Existence of Fredholm operators between two Banach spaces

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1 Remark 1

A Fredholm operator $T : X \to Y$ is nearly an isomorphism (in the category of linear continuous maps). The existence of a Fredholm operator between X and Y demands these spaces to have certain similarities. For example, if one of them is infinite dimensional, so must be the other. Other similarities include separability and reflexivity.

Let X and Y be Banach spaces and $T : X \to Y$ be a Fredholm operator. We have the following statements.

(i) X is separable $\Leftrightarrow$ Y is separable.

(ii) X is reflexive $\Leftrightarrow$ Y is reflexive.

Let Ω be a nonempty open subset of $\mathbb{R}^n$. A consequence of (i) is that there is no Fredholm operator between $L^1(\Omega)$ and $L^p(\Omega)$, $1 < p < \infty$. A consequence of (ii) is that there is no Fredholm operator between $L^1(\Omega)$ and $L^\infty(\Omega)$.

Proof. Put $X_0 = \ker T$ and $Y_1 = T(X)$. Then $\dim X_0 < \infty$, $\operatorname{codim} Y_1 < \infty$, and Y_1 is closed in Y. Since X_0 is finite dimensional, it has an algebraic topological complement X_1. Then $T|_{X_1} : X_1 \to Y_1$ is an isomorphism (in the category of linear continuous maps). Since Y_1 is closed and has finite codimension in Y, it has an algebraic topological complement Y_0. Then $\dim Y_0 < \infty$ and Y_0 is closed in Y. We have

$$\begin{aligned} X &= X_0 \oplus X_1, \\ Y &= Y_0 \oplus Y_1. \end{aligned}$$

(i)

($\Rightarrow$) Suppose X is separable. Then X_1 is also separable. Then $Y_1 = T(X)$ is also separable. Since Y_0 is finite dimensional, it is separable. Let S_0 be be countable dense subset of Y_0, and S_1 be be countable dense subset of Y_1. Then the set

$$S = \{a + b : a \in S_0, b \in S_1\}$$

is also countable. Each $y \in Y$ can be written as $y = y_0 + y_1$ with $y_0 \in Y_0$ and $y_1 \in Y_1$. There are a sequence (a_n) in S_0 converging to y_0, and a sequence (b_n) in S_1 converging to y_1. Then $(a_n + b_n)$ is a sequence in S converging to $y_0 + y_1 = y$. Thus, S is dense in Y. We have showed that Y is separable.

($\Leftarrow$) Suppose Y is separable. Then Y_1 is also separable. Then $X_1 = (T|_{X_1})^{-1}(X)$ is also separable. Since X_0 is finite dimensional, it is separable. By the same arguments as in the previous part, $X = X_0 + X_1$ is separable.

(ii)

($\Rightarrow$) Suppose X is reflexive. Because X_1 is a closed subspace of X, it is also reflexive. Because $T|_{X_1} : X_1 \to Y_1$ is an isomorphism (in the category of linear continuous maps), Y_1 is reflexive. Since Y_0 is finite dimensional, it is reflexive. We now show that $Y = Y_0 \oplus Y_1$ is also reflexive. In the following, we denote by $\langle ., . \rangle$ the duality between a space and its dual. Put

$$Y_0^{\perp} = \{f \in Y^* : f|_{Y_0} = 0\},$$
$$Y_1^{\perp} = \{g \in Y^* : g|_{Y_1} = 0\}.$$

Then the maps $L_0 : Y_1^{\perp} \to Y_0^*$, $L_0 f = f|_{Y_0}$ and $L_1 : Y_0^{\perp} \to Y_1^*$, $L_1 g = g|_{Y_1}$ are isomorphisms (in the category of linear continuous maps). Let $y^{**} \in Y^{**}$. We determine $y \in Y$ such that $\langle y^{**}, y^* \rangle = \langle y^*, y \rangle$ for every $y^* \in Y^*$.

$$Y_0^* \xrightarrow{L_0^{-1}} Y_1^{\perp} \subset Y^* \xrightarrow{y^{**}} \mathbb{R},$$
$$Y_1^* \xrightarrow{L_1^{-1}} Y_0^{\perp} \subset Y^* \xrightarrow{y^{**}} \mathbb{R}.$$

Because $y^{**} L_0^{-1} \in Y_0^{**}$ and Y_0 is reflexive, there exists $y_0 \in Y_0$ such that

$$\left\langle y^{**} L_0^{-1}, u \right\rangle = \langle u, y_0 \rangle \quad \forall u \in Y_0^*. \tag{1.1}$$

