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Math 8502: Differential Equations and Dynamical Systems

Homework #4

(2) Consider a linear operator $T: \mathbb{R}^2 \to \mathbb{R}^2$ that is diagonalized by the standard basis (e_1, e_2) of $\mathbb{R}^2$. Suppose

$$\begin{cases} Te_1 = \lambda^{uu} e_1, \\ Te_2 = \lambda^u e_2, \\ 1 < \lambda^u < \lambda^{uu}. \end{cases}$$

Denote

$$E^{uu} = \mathbb{R}e_1 = \{(a,0) : a \in \mathbb{R}\},$$

$$E^u = \mathbb{R}e_2 = \{(0,b) : b \in \mathbb{R}\},$$

$$T^{uu} : E^{uu} \to E^{uu}, \quad T^{uu}x = \lambda^{uu}x,$$

$$T^u : E^u \to E^u, \quad T^u x = \lambda^u x.$$

A graph in $E^{uu} \times E^u$ is identified with a map $\sigma : E^{uu} \to E^u$. Consider a special class of graphs

$$L = \left\{ \sigma : E^{uu} \to E^u \text{ continuous}, \ \sigma(0) = 0, \ \sup_{x \neq 0} \frac{|\sigma(x)|}{|x|} < \infty \right\}.$$

We see that L is a linear space over $\mathbb{R}$. If we define

$$\|\sigma\| = \sup_{x \neq 0} \frac{|\sigma(x)|}{|x|} \qquad \forall \sigma \in L$$

then $(L, \|.\|)$ is a normed space. Suppose (σ_n) is a Cauchy sequence in L. Then

$$|\sigma_m(x) - \sigma_n(x)| = |x| \frac{|\sigma_m(x) - \sigma_n(x)|}{|x|} \leq |x| \, \|\sigma_m - \sigma_n\| \qquad \forall x \in E^{uu}.$$

Then $(\sigma_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in E^u. Thus, it converges. Denote the limit by $\sigma(x)$. For any $R>0$,

$$\begin{aligned}|\sigma(x)-\sigma_n(x)| &\le |\sigma(x)-\sigma_m(x)| + |\sigma_m(x)-\sigma_n(x)| \\ &\le |\sigma(x)-\sigma_m(x)| + |x|\,\|\sigma_m-\sigma_n\| \qquad (1) \\ &\le |\sigma(x)-\sigma_m(x)| + R\,\|\sigma_m-\sigma_n\| \quad \forall m\in\mathbb{N},\ \forall x\in E^{uu}, |x|\le R\end{aligned}$$

For every $\varepsilon>0$, there exists $N(\varepsilon)\in\mathbb{N}$ such that $\|\sigma_m-\sigma_n\|<\varepsilon$ for all $m,n>N(\varepsilon)$. Then

$$|\sigma(x)-\sigma_n(x)| \le |\sigma(x)-\sigma_m(x)| + R\varepsilon \quad \forall m,n>N(\varepsilon), \forall x\in E^{uu}, |x|\le R$$

~~For ever~~ Letting $m\to\infty$, we get $|\sigma(x)-\sigma_n(x)|\le R\varepsilon$ for all $n>N(\varepsilon)$, $x\in E^{uu}$ such that $|x|\le R$. Thus, (σ_m) converges to σ uniformly on every compact subset of E^{uu}. Then $\sigma: E^{uu}\to E^u$ is continuous. Also, $\sigma(0)=\lim_{n\to\infty}\sigma_n(0)=0$.

$$\begin{aligned}\frac{|\sigma(x)-\sigma_n(x)|}{|x|} &\overset{(1)}{\le} \frac{|\sigma(x)-\sigma_m(x)|}{|x|} + \|\sigma_m-\sigma_n\| \\ &< \frac{|\sigma(x)-\sigma_m(x)|}{|x|} + \varepsilon \qquad \forall m,n>N(\varepsilon), \forall x\in E^{uu}\setminus\{0\}.\end{aligned}$$

Letting $m\to\infty$, we get

$$\frac{|\sigma(x)-\sigma_n(x)|}{|x|} \le \varepsilon \qquad \forall n>N(\varepsilon), \forall x\in E^{uu}\setminus\{0\}. \qquad (2)$$

Thus, $\displaystyle \frac{|\sigma(x)|}{|x|} \le \frac{|\sigma_n(x)|}{|x|} + \frac{|\sigma(x)-\sigma_n(x)|}{|x|} \le \|\sigma_n\|+\varepsilon \quad \forall n>N(\varepsilon), \forall x\in E^{uu}\setminus\{0\}.$

Then $\displaystyle \sup_{x\neq 0}\frac{|\sigma(x)|}{|x|} \le \|\sigma_{N(\varepsilon)+1}\| + \varepsilon < \infty.$

This implies $\sigma\in L$. Then (2) implies $\|\sigma-\sigma_n\|\le\varepsilon$ for all $n>N(\varepsilon)$. Thus, $\sigma_n\to\sigma$

in L. We have showed that L is a Banach space (over $\mathbb{R}$).

