0 = \sum_j (\partial_{x_k x_j} f)\, v_j .$$

Thus,

$$\sum_k v_k (\partial_{x_k x} f) \cdot v_0 = \sum_{k,j} (\partial_{x_k x_j} f)\, v_j v_k = \partial_{xx} f[v_0, v_0] = \sum_k a_k \partial_{x_k} f(x_0, \mu_0).$$

Without loss of generality, we can assume $v_1 \neq 0$. In (1), we multiply the first row with v_1, then adding to it v_2 times the second row, v_3 times the third row, ..., v_n times the n'th row, then subtracting from it a_1 times the

$(n+1)$'th row, a_2 times the $(n+2)$'th row, $\ldots$, a_n times the $(2n)$'th row. We get

$$A^* \sim \begin{array}{cl} \left(\begin{array}{c|c} \sum_k v_k \partial_{x_k} f & \sum v_k (\partial_{x_k x} f)\cdot v_0 - \sum a_k \partial_{x_k} f \\ \hline \partial_{x_2} f & (\partial_{x_2 x} f)\cdot v_0 \\ \vdots & \vdots \\ \partial_{x_n} f & (\partial_{x_n x} f)\cdot v_0 \\ \hline 0 & \partial_{x_1} f \\ \vdots & \vdots \\ 0 & \partial_{x_n} f \\ \hline \partial_\mu f & (\partial_{x\mu} f)\cdot v_0 \end{array}\right) & \begin{array}{l} \}\, 1 \\ \}\, n-1 \\ \}\, n \\ \}\, 1 \end{array} \\ \begin{array}{cc} \quad n \qquad\qquad & \qquad\qquad n \end{array} & \end{array}$$

The first row is equal to zero. Because $\operatorname{rank} \partial_x f(x_0,\mu_0) = n-1$, by suitable row reduction among n rows in the middle, we can make the $(n+1)$'th row in the above matrix zero. Exchanging this row with the second row, we get

$$A^* \sim \begin{array}{cl} \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \\ \hline * & * \end{array}\right) & \begin{array}{l} \}\, 1 \\ \}\, 1 \\ \}\, 2n-1 \end{array} \\ \begin{array}{cc} n & n \end{array} & \end{array}$$

Then $\operatorname{rank} A^* \leq 2n-1$. This contradicts the fact that $\operatorname{rank} A^* = 2n$.

(iii) Define $g : \mathbb{R}^n \times [0,1] \times \mathbb{R}^n \times \mathbb{R}^{n^2} \longrightarrow \mathbb{R}^n$, $g(x,\mu,a,B) = f(x,\mu) + a + Bx$,

$$G : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \times [0,1] \times \mathbb{R}^n \times \mathbb{R}^{n^2} \longrightarrow \mathbb{R}^n \times \mathbb{R}^n,$$

$$G(x, v, \mu, a, B) = \left(g(x,\mu,a,B),\ \partial_x g(x,\mu,a,B).v\right).$$

The function G was denoted as $\mathcal{F}$ (script F) in the printed problem, but we change the notation to avoid confusion between F and $\mathcal{F}$ in hand-writing.

We show that $DG = \partial_{x,v,\mu,a,B}\, G$ is a surjective linear map from $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} = \mathbb{R}^{n^2+3n+1}$ to $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$. Put $A = DG(x,v,\mu,a,B)$ which can be viewed as a $(2n)\times(n^2+3n+1)$ matrix. We need to show that $\operatorname{rank} A = 2n$.

$$A = \begin{array}{c} n\{ \\ n\{ \end{array} \left(\begin{array}{c|c|c|c|c} \partial_x[f(x,\mu)+a+Bx] & \partial_v[f(x,\mu)+a+Bx] & \partial_\mu[f(x,\mu)+a+Bx] & \partial_a[f(x,\mu)+a+Bx] & \partial_B[f(x,\mu)+a+Bx] \\ \hline \partial_x[(\partial_x f).v+Bv] & \partial_v[(\partial_x f).v+Bv] & \partial_\mu[(\partial_x f).v+Bv] & \partial_a[(\partial_x f).v+Bv] & \partial_B[(\partial_x f).v+Bv] \\ \hline \underbrace{\qquad}_{n} & \underbrace{\qquad}_{n} & \underbrace{\qquad}_{1} & \underbrace{\qquad}_{n} & \underbrace{\qquad}_{n^2} \end{array}\right)$$

