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Math 8502: Differential Equations & Dynamical Systems
Homework #3

(1) Identify the circle S^1 with $[0,1]/_{0\sim 1}$. Let $\alpha \in (0,1)\setminus \mathbb{Q}$, $J_0 = [0, 1-\alpha)$, $J_1 = [1-\alpha, 1)$. Let $R_\alpha : S^1 \to S^1$, $R_\alpha(x) = (x+\alpha) \bmod 1$ be the rigid circle rotation.

(i) For $x \in J_1$, we determine the smallest positive integer k such that $R_\alpha^k(x) \in J_1$. For $k \in \mathbb{N}$, write $x + k\alpha = m_k + \beta_k$ where $m_k = [x+k\alpha]$ is the integer part of $x + k\alpha$. Then

$$k = \frac{\beta_k - x + m_k}{\alpha} \in \mathbb{N}$$

• If $m_k = 0$ then $k = \frac{\beta - x}{\alpha} < \frac{1-(1-\alpha)}{\alpha} = 1$.

• If $m_k = 1$ then $k = \frac{\beta_k - x + 1}{\alpha}$. To have $\beta_k \in J_1$, we require

$$\frac{2-x}{\alpha} - 1 = \frac{(1-\alpha) - x + 1}{\alpha} \leq k < \frac{1 - x + 1}{\alpha} = \frac{2-x}{\alpha}.$$

The interval $\left[\frac{2-x}{\alpha} - 1, \frac{2-x}{\alpha}\right)$ is of length 1 and $\frac{2-x}{\alpha} > \frac{1}{\alpha} > 1$. Thus, it contains exactly one positive integer, which is $\left[\frac{2-x}{\alpha}\right]$ if $\frac{2-x}{\alpha} \notin \mathbb{Z}$, or $\left[\frac{2-x}{\alpha}\right] - 1$ if $\frac{2-x}{\alpha} \in \mathbb{Z}$.

• If $m_k \geq 2$ then

$$k = \frac{\beta_k - x + m_k}{\alpha} \geq \frac{0 - x + 2}{\alpha} = \frac{2-x}{\alpha}.$$

Therefore, the smallest $k \in \mathbb{N}$ such that $\beta_k \in J_1$ is found only in the case $m_k = 1$. We conclude

2

$$k(x) = \begin{cases} \left[\frac{2-x}{\alpha}\right] & \text{if } \frac{2-x}{\alpha} \notin \mathbb{Z}, \\ \left[\frac{2-x}{\alpha}\right] - 1 & \text{if } \frac{2-x}{\alpha} \in \mathbb{Z}. \end{cases} \quad (1)$$

Put $l = [\frac{1}{\alpha}] \in \mathbb{N}$. We show that $k(x)$ given by (1) assumes one of two values l or $l+1$. According to (1),

$$k(x) < \frac{2-x}{\alpha} \leq \frac{2-(1-\alpha)}{\alpha} = \frac{1}{\alpha} + 1,$$

$$k(x) \geq \frac{2-x}{\alpha} - 1 > \frac{2-1}{\alpha} - 1 = \frac{1}{\alpha} - 1.$$

Thus, $k(x) \in (\frac{1}{\alpha} - 1, \frac{1}{\alpha} + 1)$. Because $\frac{1}{\alpha} \notin \mathbb{Q}$, the interval contains only two integers: $[\frac{1}{\alpha}]$ and $[\frac{1}{\alpha}] + 1$. Therefore, $k(x) \in \{l, l+1\}$.

With $k(x)$ given at (1), we define the return map $\phi : J_1 \to J_1$,

$$\phi(x) = R_\alpha^{k(x)}(x) = (x + k(x)\alpha) \bmod 1$$
$$= x + k(x)\alpha - 1. \quad (2)$$

(ii) We show that $\phi(1-\alpha) = \lim_{x \to 1^-} \phi(x)$. Put $\alpha' = 1-\alpha$.

$$\frac{2-\alpha'}{\alpha} = \frac{1+\alpha}{\alpha} = \frac{1}{\alpha} + 1 \notin \mathbb{Z}.$$

Thus, $k(\alpha') = \left[\frac{2-\alpha'}{\alpha}\right] = \left[\frac{1}{\alpha}\right] + 1 = l + 1.$

By (2),

$$\phi(\alpha') = \alpha' + k(\alpha')\alpha - 1 = (1-\alpha) + (l+1)\alpha - 1 = l\alpha.$$

We see that $\frac{2-1}{\alpha} = \frac{1}{\alpha} \notin \mathbb{Z}$. Because of the discreteness of $\mathbb{Z}$ in $\mathbb{R}$, there exists $\varepsilon \in (0, \alpha)$ such that $\frac{2-x}{\alpha} \notin \mathbb{Z}$ for all $x \in (1-\varepsilon, 1+\varepsilon)$. Then $[\frac{2-x}{\alpha}]$ is constant in the interval $x \in (1-\varepsilon, 1+\varepsilon)$. It is equal to $[\frac{2-1}{\alpha}] = l$.

$$k(x) = \left[\frac{2-x}{\alpha}\right] = \ell \quad \forall x \in (1-\varepsilon, 1).$$

Then

$$\phi(x) = x + k(x)\alpha - 1 = x + \ell\alpha - 1 \quad \forall x \in (1-\varepsilon, 1).$$

Then $\lim_{x \to 1^-} \phi(x) = 1 + \ell\alpha - 1 = \ell\alpha = \phi(\alpha')$.

Next, define an equivalence relation in $\mathbb{R}$: $x \sim y \Leftrightarrow \frac{x-y}{\alpha} \in \mathbb{Z}$.