Denote by $p^u : E^{uu} \oplus E^u \to E^u$ and $p^{uu} : E^{uu} \oplus E^u \to E^{uu}$ the (orthogonal) projection maps. For a map $g : E \to F$ between two normed spaces, we denote

$$\mathrm{Lip}(g) = \sup_{x \neq y} \frac{\| g(x) - g(y) \|}{\| x - y \|}.$$

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map such that $f(0) = 0$ and $\mathrm{Lip}(f - T) < \varepsilon$ for some $\varepsilon \in \left(0, \frac{\lambda^{uu} - \lambda^u}{4}\right)$. Denote $f^u = p^u \circ f : E^{uu} \oplus E^u \to E^u$ and $f^{uu} = p^{uu} \circ f : E^{uu} \oplus E^u \to E^{uu}$. Denote

$$B = \{ \sigma \in L : \mathrm{Lip}\, \sigma \leq 1 \}.$$

It is a closed subset of L. Indeed, if (σ_n) is a sequence in B that converges to $\sigma \in L$ then (σ_n) converges to σ pointwise. Then

$$\frac{|\sigma(x) - \sigma(y)|}{|x - y|} = \lim_{n \to \infty} \underbrace{\frac{|\sigma_n(x) - \sigma_n(y)|}{|x - y|}}_{\leq \mathrm{Lip}(\sigma_n) \leq 1} \leq 1 \qquad \forall x, y \in E^{uu},\ x \neq y.$$

Then $\mathrm{Lip}(\sigma) = \sup_{x \neq y} \frac{|\sigma(x) - \sigma(y)|}{|x - y|} \leq 1$.

We define a map $\Gamma_f : B \to B$, $\Gamma_f(\sigma) = f^u \circ (id, \sigma) \circ [f^{uu} \circ (id, \sigma)]^{-1}$. We show that Γ_f is well-defined and is a contraction. A consequence is that Γ_f has a fixed point in B.

For $\sigma \in B$, we first show that

$$\mathrm{Lip}\left(p^{uu} \circ (f - T) \circ (id, \sigma)\right) \leq 2\varepsilon \quad \ldots \quad (3)$$

$\boxed{4}$

Since p^{uu} is an orthogonal projection, $Lip(p^{uu}) = 1$.

$$Lip(id, \sigma) = \sup_{x \neq y} \frac{|(x-y, \sigma(x)-\sigma(y))|}{|x-y|} \leq \sup_{x\neq y} \frac{|x-y| + |\sigma(x)-\sigma(y)|}{|x-y|}$$

$$= 1 + Lip(\sigma) \leq 2.$$

Then $Lip(p^{uu} \circ (f-T) \circ (id, \sigma)) \leq Lip(p^{uu})\, Lip(f-T)\, Lip(id, \sigma) \leq 1 \cdot \varepsilon \cdot 2 = 2\varepsilon$.

We have proved (3). Put $\varphi = f^{uu} \circ (id, \sigma) : E^{uu} \to E^{uu}$. Then

$$\varphi(x) - T^{uu}(x) = f^{uu}(x, \sigma(x)) - T^{uu}(x) = p^{uu} \circ (f-T)(x, \sigma(x)).$$

Then

$$|(\varphi(x) - T^{uu}(x)) - (\varphi(y) - T^{uu}(y))| = |p^{uu} \circ (f-T) \circ (id, \sigma)(x) - p^{uu} \circ (f-T) \circ (id, \sigma)(y)|$$

$$\leq Lip(p^{uu} \circ (f-T) \circ (id, \sigma))\, |x-y|$$

$$\overset{(3)}{\leq} 2\varepsilon |x-y|. \qquad (4)$$

Then $$|\varphi(x) - \varphi(y)| \geq |\underbrace{T^{uu}(x)}_{=\lambda^{uu}x} - \underbrace{T^{uu}(y)}_{=\lambda^{uu}y}| - 2\varepsilon|x-y|$$