$$= \left(\begin{array}{c|c|c|c|c|c|c|c|c} \partial_x f + B & 0 & \partial_\mu f & I_n & \begin{matrix} x \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ x \\ 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ x \\ \vdots \\ 0 \end{matrix} & \cdots & \begin{matrix} 0 \\ 0 \\ 0 \\ \vdots \\ x \end{matrix} \\ \hline (\partial_{xx} f).v & \partial_x f + B & (\partial_{\mu x} f).v & 0 & \begin{matrix} v \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ v \\ 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ v \\ \vdots \\ 0 \end{matrix} & \cdots & \begin{matrix} 0 \\ 0 \\ 0 \\ \vdots \\ v \end{matrix} \\ \hline \underbrace{\qquad}_{n} & \underbrace{\qquad}_{n} & \underbrace{\qquad}_{1} & \underbrace{\qquad}_{n} & \underbrace{\quad}_{n} & \underbrace{\quad}_{n} & \cdots & & \underbrace{\quad}_{n} \end{array}\right) \begin{array}{c} \}n \\ \}n \end{array}$$

The first n rows are linearly independent because of the block I_n. Those rows are also linearly independent with each row in the last n rows because of the $n \times n$ block 0. Because $v \neq 0$, the block

$$\left(\begin{array}{c|c|c|c} \begin{matrix} v \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ v \\ 0 \\ \vdots \\ 0 \end{matrix} & \cdots & \begin{matrix} 0 \\ 0 \\ 0 \\ \vdots \\ v \end{matrix} \\ \hline \underbrace{\quad}_{n} & \underbrace{\quad}_{n} & \cdots & \underbrace{\quad}_{n} \end{array}\right) \}n$$

has rank n. Hence, the last n rows of A are linearly independent. Then all rows of A are linearly independent. Thus, $\operatorname{rank} A = 2n$.

(iv) We show that $g(\cdot,\cdot, a, B)$ has only regular fold points for a dense set of $(a,B) \in \mathbb{R}^n \times \mathbb{R}^{n^2}$. By regular fold point, we mean a bifurcation point satisfying three properties listed in Part (ii).

Applying Part (ii) for $f = g(\cdot,\cdot, a, B)$, we only need to show that 0 is not a critical value of $G(\cdot,\cdot,\cdot, a, B)$ for a dense set of $(a,B) \in \mathbb{R}^n \times \mathbb{R}^{n^2}$. By Part (iii), $DG(x, v, \mu, a, B)$ is surjective. Thus, G is transverse to $\{0\}$. Let

$$\mathcal{A} = \{ (a,B) \in \mathbb{R}^n \times \mathbb{R}^{n^2} : G(\cdot,\cdot,\cdot, a, B) \text{ is transverse to } \{0\} \}.$$

By Theorem 10.14, Chow-Hale "Methods of Bifurcation Theory" 1982, page 65, $\mathcal{A}$ is residual in $\mathbb{R}^n \times \mathbb{R}^{n^2}$, i.e. equal to the countable intersection of open dense sets. By Baire category theorem, $\mathcal{A}$ is also dense in $\mathbb{R}^n \times \mathbb{R}^{n^2}$. Take $(a,B) \in \mathcal{A}$. We show that 0 is not a critical value of $G(\cdot,\cdot,\cdot, a, B)$. Suppose otherwise. Then there exists $(x_0, v_0, \mu_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \times [0,1]$ such that $G(x_0, v_0, \mu_0, a, B) = 0$ and $\partial_{x,v,\mu} G(x_0, v_0, \mu_0, a, B)$ is not full-rank. Since $G(\cdot,\cdot,\cdot, a, B)$ is transverse to $\{0\}$ and $G(x_0, v_0, \mu_0, a, B) = 0 \in \{0\}$, $\partial_{x,v,\mu} G(x_0, v_0, \mu_0, a, B)$ must be surjective. This is a contradiction.

4 Consider the system of ODEs involving parameter $\mu \in \mathbb{R}$

$$\begin{cases} x' = y, \\ y' = -x + \mu y + (x+y)z, \\ z' = -z + x^2. \end{cases} \qquad (\mathrm{I})$$

Put $f(x,y,z;\mu) = (y, -x+\mu y + (x+y)z, -z+x^2)$. Then system (I) becomes

$$(x,y,z)' = f(x,y,z;\mu).$$

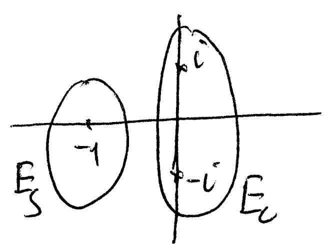
First, we determine the center manifold nearby $(x,y,z;\mu) = (0,0,0;0)$.

$$\partial_{x,y,z} f = \begin{pmatrix} 0 & 1 & 0 \\ -1+z & \mu+z & x+y \\ 2x & 0 & -1 \end{pmatrix}.$$

Put

$$A = \partial_{x,y,z} f(0,0,0;0) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then $\det(A - \lambda I_3) = -(\lambda+1)(\lambda^2+1)$. Thus, A has three eigenvalues $-1, \pm i$.