Denote $\bar{x}$ the equivalence class of x. Put $S = \{\bar{x} : x \in \mathbb{R}\}$. We can identify

$$S \equiv \mathbb{R}/\sim \; \equiv [1-\alpha, 1]/_{1-\alpha \sim 1}.$$

For this reason, S can be thought as a smaller circle (than S^1). Define a map $\tilde{\phi} : S \to S$,

$$\tilde{\phi}(\bar{x}) = \overline{\phi(x)} \quad \forall x \in J_1.$$

It is a well-defined map. We show that it is continuous.

$$\overline{\phi(x)} = \overline{x + k(x)\alpha - 1} = \overline{x-1} \quad \forall x \in J_1.$$

Thus, $\tilde{\phi}(\bar{x}) = \overline{x-1}$ for all $x \in J_1$. We see that

$$\bar{x} = \bar{y} \Leftrightarrow \frac{x-y}{\alpha} \in \mathbb{Z} \Leftrightarrow \frac{(x-1)-(y-1)}{\alpha} \in \mathbb{Z} \Leftrightarrow \overline{x-1} = \overline{y-1}.$$

Hence, $\tilde{\phi}(\bar{x}) = \overline{x-1}$ for all $x \in \mathbb{R}$. Put $T : \mathbb{R} \to \mathbb{R}$, $T(x) = x - 1$. Then

$$\tilde{\phi}(\bar{x}) = p \circ T(x) \quad \forall x \in \mathbb{R}.$$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "T"] \arrow[drr, "p"'] & \mathbb{R} \arrow[r, "p"] \arrow[dr, phantom, "\circlearrowleft"] & S \\ & & S \arrow[u, "\tilde{\phi}"'] \end{tikzcd}$$

Here p is the map $p : \mathbb{R} \to S$, $p(x) = \bar{x}$.

By the characteristic property of quotient topology (Theorem 3.70, John Lee

"Introduction to topological manifolds", 2011), $\tilde{\Phi}: S \to S$ is continuous if and only if $T \circ p: \mathbb{R} \to S$ is continuous. This is the case because both T and p are continuous.

(iii) Define a map $\Psi: S \to S^1$,

$$\Psi(\bar{x}) = \left(1 - \frac{x-(1-\alpha)}{\alpha}\right) \bmod 1 \quad \forall x \in \mathbb{R}.$$

It is well-defined because the right hand side does not depend on the choice of representative in $\bar{x}$. We compute the composite map $\Psi \circ \tilde{\Phi} \circ \Psi^{-1}$. For $y \in S^1$, the equation $\Psi(\bar{x}) = y$ is satisfied if

$$1 - \frac{x-(1-\alpha)}{\alpha} = y,$$

which is satisfied if $x = 1 - \alpha y$. Then $\tilde{\Phi}(\bar{x}) = \overline{x-1} = \overline{-\alpha y}$. By the definition of Ψ,

$$\Psi(\overline{-\alpha y}) = 1 - \frac{-\alpha y-(1-\alpha)}{\alpha} = 1 + \frac{\alpha y+(1-\alpha)}{\alpha} = y + \frac{1}{\alpha}.$$

Thus, $\quad \Psi \circ \tilde{\Phi} \circ \Psi^{-1}(y) = \Psi \circ \tilde{\Phi}(\bar{x}) = \Psi(\overline{-\alpha y}) = \left(y + \frac{1}{\alpha}\right) \bmod 1.$

Put $\beta = \frac{1}{\alpha} - \ell = \frac{1}{\alpha} - \left[\frac{1}{\alpha}\right] \in (0,1) \setminus \mathbb{Q}$. Then

$$\Psi \circ \tilde{\Phi} \circ \Psi^{-1}(y) = (y+\beta) \bmod 1 = R_\beta(y) \quad \forall y \in S^1.$$

Thus, $\Psi \circ \tilde{\Phi} \circ \Psi^{-1} = R_\beta$.

(iv) By Part (iii), $\quad \alpha = \dfrac{1}{\ell+\beta}.$

Labeling α as α_1, ℓ as ℓ_1, β as α_2, we can write

$$\alpha_1 = \frac{1}{l_1 + \alpha_2}.$$

Recall that $l_1 \in \mathbb{N}$ is "almost" the returning time of $[1-\alpha_1, 1)$ under the rigid rotation R_{α_1}. "Almost" is understood as that the returning time of each point of $[1-\alpha_1, 1)$ is either l_1 or l_1+1.

Because $\alpha_2 \in (0,1)\setminus\mathbb{Q}$, we can view α_2 as α_1 and repeat the process in Parts (i) and (ii) and (iii). Then

$$\alpha_2 = \frac{1}{l_2 + \alpha_3}$$

where $l_2 \in \mathbb{N}$ is almost the returning time of $[1-\alpha_2, 1)$ under the rigid rotation R_{α_2}, and $\alpha_3 \in (0,1)\setminus\mathbb{Q}$. Similarly,

$$\alpha_3 = \frac{1}{l_3+\alpha_4}, \qquad \alpha_4 = \frac{1}{l_4+\alpha_5}, \quad \ldots$$

Then

$$\alpha = \alpha_1 = \frac{1}{l_1+\alpha_2} = \frac{1}{l_1 + \frac{1}{l_2+\alpha_3}} = \frac{1}{l_1 + \frac{1}{l_2 + \frac{1}{l_3+\alpha_4}}} = \frac{1}{l_1 + \frac{1}{l_2 + \frac{1}{l_3 + \frac{1}{l_4+\alpha_5}}}}$$

$$= \cdots$$

The returning times $l_1, l_2, l_3, \ldots$ give us a continued fraction expression for α.