$$= (\lambda^{uu} - 2\varepsilon)\, |x-y|. \qquad (5)$$

This shows that φ is injective. Because $\dim E^{uu} = 1$, we can think of φ as a function from $\mathbb{R}$ to $\mathbb{R}$. It is then continuous and injective. Thus, it is strictly monotone. If we can show that φ is surjective then its inverse is continuous. Take $y \in E^{uu}$. The equation $\varphi(x) = y$ is equivalent to

$$\tilde{x} = y - (\varphi - T^{uu})(T^{uu})^{-1}\tilde{x}$$

where $\tilde{x} = T^{uu}(x)$. Define a map $\Psi : E^{uu} \to E^{uu}$,

$$\Psi(\tilde{x}) = y - (\varphi - T^{uu})(T^{uu})^{-1}\tilde{x}.$$

Then Ψ is continuous.

$$|\Psi(\tilde{x}_1) - \Psi(\tilde{x}_2)| = |(\varphi - T^{uu})(T^{uu})^{-1}\tilde{x}_1 - (\varphi - T^{uu})(T^{uu})^{-1}\tilde{x}_2|$$

$$\leq \mathrm{Lip}(\varphi - T^{uu})\, \underbrace{|(T^{uu})^{-1}\tilde{x}_1 - (T^{uu})^{-1}\tilde{x}_2|}_{=(\lambda^{uu})^{-1}|\tilde{x}_1 - \tilde{x}_2|}, \quad \forall \tilde{x}_1, \tilde{x}_2 \in E^{uu}.$$

A consequence of (4) is that $\mathrm{Lip}(\varphi - T^{uu}) \leq 2\varepsilon$. Then

$$|\Psi(\tilde{x}_1) - \Psi(\tilde{x}_2)| \leq \underbrace{\frac{2\varepsilon}{\lambda^{uu}}}_{<1} |\tilde{x}_1 - \tilde{x}_2| \qquad \forall \tilde{x}_1, \tilde{x}_2 \in E^{uu}.$$

Thus, Ψ is a contraction. Since E^{uu} is a Banach space, Ψ has a fixed point $\tilde{x} \in E^{uu}$. We have showed that φ is surjective. Then $\Gamma_f(\sigma): E^{uu} \to E^{u}$ is a continuous map.

$$\varphi(0) = (f^{uu} \circ (id, \sigma))(0) = f^{uu}(0, \sigma(0)) = f^{uu}(0) = p^{uu} \circ f(0) = 0.$$

Thus,

$$\Gamma_f(\sigma)(0) = f^{u} \circ (id, \sigma) \circ \varphi^{-1}(0) = f^{u} \circ (id, \sigma)(0) = f^{u}(0, 0) = p^{u} \circ f(0) = 0.$$

Also,

$$\mathrm{Lip}(\Gamma_f(\sigma)) = \mathrm{Lip}(f^{u} \circ (id, \sigma) \circ \varphi^{-1})$$

$$\leq \mathrm{Lip}(p^{u} \circ (f - T) \circ (id, \sigma) \circ \varphi^{-1}) + \mathrm{Lip}(\underbrace{p^{u} \circ T \circ (id, \sigma) \circ \varphi^{-1}}_{=\lambda^{u} \sigma \circ \varphi^{-1}})$$

$$\leq \underbrace{\mathrm{Lip}(p^{u})}_{=1}\, \underbrace{\mathrm{Lip}(f - T)}_{<\varepsilon}\, \underbrace{\mathrm{Lip}(id, \sigma)}_{\leq 1 + \mathrm{Lip}(\sigma)}\, \mathrm{Lip}(\varphi^{-1}) + \lambda^{u}\, \underbrace{\mathrm{Lip}(\sigma \circ \varphi^{-1})}_{\leq \mathrm{Lip}(\sigma)\, \mathrm{Lip}(\varphi^{-1})}$$

$$\leq \varepsilon(1 + \underbrace{\mathrm{Lip}(\sigma)}_{\leq 1})\, \mathrm{Lip}(\varphi^{-1}) + \lambda^{u}\, \underbrace{\mathrm{Lip}(\sigma)}_{\leq 1}\, \mathrm{Lip}(\varphi^{-1})$$

$$\leq (2\varepsilon + \lambda^u)\, Lip(\varphi^{-1}).$$

By (5), $Lip(\varphi^{-1}) \leq (\lambda^{uu} - 2\varepsilon)^{-1}$. Thus,

$$Lip(\Gamma_f(\sigma)) \leq (2\varepsilon + \lambda^u)(\lambda^{uu} - 2\varepsilon)^{-1} = \frac{\lambda^u + 2\varepsilon}{\lambda^{uu} - 2\varepsilon} < 1.$$

Then $\Gamma_f(\sigma) \in B$. We have showed that $\Gamma_f : B \to B$ is well-defined.