Similarly, there exists $y_1 \in Y_1$ such that

$$\left\langle y^{**} L_1^{-1}, v \right\rangle = \langle v, y_1 \rangle \quad \forall v \in Y_1^*. \tag{1.2}$$

We show that $y = y_0 + y_1$ satisfies our demand. Let $\pi_0 : Y \to Y_0$ and $\pi_1 : Y \to Y_1$ be the projection maps. Because Y_0 is finite dimensional, π_0 is continuous. Then $\pi_1 = \mathrm{id}_Y - \pi_0$ is also continuous. Let $y^* \in Y^*$. Then $y^* \pi_0 \in Y_1^{\perp}$ and $y^* \pi_1 \in Y_0^{\perp}$. Replacing u in (1.1) by $L_0(y^* \pi_0)$, we get

$$\left\langle y^{**} L_0^{-1}, L_0(y^* \pi_0) \right\rangle = \langle L_0(y^* \pi_0), y \rangle.$$

In other words,

$$\langle y^{**}, y^* \pi_0 \rangle = \langle y^* \pi_0, y_0 \rangle. \tag{1.3}$$

Similarly, replacing v in (1.2) by $L_1(y^*\pi_1)$, we get

$$\langle y^{**}, y^*\pi_1\rangle = \langle y^*\pi_1, y_1\rangle . \tag{1.4}$$

Summing (1.3) and (1.4) together, we get

$$\langle y^{**}, y^*\pi_0 + y^*\pi_1\rangle = \langle y^*\pi_0, y_0\rangle + \langle y^*\pi_1, y_1\rangle = \langle y^*, y_0\rangle + \langle y^*, y_1\rangle = \langle y^*, y\rangle .$$

We have showed that Y is reflexive.

($\Leftarrow$) Suppose Y is reflexive. Because Y_1 is a closed subspace of Y, it is also reflexive. Because $T|_{X_1} : X_1 \to Y_1$ is an isomorphism (in the category of linear continuous maps), X_1 is reflexive. Since X_0 is finite dimensional, it is reflexive. By the same arguments as in the previous part, we conclude that $X = X_0 \oplus X_1$ is reflexive. □

Comment. A result on the reflexivity of normed spaces which is more general than what we have showed is found in [Meg98, p.105]. Corollary 1.11.20 states that:

> Suppose that $X_1, X_2, ..., X_n$ are normed spaces. Then $X = X_1 \oplus X_2 \oplus ... \oplus X_n$ is reflexive if and only if each X_j is reflexive.

2 Remark 2

We recall the separability and reflexivity of the Banach spaces c_0, l^1, l^p $(1 < p < \infty)$, l^∞, $L^1(\Omega)$, $L^p(\Omega)$ $(1 < p < \infty)$ and $L^\infty(\Omega)$. Here Ω is a nonempty open subset of $\mathbb{R}^n$.

	Separable	Reflexive	Dual space
l^p	YES	YES	$l^{p'}$
l^1	YES	NO	l^∞
c_0	YES	NO	l^1
l^∞	NO	NO	Strictly larger than l^1

	Separable	Reflexive	Dual space
$L^p(\Omega)$	YES	YES	$L^{p'}(\Omega)$
$L^1(\Omega)$	YES	NO	$L^\infty(\Omega)$
$L^\infty(\Omega)$	NO	NO	Strictly larger than $L^1(\Omega)$

Remark 1 gives two necessary conditions for the existence of a Fredholm operator from a Banach space X to a Banach space Y. Within each of the above charts, we see that no two spaces from different rows have a Fredholm operator between them, except for the pair l^1 and c_0. It turns out to be also the case by the following argument.

Suppose there is a Fredholm operator $T : l^1 \to c_0$ (or $T : c_0 \to l^1$). Then the dual map $T^* : c_0^* = l^1 \to (l^1)^* = l^\infty$ (or $T^* : l^\infty \to l^1$) is also a Fredholm operator. This is a contradiction because l^1 is separable while l^∞ is not.

For $r, s \in (1, \infty), r \neq s$, our concern is whether there are Fredholm operators between l^r and l^s, between $L^r(\Omega)$ and $L^s(\Omega)$. As showed below, the answer for both cases is no.

3 Remark 3

We show that there is no Fredholm operator from l^r to l^s, where $r, s \in (1, \infty), r \neq s$. A proposition on the "maximal extension" of Fredholm operators and some background on Schauder bases are needed.

