Math 8502, Fall 2015

Dynamical Systems and Differential Equations II

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— *Homework 2* —

(1) Determine equilibria and their stability in the unfoldings of the cusp:

(i) Study the unfolding of the cusp $x' = \mu_1 + \mu_2 x - x^3$: find the region in parameter space where there exist two stable equilibria!

(ii) Consider the more general $x' = a_0(\mu) + a_1(\mu)x + a_2(\mu)x^2 + a_3(\mu)x^3 + \mathrm{O}(x^4)$, with $a_j(0) = 0$, $j = 0, 1, 2$, $a_3(0) = -1$, $\mu = (\mu_1, \mu_2)$! Show first that after a change of coordinates we can assume $a_2 \equiv 0$.

(iii) Assuming that $D_\mu(a_0, a_1)^T$ is invertible, show that a local diffeomorphism in parameter space brings the equation into the form $\tilde{x}' = \tilde{\mu}_1 + \tilde{\mu}_2 x + \tilde{a}_3(\tilde{\mu})x^3 + \mathrm{O}(x^4)$, with $\tilde{a}_3(0) = -1$. Now find the region of bistability using Newton's polygon (after solving for $\tilde{\mu}_2 \ldots$).

(2) Find the bifurcation diagram (equilibria, stability, heteroclinics) for the following normal forms:

(i) transcritical: $x' = \mu x \pm x^2$;

(ii) degenerate pitchfork: $x' = \mu_1 x + \mu_2 x^3 - x^5$ (find the region with 5 equilibria!)

(3) Show that the fold is "generic"! Therefore, consider $f(x, \mu) : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$, smooth, and the associated map

$$F : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \times [0, 1] \to \mathbb{R}^n \times \mathbb{R}^n, \quad F(x, v, \mu) = (f(x, \mu), \partial_x f(x, \mu) \cdot v).$$

(i) Notice that zeros of F correspond to bifurcation points of f, that is points, where the IFT fails.

(ii) Assume that 0 is not a critical value of F ($\partial_{x,v,\mu} F$ is onto). Conclude that all turning points are "regular" fold points, with quadratic tangency, that is, $(e_0^*, \partial_\mu f) \neq 0$ and $(e_0^*, \partial_{xx} f[e_0, e_0]) \neq 0$.

(iii) Consider the perturbed map $\mathcal{F}(x, v, \mu, a, B)$, induced by $f(x, \mu) + a + Bx$. Show that $D\mathcal{F}$ is onto and therefore $\mathcal{F} \pitchfork \{0\}$.

(iv) Conclude that $f + a + Bx$ has only regular fold points for a dense set of a, B.

(4) Consider

$$
\begin{aligned}
x' &= y \\
y' &= -x + \mu y + (x+y)z \\
z' &= -z + x^2.
\end{aligned}
$$

Find the center manifold to quadratic order and compute the coefficients $c_1, c_3 \in \mathbb{C}$ in the cubic Hopf normal form $\zeta' = -\mathrm{i}\zeta + c_1\mu\zeta + c_3\zeta|\zeta|^2$ for $\zeta \sim x + \mathrm{i}y$. Derive the scaling $|\zeta| \sim \sqrt{10\mu}$ for the amplitude of the periodic orbit. *Optional:* Compare your findings with numerical computations.

Due February 27