Next, we show that Γ_f is a contraction. For $\sigma_1, \sigma_2 \in B$, $x \in E^{uu} \setminus \{0\}$,

$$|\Gamma_f(\sigma_1)(x) - \Gamma_f(\sigma_2)(x)| = |f^s \circ (id, \sigma_1) \circ \underbrace{[f^u \circ (id, \sigma_1)]^{-1}(x)}_{y_1} - f^s \circ (id, \sigma_2) \circ \underbrace{[f^u \circ (id, \sigma_2)]^{-1}(x)}_{y_2}|$$

$$= |f^s(y_1, \sigma_1(y_1)) - f^s(y_2, \sigma_2(y_2))|$$

$$\leq \underbrace{|p^s \circ T(y_1, \sigma_1(y_1)) - p^s \circ T(y_2, \sigma_2(y_2))|}_{= \lambda^u |\sigma_1(y_1) - \sigma_2(y_2)|} + \underbrace{|p^s \circ (f - T)(y_1, \sigma_1(y_1)) - p^s \circ (f - T)(y_2, \sigma_2(y_2))|}_{\{1\}} \tag{6}$$

We have

$$\{1\} \leq Lip(p^s \circ (f - T))\, |(y_1, \sigma_1(y_1)) - (y_2, \sigma_2(y_2))|$$

$$\leq \underbrace{Lip(p^s)}_{=1} \underbrace{Lip(f - T)}_{<\varepsilon} (|y_1 - y_2| + |\sigma_1(y_1) - \sigma_2(y_2)|)$$

$$\leq \varepsilon (|y_1 - y_2| + |\sigma_1(y_1) - \sigma_2(y_2)|).$$

Then (6) implies

$$|\Gamma_f(\sigma_1)(x) - \Gamma_f(\sigma_2)(x)| \leq \lambda^u |\sigma_1(y_1) - \sigma_2(y_2)| + \varepsilon (|y_1 - y_2| + |\sigma_1(y_1) - \sigma_2(y_2)|)$$

$$= (\lambda^u + \varepsilon)|\sigma_1(y_1) - \sigma_2(y_2)| + \varepsilon |y_1 - y_2|$$

$$\leq (\lambda^u + \varepsilon) \underbrace{|\sigma_1(y_1) - \sigma_1(y_2)|}_{\leq Lip(\sigma_1)|y_1 - y_2|} + (\lambda^u + \varepsilon) \underbrace{|\sigma_1(y_2) - \sigma_2(y_2)|}_{\leq \|\sigma_1 - \sigma_2\|\, |y_2|} + \varepsilon |y_1 - y_2|$$

$$\leq (\lambda^u + 2\varepsilon)\,|y_1 - y_2| + (\lambda^u + \varepsilon)\,|y_2|\,\|\sigma_1 - \sigma_2\|. \qquad (7)$$

We have

$$0 = x - x = f^{uu}(y_1, \sigma_1(y_1)) - f^{uu}(y_2, \sigma_2(y_2))$$

$$= \underbrace{p^{uu}\circ T(y_1, \sigma_1(y_1))}_{=\lambda^{uu} y_1} - \underbrace{p^{uu}\circ T(y_2, \sigma_2(y_2))}_{=\lambda^{uu} y_2} + p^{uu}\circ(f-T)(y_1, \sigma_1(y_1)) - p^{uu}\circ(f-T)(y_2, \sigma_2(y_2)).$$

Then

$$\lambda^{uu}|y_1 - y_2| = |p^{uu}\circ(f-T)(y_1, \sigma_1(y_1)) - p^{uu}\circ(f-T)(y_2, \sigma_2(y_2))|$$

$$\leq \mathrm{Lip}(p^{uu}\circ(f-T))\,|(y_1, \sigma_1(y_1)) - (y_2, \sigma_2(y_2))|$$

$$\leq \underbrace{\mathrm{Lip}(p^{uu})}_{=1}\,\underbrace{\mathrm{Lip}(f-T)}_{<\varepsilon}\,(|y_1 - y_2| + |\sigma_1(y_1) - \sigma_2(y_2)|)$$