(v) Take any $x \in S^1$. Consider the orbit

$$x_j = R_\alpha^j(x) = (x + j\alpha) \bmod 1, \quad \forall j \in \mathbb{N}.$$

Define a coding sequence

$$a_j = \begin{cases} 0 & \text{if } x_j \in [0, 1-\alpha), \\ 1 & \text{if } x_j \in [1-\alpha, 1). \end{cases}$$

We show that starting from some index, (a_j) has the pattern: ℓ or $\ell-1$ number 0's followed by exactly one number 1. That is to show

$$(a_j) = \underbrace{\cdots\cdots}_{\substack{\text{disordered}\\ \text{portion}}} \underbrace{0\cdots0}_{\ell \text{ or } \ell-1} 1 \underbrace{0\cdots0}_{\ell \text{ or } \ell-1} 1 \underbrace{0\cdots0}_{\ell \text{ or } \ell-1} 1 \underbrace{0\cdots0}_{\ell \text{ or } \ell-1} 1 \cdots\cdots$$

Because the interval $[1-\alpha, 1)$ is of length α, there exists $j_0 \in \mathbb{N}$ such that $x_{j_0} \in [1-\alpha, 1)$. We can assume that the sequence (x_j) starts from x_{j_0}, which is equivalent to assuming $x \in [1-\alpha, 1)$.

In Part (i), we showed that the smallest positive number k such that $R_\alpha^k(x) \in [1-\alpha, 1)$ is $k(x) \in \{\ell, \ell+1\}$. Thus,

$$\begin{cases} a_1, a_2, \ldots, a_{\ell-1} = 0, \\ a_\ell = 1 \text{ or } (a_\ell = 0 \text{ and } a_{\ell+1} = 1). \end{cases}$$

If $a_\ell = 1$, we regard x_ℓ as x which starts a new sequence. Then a_ℓ is followed by $\ell-1$ or ℓ number 0's and then a number 1.

~~If $a_{\ell+1} = 1$, we regar~~

$$\underbrace{0\cdots0}_{\ell-1} \underset{\substack{\uparrow\\ a_\ell}}{1} \underbrace{0\cdots0}_{\ell \text{ or } \ell-1} 1$$

If $a_{\ell+1} = 1$, we regard $x_{\ell+1}$ as x which starts a new sequence. Then $a_{\ell+1}$ is followed by $\ell-1$ or ℓ number 0's and then a number 1.

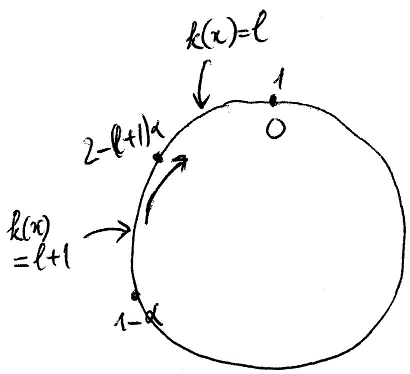
$$\underbrace{0\cdots0}_{\ell} \underset{\substack{\uparrow\\ a_{\ell+1}}}{1} \underbrace{0\cdots0}_{\ell \text{ or } \ell-1} 1$$

Continue the process of translating the sequence (x_j) by updating the value of x in $[1-\alpha, 1)$. We conclude that (a_j) has the pattern: ℓ or $\ell-1$ consecutive number 0's followed by exactly one number 1.

ℓ is about the time it takes to go (via R_α) from a point in $[1-\alpha, 1)$ back to the same interval.

(vi) Define a sequence (b_j) obtained from (a_j) by replacing each block $\underbrace{0\cdots0}_{\ell-1}1$ by number 0, and each block $\underbrace{0\cdots0}_{\ell}1$ by number 1. We show that (b_j) has the same pattern as (a_j), except that the "period" may be different from ℓ.

To do so, we show that the process of substituting each block by a single number creates an orbit on the smaller circle S which is analoguous to the orbit (x_j) on S^1.



Translating (x_j) by some index if necessary, we can assume $x \in [1-\alpha, 1)$.

$k(x) = \ell+1 \Leftrightarrow x_j$ has not returned to $[1-\alpha, 1)$ after ℓ times

$$\Leftrightarrow x_\ell = x + \ell\alpha < 1 + (1-\alpha)$$

$$\Leftrightarrow x < 2 - (\ell+1)\alpha.$$

Put $J_0' = [1-\alpha, 2-(\ell+1)\alpha)$ and $J_1' = [2-(\ell+1)\alpha, 1)$. Then

$$k(x) = \begin{cases} \ell+1 & \text{if } x \in J_0', \\ \ell & \text{if } x \in J_1'. \end{cases}$$

The sequence (a_j) starts with block $\underbrace{0\cdots0}_{\ell}1$ if $x \in J_0'$, and with block $\underbrace{0\cdots0}_{\ell-1}1$ if $x \in J_1'$. Recall from Part (ii) that $S = R/\sim \; \equiv [1-\alpha, 1]/_{1-\alpha\sim 1}$.

The point $\cancel{k(x)} = (x+\ell\alpha)$ may not lie in $[1-\alpha, 1)$, but is equivalent to a unique point in S thanks to the equivalence relation $\sim$. Then (a_j) starts with block $\underbrace{0\cdots 0}_{\ell-1}1$ if $\phi(x) = x + k(x)\alpha - 1$ is equivalent to a point in J_1', and with block $\underbrace{0\cdots 0}_{\ell}1$ if $\phi(x)$ is equivalent to a point in J_0'. Note that $\phi(x)$ is equivalent to $x-1$. Put $x_1' = x \in S$. Then

$$b_1 = \begin{cases} 1 & \text{if } x_1' \in J_0' \\ 0 & \text{if } x_1' \in J_1' \end{cases}$$

$$b_2 = \begin{cases} 1 & \text{if } x_1' - 1 \text{ is equivalent to some } x_2' \in J_0', \\ 0 & \text{if } x_1' - 1 \text{ is equivalent to some } x_2' \in J_1' \end{cases}$$

$$b_3 = \begin{cases} 1 & \text{if } x_2' - 1 \text{ is equivalent to some } x_3' \in J_0' \\ 0 & \text{if } x_2' - 1 \text{ is equivalent to some } x_3' \in J_1' \end{cases}$$

