Math 8502, Fall 2015

Dynamical Systems and Differential Equations II

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— *Homework 3* —

(1) Consider the rigid circle rotation $R_\alpha : x \mapsto x + \alpha \mod 1$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let $J_0 = [0, 1-\alpha)$ and $J_1 = [1-\alpha, 1)$.

(i) Construct the first return map $\Phi : J_1 \to J_1$, that is, $\Phi(x) = R_\alpha^{k(x)}(x)$ with $k(x) > 0$ minimal so that $\Phi(x) \in J_1$. Show that $k(x) \in \{\ell, \ell+1\}$, where $\ell = \left[\frac{1}{\alpha}\right]$, and $[z]$ denotes the integer part of z.

(ii) Show that $\Phi(1-\alpha) = \lim_{x\to 1} \Phi(x)$, so that Φ defines a continuous circle map on the "small" circle $[1-\alpha, 1]/(1-\alpha \sim 1)$.

(iii) Let $\Psi : [1-\alpha, 1) \to (0, 1], x \mapsto y = 1 - \frac{x-(1-\alpha)}{\alpha}$ be the unique affine, orientation reversing map from the "small" circle to the standard circle. Show that $\Psi \circ \Phi \circ \Psi^{-1}$ is a rigid rotation R_β of S^1 and compute β.

(iv) One can clearly iterate this procedure. Interprete the resulting return times ℓ as demoniators in the continued fraction expansion of α.

(v) Consider an (arbitrary) orbit $x_j = x + j \cdot \alpha \mod 1$. Define a coding sequence $a_j \in \{0, 1\}$ so that $a_j = k$ if $x_j \in J_k$. Show that a_j consists of ℓ or $\ell - 1$ 0's followed by precisely one 1. What is ℓ?

(vi) Define a renomalized sequence b_j, as replacing a block of ℓ 0's and a following 1 by a 0, a block of $\ell - 1$ 0's following a 1 by 1. Show that the renormalized sequence is of the same type as the first sequence. Explain the relation to the precious considerations of renormalized return maps.

(2) Show that the rotation number is a continuous function of a circle homeomorphism.

(3) Let Φ_1 be the time-one map of the flow to $\dot{x} = f(x)$, $f : \mathbb{R} \to \mathbb{R}$ a C^1-vector field with $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Show that Φ_1 induces a circle homeomorphism ϕ_1 and compute the rotation number (recall that you can solve the ODE by separation of variables). Conclude that the rotation number depends smoothly on the vector field when $f \neq 0$. Compare with the case of general families of homeomorphisms. *Optional:* Computer generic asymptotics for the rotation number near $\rho \gtrsim 0$.

Due March 25