$$\leq \varepsilon\,|y_1 - y_2| + \varepsilon\,|\sigma_1(y_1) - \sigma_2(y_2)|$$

$$\leq \varepsilon|y_1 - y_2| + \varepsilon\,\underbrace{|\sigma_1(y_1) - \sigma_1(y_2)|}_{\leq \mathrm{Lip}(\sigma_1)|y_1 - y_2|} + \varepsilon\,\underbrace{|\sigma_1(y_2) - \sigma_2(y_2)|}_{\leq \|\sigma_1 - \sigma_2\|\,|y_2|} \qquad (8)$$

$$\leq 2\varepsilon\,|y_1 - y_2| + \varepsilon\,\|\sigma_1 - \sigma_2\|\,|y_2|.$$

Thus, $$|y_1 - y_2| \leq \frac{\varepsilon|y_2|}{\lambda^{uu} - 2\varepsilon}\,\|\sigma_1 - \sigma_2\|.$$

Then (7) implies

$$|\Gamma_f(\sigma_1)(x) - \Gamma_f(\sigma_2)(x)| \leq \left(\frac{\lambda^u + 2\varepsilon}{\lambda^{uu} - 2\varepsilon}\,\varepsilon + (\lambda^u + \varepsilon)\right)|y_2|\,\|\sigma_1 - \sigma_2\|. \qquad (9)$$

We have

$$|x| = |f^{uu}(y_2, \sigma_2(y_2))| \geq \underbrace{|p^{uu}\circ T(y_2, \sigma_2(y_2))}_{=\lambda^{uu} y_2}| - |p^{uu}\circ(f-T)(y_2, \sigma_2(y_2))|$$

$$= \lambda^{uu}|y_2| - |p^{uu}\circ(f-T)(y_2,\sigma_2(y_2)) - p^{uu}\circ(f-T)(0,0)|$$

$$\geq \lambda^{uu}|y_2| - \mathrm{Lip}(p^{uu}\circ(f-T))\,|(y_2,\sigma_2(y_2))|$$

$$\geq \lambda^{uu}|y_2| - \underbrace{\mathrm{Lip}(p^{uu})}_{=1}\ \underbrace{\mathrm{Lip}(f-T)}_{<\varepsilon}\ (|y_2| + \underbrace{|\sigma_2(y_2)|}_{\leq \mathrm{Lip}(\sigma_2)|y_2|})$$

$$\geq \lambda^{uu}|y_2| - 2\varepsilon|y_2|$$

$$= (\lambda^{uu} - 2\varepsilon)|y_2|.$$

Thus, $|y_2| \leq \dfrac{|x|}{\lambda^{uu}-2\varepsilon}$.

Dividing both sides of (9) by $|x|$, we get

$$\frac{|\Gamma_f(\sigma_1)(x) - \Gamma_f(\sigma_2)(x)|}{|x|} \leq \Big(\underbrace{\frac{\lambda^u+2\varepsilon}{\lambda^{uu}-2\varepsilon}}_{<1}\varepsilon + \lambda^u + \varepsilon\Big)\frac{1}{\lambda^{uu}-2\varepsilon}\|\sigma_1-\sigma_2\|$$

$$\leq (\lambda^u+2\varepsilon)\frac{1}{\lambda^{uu}-2\varepsilon}\|\sigma_1-\sigma_2\| \quad \forall x\in E^{uu}\setminus\{0\}.$$

Thus, $$\|\Gamma_f(\sigma_1) - \Gamma_f(\sigma_2)\| \leq \underbrace{\frac{\lambda^u+2\varepsilon}{\lambda^{uu}-2\varepsilon}}_{<1}\|\sigma_1-\sigma_2\| \quad \forall \sigma_1,\sigma_2\in B.$$

We have showed that $\Gamma_f : B\to B$ is a contraction.

If we used the norm $\|\sigma\|_\infty = \sup_{x\in E^{uu}}|\sigma(x)|$ instead of the norm $\|\sigma\| = \sup_{x\in E^{uu}\setminus\{0\}}\dfrac{|\sigma(x)|}{|x|}$ for graph σ, the map Γ_T (taking $f=T$) would not be a contraction. Indeed, we have

$$\Gamma_T(\sigma)(x) = T^u \circ (id, \sigma) \circ \underbrace{(T^{uu} \circ (id, \sigma))^{-1}(x)}_{y}$$

