Math 8502, Fall 2015

Dynamical Systems and Differential Equations II

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— *Homework 4* —

(1) Consider the piecewise linear map T on $(x, y) \in \mathbb{R}^2$, defined as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

for $x \leq 1$, and

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

for $\xi = x - 3 > -2$, and $\eta = y + 1$.

(i) Show that $p_- = (0, 0)$ and $p_+ = (3, -1)$ are hyperbolic fixed points and draw a phase portrait with stable and unstable manifolds.

(ii) Show that $q = (1, 0)$ is a heteroclinic point, that is, $T^n(q) \to p_\pm$ for $n \to \pm\infty$.

(iii) Show that $\Lambda := \overline{\bigcup_{n\in\mathbb{Z}} T^n(q)}$ is a compact hyperbolic set.

(iv) Analyze the differentiability of the stable subspace at accumulation points of Λ.

(2) Consider $T = \mathrm{diag}\,(\lambda^{uu}, \lambda^u)$, with $1 < \lambda^u < \lambda^{uu}$ and the graph transform associated with the construction of a strong unstable manifold. Show that the graph transform defines a contraction on the set of continuous graphs σ with $\sigma(0) = 0$ and $\|\sigma\| = \sup_{x\neq 0} |\sigma(x)/x| < \infty$, when using the distance $d(\sigma_1, \sigma_2) = \|\sigma_1 - \sigma_2\|$. Also, explain why the distance induced by the supremum norm will not give a contraction.

(3) Find a horseshoe in the cat map (draw the picture)!

Due April 22