Let $A(-1), A(i), A(-i)$ be respectively the eigenspaces to $-1, i, -i$. Then the center subspace is $E_c = A(i) \oplus A(-i)$. The stable subspace is $E_s = A(-1)$. The unstable subspace is $E_u = \{0\}$.

$$(A - iI_3)v = \begin{pmatrix} -i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 0 & -1-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \iff \begin{cases} -iv_1 + v_2 = 0, \\ v_3 = 0. \end{cases}$$

Thus, $A(i) = \mathbb{C}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$.

$$(A+iI_3)v = \begin{pmatrix} i & 1 & 0 \\ -1 & i & 0 \\ 0 & 0 & -1+i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \iff \begin{cases} iv_1 + v_2 = 0, \\ v_3 = 0. \end{cases}$$

Thus,

$$A(-i) = \mathbb{C}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}.$$

$$(A+I_3)v = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \iff \begin{cases} v_1 = v_2 = 0, \\ v_3 \text{ arbitrary}. \end{cases}$$

Thus, $A(-1) = \mathbb{C}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Then

$$E_c = A(i) \oplus A(-i) = \mathbb{C}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix},$$

$$E_h = E_s \oplus E_u = \mathbb{C}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For $x, y, z \in \mathbb{C}$, we write

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \eta\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} + \zeta\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where (η, ζ) are coordinates in the center subspace E_c, and z is the coordinate in E_h. The inversion formula is

$$\begin{cases} \eta = \frac{1}{2}(x - iy), \\ \zeta = \frac{1}{2}(x + iy). \end{cases}$$

For $x, y \in \mathbb{R}$, we see that $\eta = \bar{\zeta}$. We rewrite system (I) in terms of new coordinates $(\bar{\zeta}, \zeta, z)$.

$$\zeta' = \frac{1}{2}(x'+iy') = \frac{1}{2}\left[y + i(-x+\mu y+(x+y)z)\right]$$

$$= \frac{1}{2}\left\{(i\bar{\zeta}-i\zeta) + i\left[-\bar{\zeta}-\zeta+\mu i\bar{\zeta}-\mu i\zeta+(\bar{\zeta}+\zeta+i\bar{\zeta}-i\zeta)z\right]\right\}$$

$$= \frac{1}{2}\left\{(-2i+\mu)\zeta - \mu\bar{\zeta} + \left[(i-1)\bar{\zeta}+(i+1)\zeta\right]z\right\}. \qquad (1)$$

The differential equation that $\bar{\zeta}$ satisfies is obtained by taking the complex conjugate of both sides of (1). The differential equation that z satisfies is

$$z' = -z + x^2 = -z + (\bar{\zeta}+\zeta)^2 = -z + \bar{\zeta}^2 + 2\bar{\zeta}\zeta + \zeta^2. \qquad (2)$$

The Center Manifold theorem (Theorem 3.3, page 47, Haragus-Iooss, "Local Bifurcations, Center Manifolds, and Normal Forms in Infinite-dimensional Dynamical Systems", 2011) says that there exists a smooth function $h : (E_c \times \mathbb{R}) \cap U(0) \to E_h$, where $U(0)$ is a neighborhood of 0 in $\mathbb{R}^3 \times \mathbb{R}$, such that $h(0;0)=0$, $\partial_{(\bar{\zeta},\zeta)} h(0;0) = 0$, and $z = h(\bar{\zeta},\zeta;\mu)$. Write

$$z = h(\bar{\zeta},\zeta;\mu) = a_1\bar{\zeta}^2 + a_2\bar{\zeta}\zeta + a_3\zeta^2 + O(|\zeta|^3+|\mu|).$$

The residue is actually $O(|\zeta|^3+|\zeta||\mu|)$ because $h(0;\mu)=0$ (i.e. $\bar{\zeta}=\zeta=z=0$ is always an equilibrium). We rewrite

$$z = h(\bar{\zeta},\zeta;\mu) = a_1\bar{\zeta}^2 + a_2\bar{\zeta}\zeta + a_3\zeta^2 + O(|\zeta|^3+|\zeta||\mu|). \qquad (3)$$