Because $x = T^{uu} \circ (id, \sigma)(y) = T^{uu}(y, \sigma(y)) = \lambda^{uu} y$,

$$\Gamma_T(\sigma)(x) = T^u \circ (id, \sigma)\left(\frac{1}{\lambda^{uu}} x\right) = T^u\left(\frac{1}{\lambda^{uu}} x, \sigma\left(\frac{1}{\lambda^{uu}} x\right)\right) = \lambda^u \sigma\left(\frac{1}{\lambda^{uu}} x\right).$$

Then

$$|\Gamma_T(\sigma_1)(x) - \Gamma_T(\sigma_2)(x)| = \lambda^u \left| \sigma_1\left(\frac{1}{\lambda^{uu}} x\right) - \sigma_2\left(\frac{1}{\lambda^{uu}} x\right)\right|$$

Taking the supremum over $x \in E^{uu}$, we get

$$\| \Gamma_T(\sigma_1) - \Gamma_T(\sigma_2)\|_\infty = \underbrace{\lambda^u}_{>1} \|\sigma_1 - \sigma_2\|_\infty .$$

Thus, Γ_T is not a contraction.

1) Consider the map $T : \mathbb{R}^2 \to \mathbb{R}^2$,

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} & \text{if } x \leq 1, \quad (1) \\[2ex] \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}\begin{pmatrix} x-3 \\ y+1 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} & \text{if } x > 1. \quad (2) \end{cases}$$

(i) We show that $p_- = (0,0)$ and $p_+ = (3,-1)$ are hyperbolic fixed points of T.

$$Tp_- = T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = p_-,$$

$$Tp_+ = T\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} = p_+.$$

Thus, p_- and p_+ are fixed points of T.

Put $\quad T_1 = DT(p_-) = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

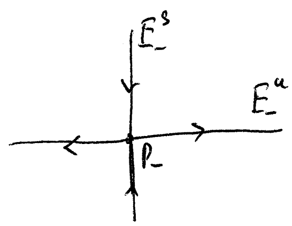
T_1 has two eigenvalues 2 and 1/2. Two respective eigenvectors are $(1,0)$ and $(0,1)$. The stable manifold is

$$E_-^s = \text{linear span } \{(0,1)\} = \{(0,b) : b \in \mathbb{R}\}.$$

The unstable manifold is

$$E_-^u = \text{linear span } \{(1,0)\} = \{(a,0) : a \in \mathbb{R}\}.$$

Both E_-^s and E_-^u are invariant under T_1.



For $p \in E_-^s$ close to p_-, the point $T^n p = \underbrace{T \circ \cdots \circ T}_{n}(p)$ gets closer to p_- as n increases. Thus, we use formula (1) to compute $T^n|_{E_-^s}$. A point on E_-^s is of the form $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}$. Then

$$T^n \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}^n \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2^n} y \end{pmatrix} = \frac{1}{2^n} \begin{pmatrix} 0 \\ y \end{pmatrix}. \qquad (3)$$

Thus, $\quad \| DT^n|_{E_-^s} \| = \left(\frac{1}{2}\right)^n$.

For each $p \in E_-^u$ close to p_-, the point $T^{-n} p = T^{-1} \circ T^{-1} \circ \cdots \circ T^{-1}(p)$ gets closer to p_- as n increases. Thus, we use formula (1) to compute $T^{-n}|_{E_-^u}$. A point on E_-^u is of the form $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$. Then

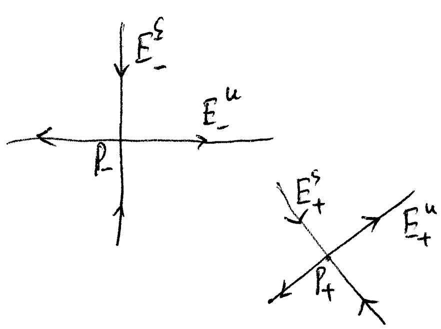
$$T^{-n} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}^{-n} \begin{pmatrix} x \\ 0 \end{pmatrix} = \frac{1}{2^n} \begin{pmatrix} x \\ 0 \end{pmatrix}. \qquad (4)$$

Thus, $\| DT^{-n}|_{E_-^u} \| = \left(\frac{1}{2}\right)^n$.

Therefore, p_- is a hyperbolic fixed point of T.

Put $T_2 = DT(p_+) = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$. It has eigenvalues $\frac{3}{2}$ and $\frac{1}{2}$.