$\cdots\cdots$

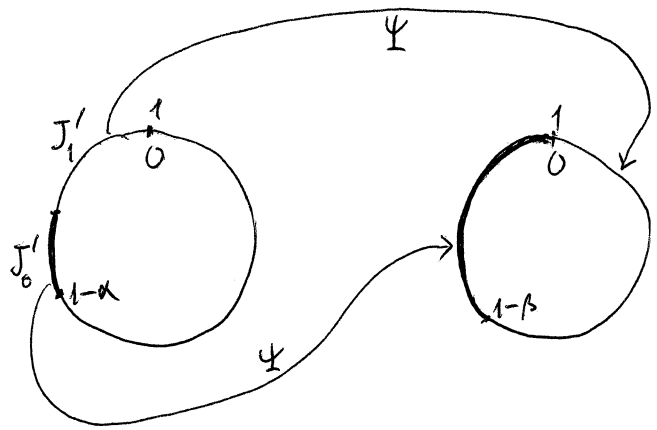
We can view (x_j') as the orbit of x on the small circle S under the rigid rotation R_{-1}. The map Ψ in Part (iii) transforms (x_j') into an orbit on the circle S^1. Put

$$x_j'' = \Psi(x_j') = \left(1 - \frac{x_j' - (1-\alpha)}{\alpha}\right) \bmod 1.$$

Then

$$\begin{aligned} x_{j+1}'' - x_j'' &= -\frac{1}{\alpha}(x_{j+1}' - x_j') \bmod 1 \\ &= \frac{1}{\alpha} \bmod 1 \\ &= \beta. \end{aligned}$$

Put $y = \Psi(x) \in S^1$. Then (x_j'') is the orbit of y on the circle S^1 under the rigid rotation R_β.



$$\Psi((1-\alpha)^+) = \left(1 - \frac{(1-\alpha)^+ - (1-\alpha)}{\alpha}\right) \bmod 1 = 1^-,$$

$$\Psi(2-(\ell+1)\alpha) = \left(1 - \frac{2-(\ell+1)\alpha-(1-\alpha)}{\alpha}\right) \bmod 1 = 1-\beta,$$

$$\Psi(1^-) = \left(1 - \frac{1^- - (1-\alpha)}{\alpha}\right) \bmod 1 = 0^+.$$

Thus, $\Psi(J_0') = [1-\beta, 1)$ and $\Psi(J_1') = [0, 1-\beta)$. We have

$$b_1 = \begin{cases} 1 & \text{if } x_1'' = \Psi(x_1') \in \Psi(J_0') = [1-\beta,1) \\ 0 & \text{if } x_1'' = \Psi(x_1') \in \Psi(J_1') = [0,1-\beta) \end{cases}$$

$$b_2 = \begin{cases} 1 & \text{if } x_2'' \in [1-\beta, 1) \\ 0 & \text{if } x_2'' \in [0, 1-\beta) \end{cases}$$

$\ldots\ldots$

We see that the sequence (b_j) is formed the same way as is (b_j), except that α is replaced by β. Thus, (b_j) also has the pattern

$$(b_j) = \underbrace{\ldots\ldots}_{\substack{\text{disordered}\\\text{portion}}} \underbrace{0\ldots0}_{\ell' \text{ or } \ell'-1}1\underbrace{0\ldots0}_{\ell' \text{ or } \ell'-1}1\underbrace{0\ldots0}_{\ell' \text{ or } \ell'-1}1\underbrace{0\ldots0}_{\ell' \text{ or } \ell'-1}1\ldots\ldots$$

where $\ell' = \left[\frac{1}{\beta}\right]$.

If we continue to replace each block $\underbrace{0\ldots0}_{\ell-1}1$ with number 0, each

block $\underbrace{0\ldots0}_{\ell}1$ with number 1, the resulting sequence still have the same pattern. The new "period" is $\ell'' = \left[\frac{1}{\gamma}\right]$ where $\gamma = \frac{1}{\beta} - \left[\frac{1}{\beta}\right]$.
The sequence of periods $\ell, \ell', \ell'', \ldots$ is exactly the sequence $\ell_1, \ell_2, \ell_3, \ldots$ in the continued fractional expression of α.

$$\alpha = \cfrac{1}{\ell_1 + \cfrac{1}{\ell_2 + \cfrac{1}{\ell_3 + \cdots}}}$$

② We view the circle S^1 as a metric subspace of $\mathbb{R}^2$. Then

$$C^0(S^1; \mathbb{R}^2) = \{h : S^1 \to \mathbb{R}^2 \text{ continuous}\}$$

is a normed vector space with $\|h\|_{C^0} = \sup_{x \in S^1} |h(x)|$. Each homeomorphism from S^1 to S^1 is an element of $C^0(S^1; \mathbb{R}^2)$.

Let $f : S^1 \to S^1$ be a homeomorphism. The topology on S^1 is the topology induced by the map $p : \mathbb{R} \to S^1$, $p(t) = \exp(i2\pi t)$. Both $f \circ p : \mathbb{R} \to S^1$ and $p : \mathbb{R} \to S^1$ are covering maps of S^1. Because $\mathbb{R}$ is simply connected, there exists a homeomorphism $F : \mathbb{R} \to \mathbb{R}$ such that the following diagram commutes (see Proposition 11.41, page 297, John Lee "Introduction to Topological Manifolds", 2011).

$$\begin{tikzcd} \mathbb{R} \arrow[rr, "F"] \arrow[dr, "f \circ p"'] & & \mathbb{R} \arrow[dl, "p"] \\ & S^1 & \end{tikzcd}$$

F is called a lift of S^1 according to terminology in Definition 14.1, page 103, Devaney "An Introduction to Chaotic Dynamical Systems", 1989.