Proposition 3.1. *Let X, Y be Banach spaces and $T : X \to Y$ be a Fredholm operator. We have the following statements.*

(i) If $\mathrm{ind}(T) < 0$ then X is isomorphic to a closed, finite codimensional subspace of Y.

(ii) If $\mathrm{ind}(T) = 0$ then X is isomorphic to Y.

(iii) If $\mathrm{ind}(T) > 0$ then Y is isomorphic to a closed, finite codimensional subspace of X.

Here the isomorphisms are understood in the category of topological vector spaces, i.e. bijective, linear, continuous, having continuous inverse.

Proof of Proposition 3.1. Put $X_0 = \mathrm{ker}T$ and $Y_1 = T(X)$. Then $\mathrm{dim}X_0 < \infty$, $\mathrm{codim}Y_1 < \infty$, and Y_1 is closed in Y. Since X_0 is finite dimensional, it has an algebraic topological complement X_1. Then $T|_{X_1} : X_1 \to Y_1$ is an isomorphism. Since Y_1 is closed and has finite codimension in Y, it has an algebraic topological complement Y_0. Then Y_0 is finite dimensional and closed in Y.

$$\begin{aligned} X &= X_0 \oplus X_1, \\ Y &= Y_0 \oplus Y_1. \end{aligned}$$

Put $n = \mathrm{dim}X_0$ and $m = \mathrm{dim}Y_0$. The index of T is $n - m$. Condider the following cases.

- $n < m$

If $n = 0$ then $X_1 = X$; then X is isomorphic to Y_1, which is a closed, finite codimensional subspace of Y.

Suppose $n \geq 1$. Then $X_1 \neq X$ and $Y_1 \neq Y$. By Lemma 3.4 below, there exist $u_1 \in X \backslash X_1$, $v_1 \in Y \backslash Y_1$ and an isomorphism $T_1 : X_1 \oplus \mathbb{R}u_1 \to Y_1 \oplus \mathbb{R}v_1$. Put $X_2 = X_1 + \mathbb{R}u_1$ and $Y_2 = Y_1 + \mathbb{R}v_1$. Then X_2 is closed in X because X_1 is closed in X. Similarly, Y_2 is closed in Y. If $n = 1$ then $X_2 = X$; then X is isomorphic to Y_2, which is a closed, finite codimensional subspace of Y.

Suppose $n \geq 2$. Then $X_2 \neq X$ and $Y_2 \neq Y$. By Lemma 3.4 below, there exist $u_2 \in X \backslash X_2$, $v_2 \in Y \backslash Y_2$ and an isomorphism $T_2 : X_2 \oplus \mathbb{R}u_2 \to Y_2 \oplus \mathbb{R}v_2$. Put $X_3 = X_2 + \mathbb{R}u_2$ and $Y_3 = Y_2 + \mathbb{R}v_2$. Then X_3 is closed in X because X_2 is closed in X. Similarly, Y_3 is closed in Y. If $n = 2$ then $X_3 = X$; then X is isomorphic to Y_3, which is a closed, finite codimensional subspace of Y.

Suppose $n \geq 3$. We continue the above procedure. The process must stop after n steps. The conclusion is that X is isomorphic to a closed, finite codimensional subspace of Y.

- $\underline{n = m}$

We apply the same procedure as for the case $n < m$. The process must stop after n steps. The conclusion is that X is isomorphic to Y.

- $\underline{n > m}$

We apply the same procedure as for the case $n < m$. The process must stop after n steps. The conclusion is that X is isomorphic to a closed, finite codimensional subspace of Y. □

Below are some concepts relating to Schauder bases [Meg98, Chapter 4].

Definition 3.2. Let X be a Banach space and (x_n) be a sequence in X.

(i) (x_n) is called a *Schauder basis* of X if for every $x \in X$, there exists a unique sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ in $\mathbb{R}$ such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$.

(ii) (x_n) is called a *basic sequence* if it is a Schauder basis of the closure of its linear span in X.

(iii) Suppose (x_n) and (y_n) are two basic sequences such that the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges if and only if the series $\sum_{n=1}^{\infty} \alpha_n y_n$ converges. Then (x_n) and (y_n) are said to be *equivalent*.