Plugging z into (1), we get

$$\zeta' = \frac{1}{2}\left\{(-2i+\mu)\zeta - \mu\bar{\zeta} + O(|\zeta|^4+|\zeta|^2|\mu|)\right\} \qquad (4)$$

We have

$$z' = \bar{\zeta}'\,\partial_{\bar{\zeta}} h(\bar{\zeta},\zeta;\mu) + \zeta'\,\partial_{\zeta} h(\bar{\zeta},\zeta;\mu)$$

$$\overset{(3),(4)}{=} \frac{1}{2}\left\{(2i+\mu)\bar{\zeta} - \mu\zeta + O(|\zeta|^4 + |\zeta|^2|\mu|)\right\}(2a_1\bar{\zeta} + a_2\zeta)$$

$$+ \frac{1}{2}\left\{(-2i+\mu)\zeta - \mu\bar{\zeta} + O(|\zeta|^4 + |\zeta|^2|\mu|)\right\}(a_2\bar{\zeta} + 2a_3\zeta) + O(|\zeta|^4 + |\zeta|^2|\mu|)$$

$$= 2ia_1\bar{\zeta}^2 - 2ia_3\zeta^2 + O(|\zeta|^4 + |\zeta|^2|\mu|). \qquad (5)$$

Substituting z given at (3) into the right hand side of (2), we get

$$z' = (1-a_1)\bar{\zeta}^2 + (2-a_2)\bar{\zeta}\zeta + (1-a_3)\zeta^2 + O(|\zeta|^3 + |\zeta||\mu|). \qquad (6)$$

Comparing (5) and (6), we deduce

$$\begin{cases} 2ia_1 = 1 - a_1, \\ 0 = 2 - a_2, \\ -2ia_3 = 1 - a_3. \end{cases}$$

Thus,

$$\begin{cases} a_1 = \dfrac{1}{1+2i}, \\[2mm] a_2 = 2, \\[2mm] a_3 = \dfrac{1}{1-2i}. \end{cases} \qquad (\mathrm{II})$$

We have computed the center manifold $z = h(\bar{\zeta},\zeta;\mu)$ to quadratic order. Now we compute a cubic Hopf normal form for ζ. Substituting z given by (3) into (1), we get

$$\zeta' = \frac{1}{2}\Big\{(-2i+\mu)\zeta - \mu\bar{\zeta} + [(i-1)\bar{\zeta} + (i+1)\zeta](a_1\bar{\zeta}^2 + a_2\bar{\zeta}\zeta + a_3\zeta^2) + O(|\zeta|^4 + |\zeta|^2|\mu|)\Big\}.$$

$$= \underbrace{\tfrac{1}{2}(-2i+\mu)}_{b_1}\zeta + \underbrace{\left(-\tfrac{\mu}{2}\right)}_{b_2}\bar\zeta + \underbrace{\tfrac{1}{2}(i-1)a_1}_{b_3}\bar\zeta^3 + \underbrace{\tfrac{1}{2}[(i+1)a_2+(i-1)a_3]}_{b_4}\bar\zeta\zeta^2$$

$$+ \underbrace{\tfrac{1}{2}[(i+1)a_1+(i-1)a_2]}_{b_5}\bar\zeta^2\zeta + \underbrace{\tfrac{1}{2}(i+1)a_3}_{b_6}\zeta^3 + O(|\zeta|^4+|\zeta|^2|\mu|)$$

$$= b_1\zeta + b_2\bar\zeta + b_3\bar\zeta^3 + b_4\bar\zeta\zeta^2 + b_5\bar\zeta^2\zeta + b_6\zeta^3 + O(|\zeta|^4+|\zeta|^2|\mu|). \qquad (7)$$

By (II),

$$b_1 = \tfrac{1}{2}(-2i+\mu), \qquad b_2 = -\tfrac{\mu}{2},$$

$$b_3 = \frac{+1+3i}{10}, \qquad b_4 = \frac{7+9i}{10},$$

$$b_5 = \cancel{\frac{-13+11i}{10}}\ \frac{-7+9i}{10}, \qquad b_6 = \frac{-1+3i}{10}.$$

To get rid of the term $b_2\bar\zeta$ on RHS (7), we introduce $\zeta = \alpha\zeta_1 + \beta\bar\zeta_1$.