Then $\exp(i2\pi F(x)) = f(\exp(i2\pi x))$ for all $x \in \mathbb{R}$. Then

$$\exp(i2\pi F(x+1)) = f(\exp(i2\pi(x+1))) = f(\exp(i2\pi x)) = \exp(i2\pi F(x)) \ \forall x \in \mathbb{R}.$$

Thus, $F(x+1) - F(x) \in \mathbb{Z}$. Because $F(x+1) - F(x)$ is a continuous function, there exists $l \in \mathbb{Z}$ such that

$$F(x+1) - F(x) = l \quad \forall x \in \mathbb{R}.$$

Once $F(x)$ is known for all $x \in [0,1]$, the value of F elsewhere will be known by adding suitable multiples of l. Then F is uniformly continuous in $\mathbb{R}$.

Recall that the rotation number of f is defined by $\rho(f) = \rho_0(F) \bmod 1$, where $\rho_0(F) = \lim_{n\to\infty} \frac{F^n(0)}{n}$ and $F^n = \underbrace{F \circ F \circ \cdots \circ F}_{n \text{ times}}$. We show that ρ is continuous in f.

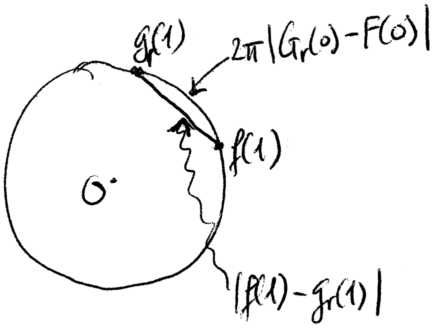
Let (g_r) be a sequence of homeomorphisms from S^1 to S^1 such that $\|g_r - f\|_{C^0} \to 0$ as $r \to \infty$. We show that $\rho(g_r) \to \rho(f)$ in modulo 1. Without loss of generality, we can assume $\|g_r - f\|_{C^0} < 2$ for all $r \in \mathbb{N}$. Let $G_r : \mathbb{R} \to \mathbb{R}$ be a lift of g_r. Then

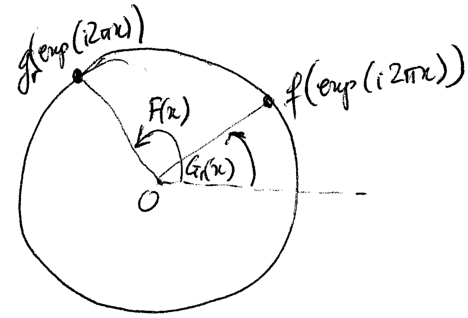
$$\exp(i2\pi G_r(x)) = g_r(\exp(i2\pi x)) \quad \forall x \in \mathbb{R}.$$

Then

$$|\exp(i2\pi G_r(x)) - \exp(i2\pi F(x))| = |g_r(\exp(i2\pi x)) - f(\exp(i2\pi x))| < 2 \quad \forall x \in \mathbb{R}. \tag{1}$$

In the definition of G_r, we ~~se~~ see that $G_r(0)$ can be determined up to an integer. We can choose $G(0)$ such that the length of the smaller arc on S^1 which joins $g_r(1)$ and $f(1)$ is $2\pi\,|G_r(0) - F(0)|$. Because G_r and F are continuous and injective maps from $\mathbb{R}$ to $\mathbb{R}$, they are monotone.





Because of (1), they must have the same monotonicity. In addition, the length of the smaller arc which joins $g_r(\exp(i2\pi x))$ and $f(\exp(i2\pi x))$ is $2\pi|G_r(x)-F(x)|$. The length is not longer than $\frac{\pi}{2}$ times the length of the chord joining these points, which is $\frac{\pi}{2}|g_r(\exp(i2\pi x))-f(\exp(i2\pi x))|$.

Thus,

$$\begin{aligned} |G_r(x)-F(x)| &\le \frac{1}{4}|g_r(\exp(i2\pi x))-f(\exp(i2\pi x))| \\ &\le \frac{1}{4}\|g_r-f\|_{C^0} \qquad \forall x\in\mathbb{R}. \qquad (2) \end{aligned}$$

Next, we show by induction in $n\in\mathbb{N}$ that

$$\lim_{r\to\infty}\|G_r^n-F^n\|_{C^0}=0. \qquad (3)$$

(3) is true for $n=1$ thanks to (2). Suppose (3) is true for some $n\in\mathbb{N}$. Then

$$\begin{aligned} |G_r^{n+1}(x)-F^{n+1}(x)| &\le |G_r(G_r^n(x))-F(G_r^n(x))|+|F(G_r^n(x))-F(F^n(x))| \\ &\le \underbrace{\|G_r-F\|_{C^0}}_{\{1\}}+\underbrace{\sup\{|F(y)-F(z)|: y,z\in\mathbb{R},\ |y-z|\le\|G_r^n-F^n\|_{C^0}\}}_{\{2\}}. \qquad (4) \end{aligned}$$

$\lim_{r\to\infty}\{1\}=0$ because of (2). By the induction hypothesis, $\lim_{r\to\infty}\|G_r^n-F^n\|_{C^0}=0$. Then by the uniform continuity of F, $\lim_{r\to\infty}\{2\}=0$. Because the estimate (4) holds for every $x\in\mathbb{R}$, $\lim_{r\to\infty}\|G_r^{n+1}-F^{n+1}\|_{C^0}=0$. We have proved (3).

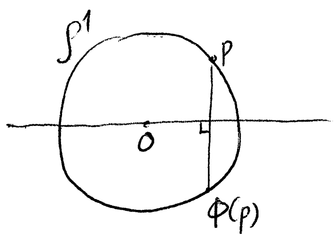
An earlier result is that there exists $l \in \mathbb{Z}$ such that $F(x+1)-F(x)$ for all $x \in \mathbb{R}$. Now we show that $l \in \{\pm 1\}$. First, $l \neq 0$ because F is injective. Suppose by contradiction that $l > 1$. Then the function $h: \mathbb{R} \to \mathbb{R}$, $h(t) = F(t) - F(0)$ is continuous, $h(0) = 0$ and $h(1) > 1$. There exists $c \in (0,1)$ such that $h(c) = 1$. Then

$$f(\exp(i2\pi c)) = \exp(i2\pi F(c)) = \exp(i2\pi F(0)) = f(\exp(i2\pi 0)).$$

This is a contradiction because f is injective on S^1. The case $l < -1$ is dealt similarly. Therefore,

$$F(x+1) - F(x) = \begin{cases} 1 & \text{if } F \text{ is increasing}, \\ -1 & \text{if } F \text{ is decreasing}. \end{cases}$$

We showed earlier that F and each G_r have the same monotonicity. Denote by $\phi: S^1 \to S^1$, $\phi(\exp(i2\pi x)) = \exp(-i2\pi x)$ the reflection map.