(iv) Suppose (x_n) is a Schauder basis of X. A sequence of nonzero elements (u_n) in X of the form $u_j = \sum_{n=p_j+1}^{p_{j+1}} \beta_n x_n$ with $\beta_1, \beta_2, \beta_3, \ldots \in \mathbb{R}$ and $1 \leq p_1 < p_2 < p_3 < \ldots$ is called a *block basis* of (x_n).

We observe that if (x_n) and y_n are equivalent basic sequences then $\tilde{X}$, the closure of the linear span of (x_n), and $\tilde{Y}$, the closure of the linear span of (y_n) are isomorphic. Indeed, consider the map $L : \tilde{X} \to \tilde{Y}$,

$$L\left(\sum_{k=1}^{\infty} \alpha_k x_k\right) = \sum_{k=1}^{\infty} \alpha_k y_k.$$

It is well-defined and linear. Because (x_n) is a basic sequence, $\ker L = \{0\}$. Because (y_n) is a basic sequence equivalent to X, $L(\tilde{X}) = \tilde{Y}$. The graph of L is

$$\Gamma(L) = \left\{(x, Lx) : x \in \tilde{X}\right\} = \left\{\left(\sum_{k=1}^{\infty} \alpha_k x_k, \sum_{k=1}^{\infty} \alpha_k y_k\right) : \sum_{k=1}^{\infty} \alpha_k x_k \text{ converges}\right\}.$$

This is a closed subset of $\tilde{X} \times \tilde{Y}$. Also, $\tilde{X}$ and $\tilde{Y}$ are Banach spaces. By the Closed Graph theorem, L is continuous. Then by the Open Mapping theorem, L is an isomorphism.

Here is an important result about Schauder bases that is needed for our problem. It is called the ***Bessaga-Pełczyňski Selection Principle*** [Meg98, p.396], [LT77, p.7].

> Let (x_n) be a Schauder basis of a Banach space X, and (y_n) be a sequence in X such that $y_n \rightharpoonup 0$ and $y_n \nrightarrow 0$. Then (y_n) has a subsequence (y_{n_k}) that is equivalent to a block basis of (x_n).

Now we have enough tools to prove the following result.

Proposition 3.3. Let $r, s \in (1, \infty), r \neq s$. Then there is no Fredholm operator from l^r to l^s.

Proof. Suppose by contradiction that there exists a Fredholm operator from l^r to l^s. By Proposition 3.1, either l^r is isomorphic to a closed subspace of l^s, or l^s is isomorphic to a closed subspace of l^r. By switching the roles of r and s if necessary, we can assume there is an isomorphism from l^r to a closed subspace Y of l^s. Denote it by $T : l^r \to Y \subset l^s$. Because T is an isomorphism, there exists a number $C > 0$ such that

$$C^{-1}\|x\|_r \leq \|Tx\|_s \leq C\|x\|_r \quad \forall x \in l^r.$$

For each $n \in \mathbb{N}$, let e_n be the sequence with value 1 at the n'th position and value 0 at other positions. Then (e_n) is a Schauder basis of l^r and l^s.

Put $v_n = Te_n \in l^s$. Then $v_n \nrightarrow 0$ because $C^{-1} \leq \|v_n\|_s \leq C$. We now show that $v_n \rightharpoonup 0$ in l^s. Denote by $\langle ., . \rangle$ the duality between a normed space and its dual. Let $f \in (l^s)^*$.

$$\langle f, v_n \rangle = \langle f, Te_n \rangle = \langle T^* f, e_n \rangle \quad \forall n \in \mathbb{N} \tag{3.1}$$

where $T^* : (l^s)^* \to (l^r)^*$ is the dual map of T. Let (e_m^*) be the sequence of coordinate functionals associate with (e_n), i.e.

$$\langle e_m^*, e_n \rangle = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Then

$$(l^r)^* = \left\{ \sum_{k=1}^{\infty} \alpha_k e_k^* : (\alpha_k) \in l^{r'} \right\}.$$

Because $T^* f \in (l^r)^*$, we can write $T^* f = \sum_{k=1}^{\infty} \alpha_k e_k^*$ for some $(\alpha_k) \in l^{r'}$. Then (3.1) becomes

$$\langle f, v_n \rangle = \left\langle \sum_{k=1}^{\infty} \alpha_k e_k^*, e_n \right\rangle = \alpha_n \quad \forall n \in \mathbb{N}.$$

This term converges to 0 as $n \to \infty$ because $\sum_{k=1}^{\infty} |\alpha_k|^{r'} < \infty$.