Then (7) becomes

$$\alpha\zeta_1' + \beta\bar\zeta_1' = b_1(\alpha\zeta_1 + \beta\bar\zeta_1) + b_2(\bar\alpha\bar\zeta_1 + \bar\beta\zeta_1) + O(|\zeta|^3+|\zeta|^2|\mu|)$$

$$= (\alpha b_1 + b_2\bar\beta)\zeta_1 + (\beta b_1 + \bar\alpha b_2)\bar\zeta_1 + O(|\zeta|^3+|\zeta|^2|\mu|). \qquad (9)$$

Taking the complex conjugate of both sides, we get

$$\bar\alpha\bar\zeta_1' + \bar\beta\zeta_1' = (\bar\alpha\bar b_1 + \bar b_2\beta)\bar\zeta_1 + (\bar\beta\bar b_1 + \alpha\bar b_2)\zeta_1 + O(|\zeta|^3+|\zeta|^2|\mu|). \qquad (10)$$

Multiplying (9) by $\bar\alpha$ and multiplying (10) by β, then subtracting the later result from the former, we get

$$(\alpha\bar\alpha-\beta\bar\beta)\dot\zeta_1 = (\bar\alpha\alpha b_1+\bar\alpha\bar\beta b_2-\beta\bar\beta\bar b_1-\beta\alpha\bar b_2)\zeta_1$$
$$+(\bar\alpha\beta b_1+\bar\alpha\bar\alpha b_2-\bar\alpha\beta\bar b_1-\beta\beta\bar b_2)\bar\zeta_1+O(|\zeta|^3+|\zeta|^2|\mu|).$$

We would like to have

$$\begin{cases} \dfrac{\bar\alpha\alpha b_1+\bar\alpha\bar\beta b_2-\beta\bar\beta\bar b_1-\beta\alpha\bar b_2}{\alpha\bar\alpha-\beta\bar\beta} = b_1+O(|\mu|^2), \\ \dfrac{\bar\alpha\beta b_1+\bar\alpha\bar\alpha b_2-\bar\alpha\beta\bar b_1-\beta\beta\bar b_2}{\alpha\bar\alpha-\beta\bar\beta} = O(\mu^2). \end{cases}$$

That is achieved by choosing $\alpha=4i$ and $\beta=\mu$.

From the relations $\begin{cases}\zeta = 4i\zeta_1+\mu\bar\zeta_1, \\ \bar\zeta = -4i\bar\zeta_1+\mu\zeta_1,\end{cases}$ we get

$$\zeta_1=\frac{1}{\mu^2-16}(4i\zeta+\mu\bar\zeta)=\frac{1}{-16}(4i\zeta+\mu\bar\zeta)+O(|\zeta||\mu|^2).$$

Taking derivative both sides, we get

$$\zeta_1' = -\frac{i}{4}\zeta'-\frac{\mu}{16}\bar\zeta' \overset{(7)}{=\!=} -\frac{i}{4}(b_1\zeta+b_2\bar\zeta+b_3\bar\zeta^3+b_4\bar\zeta\zeta^2+b_5\bar\zeta^2\zeta+b_6\zeta^3)$$
$$-\frac{\mu}{16}(\bar b_1\bar\zeta+\bar b_2\zeta+\bar b_3\zeta^3+\bar b_4\zeta\bar\zeta^2+\bar b_5\zeta^2\bar\zeta+\bar b_6\bar\zeta^3)$$
$$+O(|\zeta|^4+|\zeta|^2|\mu|+|\zeta||\mu|^2)$$

$$=b_1\zeta_1+\left(-\frac{i}{4}b_3-\frac{\mu}{16}\bar b_6\right)\bar\zeta^3+\left(-\frac{i}{4}b_4-\frac{\mu}{16}\bar b_5\right)\zeta^2\bar\zeta+$$
$$+\left(-\frac{i}{4}b_5-\frac{\mu}{16}\bar b_4\right)\zeta\bar\zeta^2+\left(-\frac{i}{4}b_6-\frac{\mu}{16}\bar b_3\right)\zeta^3$$
$$+O(|\zeta|^4+|\zeta|^2|\mu|+|\zeta||\mu|^2). \qquad (11)$$