$\phi \circ f$ and $\phi \circ g_r$ are homeomorphisms from S^1 to S^1 and $\| \phi \circ f - \phi \circ g_r \|_{C^0} \to 0$ as $r \to \infty$.

A lift of $\phi \circ f$ is $-F$ because

$$\exp(-i2\pi F(x)) = \phi(\exp(i2\pi F(x))) = \phi(f(\exp(i2\pi x))).$$

Therefore, replacing f by $\phi \circ f$, each g_r by $\phi \circ g_r$ if necessary, we can assume F (and thus each G_r) is increasing. Thus, $F(x+1) = F(x) + 1$ for every $x \in \mathbb{R}$. We now show that

$$|F^n(x) - F^n(y)| < 1 \quad \forall x, y \in [0,1) \quad \forall n \in \mathbb{N}. \qquad (5)$$

Take $x, y \in [0,1)$, $x<y$. Because F is increasing, F^n is too. Since $x<y<x+1$,

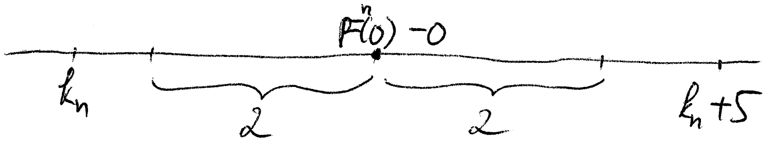
$$F^n(x) < F^n(y) < F^n(x+1) = F^n(x)+1.$$

Then $0 < F^n(y) - F^n(x) < 1$. Consequently,

$$|(F^n(x)-x)-(F^n(y)-y)| \leq |F^n(x)-F^n(y)| + |x-y| < 1+1 \quad \forall x,y \in [0,1), n \in \mathbb{N}.$$

Therefore, for each $n \in \mathbb{N}$ there exists $p_n \in \mathbb{Z}$ such that

$$p_n < F^n(x) - x < p_n + 5 \quad \forall x \in [0,1).$$



Because $F^n(x+m) = F^n(x)+m$ for every $m \in \mathbb{Z}$,

$$p_n < F^n(x) - x < p_n + 5 \quad \forall x \in \mathbb{R}. \qquad (6)$$

Because $\|G_r^n - F^n\|_{C^0} \to 0$ as $r \to \infty$, there exists $N(n) \in \mathbb{N}$ such that $\|G_r^n - F^n\|_{C^0} < 1$ for all $r > N(n)$. Then

$$|(G_r^n(x)-x)-(F^n(x)-x)| = |G_r^n(x) - F^n(x)| < 1 \quad \forall x \in \mathbb{R}, n \in \mathbb{N}, r > N(n).$$

Thus,

$$p_n - 1 < G_r^n(x) - x < p_n + 6 \quad \forall x \in \mathbb{R}, n \in \mathbb{N}, r > N(n). \qquad (7)$$

For each $m \in \mathbb{N}$,

$$F^{mn}(0) = \sum_{k=1}^{m} \left(F^n(F^{(k-1)n}(0)) - F^{(k-1)n}(0) \right).$$

Applying (6), we get $m p_n < F^{mn}(0) < m(p_n+5)$. Divide both sides by mn,

$$\frac{p_n}{n} < \frac{F^{mn}(0)}{mn} < \frac{p_n+5}{n}. \qquad (8)$$

Similarly,

$$G_r^{mn}(0) = \sum_{k=1}^{m} \left(G_r^n\left(G_r^{(k-1)n}(0)\right) - G_r^{(k-1)n}(0) \right).$$

Applying (7), we get $m(p_n - 1) < G_r^{mn}(0) < m(p_n + 6)$. Divide both sides by mn,

$$\frac{p_n - 1}{n} < \frac{G_r^{mn}(0)}{mn} < \frac{p_n + 6}{n}. \qquad (9)$$

By (8) and (9),

$$-\frac{6}{n} < \frac{F^{mn}(0)}{mn} - \frac{G_r^{mn}(0)}{mn} < \frac{6}{n} \qquad \forall m, n \in \mathbb{N},\ r > N(n).$$

Let $m \to \infty$,

$$-\frac{6}{n} \leq \rho_0(F) - \rho_0(G_r) \leq \frac{6}{n} \qquad \forall n \in \mathbb{N},\ r > N(n).$$

Let $n \to \infty$, $\lim_{r\to\infty} \rho_0(G_r) = \rho_0(F)$. Taking modulo 1 both sides, we get

$$\lim_{r\to\infty} \rho(g_r) = \rho(f) \qquad (\mathrm{mod}\ 1).$$

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-function satisfying $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Because f is locally Lipschitz, by Picard-Lindelöf's theorem, the problem $\dot{x} = f(x)$, $x(0) = x_0$ has a unique local solution. Since f is continuous and periodic, it is bounded in $\mathbb{R}$. Then a global solution exists and is unique. Denote by $x(t; x_0)$ this solution. Then $(\phi_t)_{t\in\mathbb{R}}$, $\phi_t(x_0) = x(t; x_0)$ for $t, x_0 \in \mathbb{R}$, is a flow.