We have showed that (v_n) converges weakly to 0 in l^s. By Bessaga-Pełczyňski Selection Principle, (v_n) has a subsequence (v_{n_k}) that is equivalent to a block basis of (e_n) in l^s. Denote by

$$w_j = \sum_{n=p_j+1}^{p_{j+1}} a_n e_n$$

the block basis of (e_n). Because (v_{n_k}) and (w_k) are equivalent basic sequences, the closure of the linear span of (v_{n_k}) in l^s is isomorphic to the closure of the linear span of (w_k) in l^s.[†] Then there exists a number $C_1 > 0$ such that

$$C_1^{-1} \left\| \sum_{k=1}^{\infty} \alpha_k w_k \right\|_s \leq \left\| \sum_{k=1}^{\infty} \alpha_k v_{n_k} \right\|_s \leq C_1 \left\| \sum_{k=1}^{\infty} \alpha_k w_k \right\|_s \tag{3.2}$$

whenever the series $\sum \alpha_k v_{n_k}$ converges in l^s. In particular,

$$C_1^{-1} C^{-1} \leq C_1^{-1} \|v_{n_k}\|_s \leq \|w_k\|_s \leq C_1 \|v_{n_k}\|_s \leq C_1 C. \tag{3.3}$$

$$\begin{aligned}
\left\| \sum_{k=1}^{\infty} \alpha_k w_k \right\|_s &= \left\| \sum_{k=1}^{\infty} \alpha_k \sum_{n=p_k+1}^{p_{k+1}} a_n e_n \right\|_s = \left\| \sum_{n=p_1}^{\infty} \sum_{k: p_k < n \leq p_{k+1}} \alpha_k a_n e_k \right\|_s \\
&= \left(\sum_{n=p_1}^{\infty} \sum_{k: p_k < n \leq p_{k+1}} |\alpha_k|^s |a_n|^s \right)^{\frac{1}{s}} = \left(\sum_{k=1}^{\infty} \sum_{n=p_k+1}^{p_{k+1}} |\alpha_k|^s |a_n|^s \right)^{\frac{1}{s}} \\
&= \left(\sum_{k=1}^{\infty} |\alpha_k|^s \, \|w_k\|_s^s \right)^{\frac{1}{s}}.
\end{aligned}$$

Applying the estimates for $||w_k||_s$ given by (3.3), we get

$$C_1^{-1} C^{-1} \left(\sum_{k=1}^{\infty} |\alpha_k|^s \right)^{\frac{1}{s}} \leq \left\| \sum_{k=1}^{\infty} \alpha_k w_k \right\|_s \leq C_1 C \left(\sum_{k=1}^{\infty} |\alpha_k|^s \right)^{\frac{1}{s}}.$$

Substituting this estimate into (3.2), we get

$$C_1^{-2} C^{-1} \left(\sum_{k=1}^{\infty} |\alpha_k|^s \right)^{\frac{1}{s}} \leq \left\| \sum_{k=1}^{\infty} \alpha_k v_{n_k} \right\|_s \leq C_1^2 C \left(\sum_{k=1}^{\infty} |\alpha_k|^s \right)^{\frac{1}{s}} \tag{3.4}$$

whenever the series $\sum \alpha_k v_{n_k}$ converges in l^s. By the continuity of T,

$$\sum_{k=1}^{\infty} \alpha_k v_{n_k} = \sum_{k=1}^{\infty} \alpha_k T e_{n_k} = T \left(\sum_{k=1}^{\infty} \alpha_k e_{n_k} \right)$$

[†] See explanation on page 6.

whenever the series $\sum \alpha_k e_{n_k}$ converges in l^r. Taking the norm in l^s, we get

$$C^{-1}\left\|\sum_{k=1}^{\infty}\alpha_k e_{n_k}\right\|_r \leq \left\|\sum_{k=1}^{\infty}\alpha_k v_{n_k}\right\|_s \leq C\left\|\sum_{k=1}^{\infty}\alpha_k e_{n_k}\right\|_r$$

whenever the series $\sum \alpha_k e_{n_k}$ converges in l^r. In other words,

$$C^{-1}\left(\sum_{k=1}^{\infty}|\alpha_k|^r\right)^{\frac{1}{r}} \leq \left\|\sum_{k=1}^{\infty}\alpha_k v_{n_k}\right\|_s \leq C\left(\sum_{k=1}^{\infty}|\alpha_k|^r\right)^{\frac{1}{r}}$$

whenever $\sum_{k=1}^{\infty}|\alpha_k|^r < \infty$. Substituting this estimate into (3.4), we get