From the identity $\zeta = 4i\zeta_1 + \mu\bar{\zeta}_1$, we get

$$\zeta^2 = -16\zeta_1^2 + 8i\mu\zeta_1\bar{\zeta}_1 + O(|\zeta_1||\mu|^2)$$

$$\zeta^3 = -64i\zeta_1^3 - 48\mu\zeta_1^2\bar{\zeta}_1 + O(|\zeta_1||\mu|^2)$$

Substituting these quantities into (11), we get

$$\zeta_1' = b_1\zeta_1 + \left(\frac{-i}{4}b_3 - \frac{\mu}{16}\bar{b}_6\right)(64i\zeta_1^3 - 48\mu\zeta_1^2\bar{\zeta}_1) +$$

$$+\left(\frac{-i}{4}b_4 - \frac{\mu}{16}\bar{b}_5\right)(-16\zeta_1^2 + 8i\mu\zeta_1\bar{\zeta}_1)(-4i\bar{\zeta}_1 + \mu\zeta_1) +$$

$$+\left(-\frac{i}{4}b_5 - \frac{\mu}{16}\bar{b}_4\right)(4i\zeta_1 + \mu\bar{\zeta}_1)(-16\bar{\zeta}_1^2 + 8i\mu\zeta_1\bar{\zeta}_1) +$$

$$+\left(\frac{-i}{4}b_6 - \frac{\mu}{16}\bar{b}_3\right)(-64i\bar{\zeta}_1^3 - 48\mu\zeta_1\bar{\zeta}_1) + O(|\zeta_1|^4 + |\zeta_1|^2|\mu| + |\zeta_1||\mu|^2)$$

$$= b_1\zeta_1 - c_3\zeta_1^2\bar{\zeta}_1 + d_1\bar{\zeta}_1^2\zeta_1 + d_2\bar{\zeta}_1^3 + d_3\zeta_1^3 + O(|\zeta_1|^4 + |\zeta_1|^2|\mu| + |\zeta_1||\mu|^2), \tag{12}$$

where
$$-c_3 = \left(\frac{-i}{4}b_3\right)(-48\mu) + \left(-\frac{i}{4}b_4 - \frac{\mu}{16}\bar{b}_5\right)64i + \frac{-i}{4}b_5(-32\mu) + \frac{-i}{4}b_5(-16\mu)$$
$$+ \frac{-i}{4}b_6(-48\mu)$$

$$= \frac{56-144\mu}{5} + \frac{72-28\mu}{5}i.$$

To get rid of the terms $\bar{\zeta}_1^2\zeta_1$, $\bar{\zeta}_1^3$, ζ_1^3 in RHS (12), we introduce

$$\zeta_1 = \zeta_2 + \gamma_1\bar{\zeta}_2^2\zeta_2 + \gamma_2\bar{\zeta}_2^3 + \gamma_3\zeta_2^3$$

where $\gamma_1, \gamma_2, \gamma_3$ are appropriately chosen constants. The equation that ζ_2

satisfies is

$$\zeta_2' = b_1 \zeta_2 - c_3 \zeta_2^2 \bar{\zeta}_2 + O(|\zeta_2|^4 + |\zeta_2|^2 |\mu| + |\zeta_2||\mu|^2)$$

$$= -i\zeta_2 + c_1 \mu \zeta_2 - c_3 \zeta_2^2 \bar{\zeta}_2 + O(|\zeta_2|^4 + |\zeta_2|^2 |\mu| + |\zeta_2||\mu|^2). \qquad (13)$$

where $c_1 = \frac{1}{2}$ and $c_3 = -\frac{56-144\mu}{5} - \frac{72-28\mu}{5} i$.

Write $\zeta_2 = re^{i\varphi}$. Then (13) becomes

$$(\dot{r} + ir\dot{\varphi})e^{i\varphi} = -ire^{i\varphi} + c_1\mu re^{i\varphi} - c_3 r^3 e^{i\varphi} + O(r^4 + r^2|\mu| + r|\mu|^2)$$

$$\iff \dot{r} + ir\dot{\varphi} = -ir + c_1\mu r - c_3 r^3 + O(r^4 + r^2|\mu| + r|\mu|^2).$$

Taking the real part of both sides, we get

$$\dot{r} = c_1\mu r - \mathrm{Re}(c_3) r^3 + O(r^4 + r^2|\mu| + r|\mu|^2).$$

For a circular orbit about 0, $r \equiv \text{const}$. Then $c_1\mu r - \mathrm{Re}(c_3) r^3 = 0$.

Then $r = \sqrt{\frac{c_1\mu}{\mathrm{Re}(c_3)}} = \sqrt{\frac{-5\mu}{112}} + O(|\mu|^{3/2})$.