Each $\phi_t : \mathbb{R} \to \mathbb{R}$ is a homeomorphism and $\phi_{t+s} = \phi_t \circ \phi_s$ for all $t, s \in \mathbb{R}$.

Because f is 1-periodic,

$$\frac{d}{dt}[x(t; x_0) + 1] = \frac{d}{dt} x(t; x_0) = f(x(t; x_0)) = f(x(t; x_0) + 1).$$

Thus, $x(t;x_0)+1$ is a global solution to the problem

$$\begin{cases} \dot{x} = f(x), \\ x(0) = x_0 + 1. \end{cases}$$

By the uniqueness of solutions, $x(t;x_0)+1 = x(t;x_0+1)$. In other words,

$$\phi_t(x_0+1) = \phi_t(x_0)+1 \quad \forall x_0 \in \mathbb{R}, t \in \mathbb{R}.$$

Define a map $p: \mathbb{R} \to S^1$, $p(x) = \exp(i2\pi x)$. This is a quotient map, i.e. the topology on S^1 is the one induced by p. We see that $p \circ \phi_t : \mathbb{R} \to S^1$ is continuous and is constant in each fiber of p. Indeed, suppose $p(x) = p(x')$. Then $x = x' + l$ for some $l \in \mathbb{Z}$. Then

$$p(\phi_t(x)) = p(\phi_t(x'+l)) = p(\phi_t(x')+l) = p(\phi_t(x')).$$

For this reason, there exists a continuous map $g_t : S^1 \to S^1$ such that the following diagram commutes (Theorem 3.73, page 72, John Lee "Introduction to Topological Manifolds", 2011). We have

$$\begin{tikzcd} \mathbb{R} \arrow[d, "p"'] \arrow[dr, "p\circ\phi_t"] & \\ S^1 \arrow[r, dashed, "g_t"'] & S^1 \end{tikzcd}$$

$$g_t(\exp(i2\pi x)) = \exp(i2\pi \phi_t(x)) \quad \forall x \in \mathbb{R}. \quad (1)$$

Because ϕ_t is continuous and injective on $\mathbb{R}$, it is monotone. Since $\phi_t(x+1) = \phi_t(x)+1$, ϕ_t is increasing. Then

$$\phi_t([0,1)) = [\phi_t(0), \phi_t(1)) = [\phi_t(0), \phi_t(0)+1).$$

Then $\phi_t|_{[0,1)} : [0,1) \to [\phi_t(0), \phi_t(0)+1)$ is bijective. This implies g_t is a bijection. Because $g_t : S^1 \to S^1$ is continuous, bijective and S^1 is compact, we conclude that g_t is a homeomorphism.

Next, we compute the rotation number of g_t. The identity (1) shows that ϕ_1 is a lift of g_1.

$$\rho_0(\phi_1) = \lim_{n\to\infty} \frac{\phi_1^n(0)}{n} = \lim_{n\to\infty} \frac{\phi_n(0)}{n}. \qquad (2)$$

The number $\rho_0(\phi_t)$ is expressed by the same manner.

$$\rho_0(\phi_t) = \lim_{n\to\infty} \frac{\phi_t^n(0)}{n} = \lim_{n\to\infty} \frac{\phi_{nt}(0)}{n}. \qquad (3)$$

Put $M = \sup_{x\in\mathbb{R}} |f(x)| < \infty$. Then

$$\phi_t(x_0) - \phi_1(x_0) = \int_1^t \frac{d}{ds}\phi_s(x_0)\,ds = \int_1^t f(\phi_s(x_0))\,ds.$$

Thus,

$$|\phi_t(x_0) - \phi_1(x_0)| \leq \left| \int_1^t |f(\phi_s(x_0))|\,ds \right| \leq M|t-1| \quad \forall x_0 \in \mathbb{R} \qquad (4)$$

By (1) we have

$$\begin{aligned} \left| g_t(\exp(i2\pi x_0)) - g_1(\exp(i2\pi x_0)) \right| &= \left| \exp(i2\pi\phi_t(x_0)) - \exp(i2\pi\phi_1(x_0)) \right| \\ &\leq |\phi_t(x_0) - \phi_1(x_0)| \sup_{y\in\mathbb{R}} \left| \frac{d}{dy} \exp(i2\pi y) \right| \\ &= 2\pi\, |\phi_t(x_0) - \phi_1(x_0)| \\ &\leq 2\pi M\, |t-1| \qquad \forall x_0 \in \mathbb{R}. \end{aligned}$$

Thus, $\|g_t - g_1\|_{C^0(S^1;\mathbb{R}^2)} \leq 2\pi M|t-1|$.

This implies $g_t \to g$ in $C^0(S^1;\mathbb{R}^2)$ as $t \to 1$. By Problem 2, we conclude that $\rho_0(\phi_t) \to \rho_0(\phi_1)$ and $\rho(g_t) \to \rho(g_1)$ as $t \to 1$. A direct proof can be obtained by using (2) and (3) and (4).

Assume $f(x) \neq 0$ for all $x \in [0,1]$. We now compute $\rho_0(\phi_t)$, $t \in \mathbb{R}$, explicitly in terms of f. That is to compute $\lim_{n\to\infty} \frac{\phi_{nt}(0)}{n}$ in terms of f. Recall that $\phi_t(0)$ is the solution to the problem

$$\begin{cases} \dot{x} = f(x), \\ x(0) = 0. \end{cases}$$

Because f is continuous and nonzero on $[0,1]$, it does not change its sign. There exist $m, M > 0$ such that

$$m \leq |f(x)| \leq M \quad \forall x \in [0,1].$$

Integrating the equation $1 = \frac{\dot{x}}{f(x)}$ from 0 to t, we get

$$t = \int_0^t \frac{\dot{x}(\tau)}{f(x(\tau))} d\tau = \int_0^{x(t)} \frac{ds}{f(s)}. \qquad (5)$$

Define a function $F: \mathbb{R} \to \mathbb{R}$, $F(t) = \int_0^t \frac{ds}{f(s)}$.