Two respective eigenvectors are $(1,1)$ and $(1,-1)$. The stable manifold is



$$E_+^s = p_+ + \text{linear span } \{(1,-1)\}$$
$$= \{(b+3, -b-1) : b \in \mathbb{R}\}.$$

The unstable manifold is

$$E_+^u = p_+ + \text{linear span } \{(1,1)\}$$
$$= \{(a+3, a-1) : a \in \mathbb{R}\}.$$

Both E_+^s and E_+^u are invariant under DT.

For $p \in E_+^s$ close to p_+, $T^n p$ gets closer to p_+ as n increases. Thus, we use formula (2) to compute $T^n|_{E_+^s}$.

$$T^n \begin{pmatrix} b+3 \\ -b-1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2^n} b + 3 \\ -\frac{1}{2^n} b - 1 \end{pmatrix} = \frac{1}{2^n} \begin{pmatrix} b+3 \\ -b-1 \end{pmatrix} + \begin{pmatrix} 3 - \frac{3}{2^n} \\ -1 + \frac{1}{2^n} \end{pmatrix}. \qquad (5)$$

Then $\| DT^n|_{E_+^s} \| = \left(\frac{1}{2}\right)^n$.

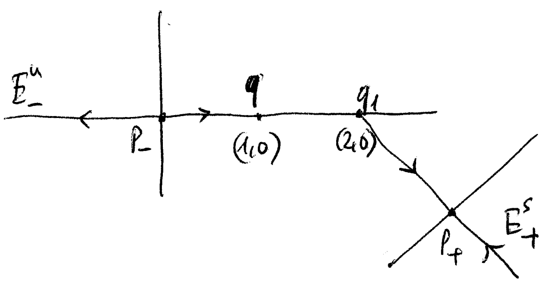
For $p \in E_+^u$ close to p_+, $T^{-n} p$ gets closer to p_+ as n increases. Thus, we use formula (2) to compute $T^{-n}|_{E_+^u}$.

$$T^{-n}\begin{pmatrix} a+3 \\ a-1 \end{pmatrix} = \begin{pmatrix} \left(\frac{2}{3}\right)^n a + 3 \\ \left(\frac{2}{3}\right)^n a - 1 \end{pmatrix} = \left(\frac{2}{3}\right)^n \begin{pmatrix} a+3 \\ a-1 \end{pmatrix} + \begin{pmatrix} 3 - 3\left(\frac{2}{3}\right)^n \\ -1 + \left(\frac{2}{3}\right)^n \end{pmatrix}. \quad (6)$$

Then
$$\left\| DT^{-n}|_{E_+^u} \right\| = \left(\frac{2}{3}\right)^n.$$

Therefore, p_+ is a hyperbolic fixed point of T.

(ii) We show that $q = (1,0)$ is a heteroclinic point. That is to show
$$\lim_{n\to\infty} T^n(q) = p_+, \quad \lim_{n\to\infty} T^{-n}(q) = p_-.$$



To compute $T^{-n}(q) = T^{-n}\begin{pmatrix}1\\0\end{pmatrix}$, we use formula (4):
$$T^{-n}(q) = T^{-n}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2^n}\begin{pmatrix}1\\0\end{pmatrix}.$$
Then $\lim_{n\to\infty} T^{-n}(q) = \begin{pmatrix}0\\0\end{pmatrix} = p_-$.

Put $q_1 = T(q) = T\begin{pmatrix}1\\0\end{pmatrix} \overset{(1)}{=\!=} \begin{pmatrix}2\\0\end{pmatrix} \in E_+^s$.

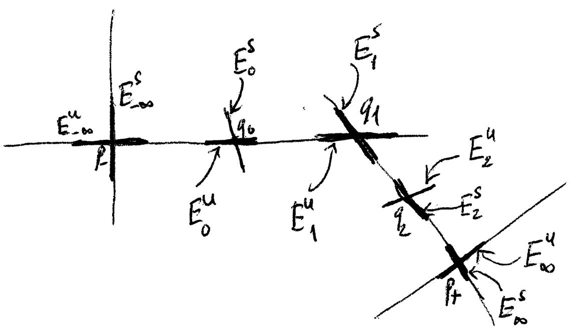
To compute $T^n(q_1)$, we use formula (5) with $b=-1$. Then
$$T^{n+1}(q) = T^n(q_1) = T^n\begin{pmatrix}2\\0\end{pmatrix} = \frac{1}{2^n}\begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix} 3 - \frac{3}{2^n} \\ -1 + \frac{1}{2^n} \end{pmatrix}$$
$$\longrightarrow \begin{pmatrix}3\\-1\end{pmatrix} = p_+ \quad \text{as } n\to\infty.$$

Therefore, q is a heteroclinic point.