We show that F is a bijection.

If f is positive then F is increasing. Moreover,

$$F(t) = \int_0^t \frac{ds}{f(s)} \geq \int_0^t \frac{ds}{M} = \frac{t}{M} \quad \forall t > 0,$$

$$F(t) = -\int_t^0 \frac{ds}{f(s)} \leq -\int_t^0 \frac{ds}{M} = \frac{t}{M} \quad \forall t < 0.$$

Then $\lim_{t\to\pm\infty} F(t) = \pm\infty$. Thus, F is a bijection.

If f is negative then F is decreasing. Moreover,

$$F(t) = \int_0^t \frac{ds}{f(s)} \leq \int_0^t \frac{ds}{-M} = -\frac{t}{M} \quad \forall t > 0,$$

$$F(t) = -\int_t^0 \frac{ds}{f(s)} \geq -\int_t^0 \frac{ds}{-M} = \frac{-t}{M} \quad \forall\, t<0.$$

Then $\lim\limits_{t\to\pm\infty} F(t) = \mp\infty$. Thus, F is a bijection.

Now (5) becomes $t = F(x(t))$, which gives $x(t) = F^{-1}(t)$. Then

$$\lim_{t\to\pm\infty} \frac{x(t)}{t} = \lim_{t\to\pm\infty} \frac{F^{-1}(t)}{t} = \begin{cases} \lim\limits_{s\to\pm\infty} \dfrac{s}{F(s)} & \text{if } f \text{ is positive}, \quad (6) \\[2ex] \lim\limits_{s\to\mp\infty} \dfrac{s}{F(s)} & \text{if } f \text{ is negative}. \quad (7) \end{cases}$$

Consider the case f is positive. For $s \neq 0$,

$$F(s) = \int_0^s \frac{d\tau}{f(\tau)} = \int_0^{[s]} \frac{d\tau}{f(\tau)} + \int_{[s]}^s \frac{d\tau}{f(\tau)}, \qquad (8)$$

where $[s]$ is the integer part of s. Since f is 1-periodic,

$$\int_0^{[s]} \frac{d\tau}{f(\tau)} = [s] \int_0^1 \frac{d\tau}{f(\tau)},$$

$$\int_{[s]}^s \frac{d\tau}{f(\tau)} = \int_0^{\{s\}} \frac{d\tau}{f(\tau+[s])} = \int_0^{\{s\}} \frac{d\tau}{f(\tau)},$$

where $\{s\}$ is the fractional part of s. Then (8) becomes

$$F(s) = [s] \int_0^1 \frac{d\tau}{f(\tau)} + \int_0^{\{s\}} \frac{d\tau}{f(\tau)}.$$

Thus,

$$\frac{F(s)}{s} = \underbrace{\frac{[s]}{s} \int_0^1 \frac{d\tau}{f(\tau)}}_{A} + \underbrace{\frac{1}{s} \int_0^{\{s\}} \frac{d\tau}{f(\tau)}}_{B}.$$

$\lim\limits_{s\to\pm\infty} A = \int_0^1 \frac{d\tau}{f(\tau)}$ because $\lim\limits_{s\to\pm\infty} \frac{[s]}{s} = 1$.

$\lim\limits_{s\to\pm\infty} B = 0$ because $0 \leq \int_0^{\{s\}} \frac{d\tau}{f(\tau)} \leq \int_0^1 \frac{d\tau}{f(\tau)}$ for all $s \neq 0$.

Therefore, $\lim_{s\to\pm\infty} \frac{F(s)}{s} = \int_0^1 \frac{d\tau}{f(\tau)}$. Then (6) gives us

$$\lim_{t\to\pm\infty} \frac{x(t)}{t} = \left(\int_0^1 \frac{d\tau}{f(\tau)}\right)^{-1}. \qquad (9)$$

Consider the case f is negative. For $s \neq 0$,

$$F(s) = \int_0^s \frac{d\tau}{f(\tau)} = \int_0^{[s]} \frac{d\tau}{f(\tau)} + \int_{[s]}^{s} \frac{d\tau}{f(\tau)} = [s]\int_0^1 \frac{d\tau}{f(\tau)} + \int_0^{\{s\}} \frac{d\tau}{f(\tau)}.$$

Then

$$\lim_{s\to\pm\infty} \frac{F(s)}{s} = \lim_{s\to\pm\infty} \frac{[s]}{s}\int_0^1 \frac{d\tau}{f(\tau)} + \lim_{s\to\pm\infty} \frac{1}{s}\int_0^{\{s\}} \frac{d\tau}{f(\tau)} = \int_0^1 \frac{d\tau}{f(\tau)}.$$

Substituting this result into (7), we also get (9). Therefore,

$$\lim_{t\to\pm\infty} \frac{\phi_t(0)}{t} = \left(\int_0^1 \frac{d\tau}{f(\tau)}\right)^{-1}.$$

Then

$$\begin{aligned}
\rho_0(\phi_t) = \lim_{n\to\infty} \frac{\phi_{nt}(0)}{n} &= t \lim_{n\to\infty} \frac{\phi_{nt}(0)}{nt} \\
&= \begin{cases} t \lim_{s\to\infty} \frac{\phi_s(0)}{s} & \text{if } t>0, \\ t \lim_{s\to-\infty} \frac{\phi_s(0)}{s} & \text{if } t<0, \end{cases} \\
&= t\left(\int_0^1 \frac{d\tau}{f(\tau)}\right)^{-1} \quad \forall t \neq 0.
\end{aligned}$$

This formula is also true for $t=0$ because ϕ_0 is the identity map on $\mathbb{R}$.

We conclude that

$$\rho_0(\phi_t) = t\left(\int_0^1 \frac{d\tau}{f(\tau)}\right)^{-1} \quad \forall t \in \mathbb{R},$$

$$\rho(g_t) = \rho_0(\phi_t) \mod 1.$$