(iii) For $q=(1,0)$, put $\Lambda = \overline{\bigcup_{n\in\mathbb{Z}} T^n(q)}$. We show that Λ is a compact hyperbolic set (for T). For each $n \in \mathbb{Z}$, denote $q_n = T^n(q)$. We computed in Part (ii) that

$$q_n = \begin{cases} \begin{pmatrix} \frac{1}{2^n} \\ 0 \end{pmatrix} & \text{for } n \leq 0, \\ \begin{pmatrix} 3 - \frac{1}{2^{n-1}} \\ -1 + \frac{1}{2^{n-1}} \end{pmatrix} & \text{for } n \geq 1. \end{cases}$$

Then $\Lambda = \overline{\{q_n : n \in \mathbb{Z}\}} = \{p_-, p_+\} \cup \{q_n : n \in \mathbb{Z}\}$ is a compact set. Denote



$E^s_1 = \text{linear span } \{(1,-1)\}$,

$E^u_1 = \text{linear span } \{(1,0)\}$.

Then $\mathbb{R}^2 = E^s_1 \oplus E^u_1$. Suppose E^s_k and E^u_k are defined for some $k \geq 1$. We then define $E^s_{k+1} = DT(q_k)E^s_k$, $E^u_{k+1} = DT(q_k)E^u_k$.

Suppose E^s_k and E^u_k are defined for some $k \leq 1$. We define

$$E^s_{k-1} = DT^{-1}(q_k)\, E^s_k, \qquad E^u_{k-1} = DT^{-1}(q_k)\, E^u_k,$$

with the convention that $DT^{-1}(q_0) = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$, i.e. using the formula (1) to compute the derivative of T at $q_0 = \binom{1}{0}$. By this convention and the definition of T, the derivative of T at any point is an invertible matrix.

If $E^s_k \oplus E^u_k = \mathbb{R}^2$ for some $k \geq 1$ then

$$E^s_{k+1} \cap E^u_{k+1} = DT(q_k)\, E^s_k \cap DT(q_k)\, E^u_k = \{0\},$$

$$E^s_{k+1} + E^u_{k+1} = DT(q_k)E^s_k + DT(q_k)E^u_k = \mathbb{R}^2.$$

If $E^s_k \oplus E^u_k = \mathbb{R}^2$ for some $k \leq 1$ then

$$E^s_{k-1} \cap E^u_{k-1} = DT^{-1}(q_k)E^s_k \cap DT^{-1}(q_k)E^u_k = \{0\},$$

$$E^s_{k-1} + E^u_{k-1} = DT^{-1}(q_k)E^s_k + DT^{-1}(q_k)E^u_k = \mathbb{R}^2.$$

Because $E^s_1 \oplus E^u_1 = \mathbb{R}^2$, we have $E^s_k \oplus E^u_k = \mathbb{R}^2$ for all $k \in \mathbb{Z}$. By the definition of E^s_k and E^u_k,

$$DT(q_k)E^s_k = E^s_{k+1}, \quad DT(q_k)E^u_k = E^u_{k+1} \qquad \forall k \in \mathbb{Z}.$$

Denote $E^s_{-\infty} =$ linear span $\{(0,1)\}$, $E^u_{-\infty} =$ linear span $\{(1,0)\}$,

$E^s_{\infty} =$ linear span $\{(1,-1)\}$, $E^u_{\infty} =$ linear span $\{(1,1)\}$.

Then

$$DT(p_-)E^s_{-\infty} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} E^s_{-\infty} = E^s_{-\infty},$$

$$DT(p_-)E^u_{-\infty} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} E^u_{-\infty} = E^u_{-\infty},$$

$$DT(p_+)E^s_{\infty} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} E^s_{\infty} = E^s_{\infty},$$

$$DT(p_+)E^u_{\infty} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} E^u_{\infty} = E^u_{\infty}.$$

We show that $E^u_k =$ linear span $\{(1,0)\}$ for all $k \leq 1$. This is true for $k=1$ and $k=0$. Suppose $E^u_k =$ linear span $\{(1,0)\}$ for some $k \leq 0$. We use formula (1) to compute T^{-1} in neighborhoods of q_k and q_0. Thus,

$DT^{-1}(q_k) = DT^{-1}(q_0) = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$. Then $E^u_{k-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} E^u_k = E^u_k =$ linear span $\{(1,0)\}$.