$$\left(\sum_{k=1}^{\infty}|\alpha_k|^s\right)^{\frac{1}{s}} \leq C^2C_1^2\left(\sum_{k=1}^{\infty}|\alpha_k|^r\right)^{\frac{1}{r}}, \tag{3.5}$$

$$\left(\sum_{k=1}^{\infty}|\alpha_k|^r\right)^{\frac{1}{r}} \leq C^2C_1^2\left(\sum_{k=1}^{\infty}|\alpha_k|^s\right)^{\frac{1}{s}} \tag{3.6}$$

whenever $\sum_{k=1}^{\infty}|\alpha_k|^r < \infty$.

Consider the case $r < s$. Take $\alpha_k = \frac{1}{k^\delta}$ for $\delta > \frac{1}{r}$. Then (3.6) becomes

$$\left(\sum_{k=1}^{\infty}\frac{1}{k^{r\delta}}\right)^{\frac{1}{r}} \leq C^2C_1^2\left(\sum_{k=1}^{\infty}\frac{1}{k^{s\delta}}\right)^{\frac{1}{s}} \quad \forall\delta > \frac{1}{r}. \tag{3.7}$$

By Fatou's lemma,

$$\liminf_{\delta\to\left(\frac{1}{r}\right)^+} \mathrm{LHS}(3.7) \geq \left(\sum_{k=1}^{\infty}\frac{1}{k}\right)^{\frac{1}{r}} = \infty.$$

By the Dominated Convergence theorem,

$$\lim_{\delta\to\left(\frac{1}{r}\right)^+} \mathrm{RHS}(3.7) = C^2C_1^2\left(\sum_{k=1}^{\infty}\frac{1}{k^{s/r}}\right)^{\frac{1}{s}} < \infty.$$

This is a contradiction.

Consider the case $r > s$. Take $\alpha_k = \frac{1}{k^\delta}$ for $\delta > \frac{1}{s}$. Then (3.5) becomes

$$\left(\sum_{k=1}^{\infty}\frac{1}{k^{s\delta}}\right)^{\frac{1}{s}} \leq C^2C_1^2\left(\sum_{k=1}^{\infty}\frac{1}{k^{r\delta}}\right)^{\frac{1}{r}} \quad \forall\delta > \frac{1}{s}. \tag{3.8}$$

Similar to the previous case,

$$\liminf_{\delta\to\left(\frac{1}{r}\right)^+} \mathrm{LHS}(3.8) \geq \left(\sum_{k=1}^{\infty}\frac{1}{k}\right)^{\frac{1}{s}} = \infty,$$

$$\lim_{\delta\to\left(\frac{1}{r}\right)^+} \text{RHS}(3.8) = C^2C_1^2\left(\sum_{k=1}^{\infty}\frac{1}{k^{r/s}}\right)^{\frac{1}{r}} < \infty.$$

This is a contradiction. □

Lemma 3.4. *Let X and Y be Banach spaces. Let X_1 (respectively Y_1) be a closed proper subspace of X (respectively Y). Suppose there is an isomorphism $T : X_1 \to Y_1$. Then there exist $u_1 \in X\backslash X_1$, $v_1 \in Y\backslash Y_1$ and an isomorphism $\tilde{T} : X_1 \oplus \mathbb{R}u_1 \to Y_1 \oplus \mathbb{R}v_1$ such that $\tilde{T}|_{X_1} = T$.*

Proof. If $T = 0$ then $X_1 = Y_1 = \{0\}$; take $u_1 \in X\backslash\{0\}$ and $v_1 \in Y\backslash\{0\}$ arbitrarily; the map $\tilde{T} : \mathbb{R}u_1 \to \mathbb{R}v_1$ is an isomorphism.

Consider the case $T \neq 0$. By replacing T with $T/||T||$, we can assume $||T|| = 1$. Because X_1 is a closed proper subspace of X, by Riesz's lemma [Meg98, p.325] there exists $u_1 \in X\backslash X_1$ such that $||u_1|| = 1$ and $\text{dist}(u_1, x) \geq \frac{1}{2}$ for every $x \in X_1$. Then

$$||x + u_1|| \geq \frac{1}{2} \quad \forall x \in X_1.$$

Take any $v_1 \in Y\backslash Y_1$, $||v_1|| = 1$. For each $x \in X_1$,

$$||x + u_1|| \geq ||x|| - ||u_1|| = ||x|| - 1 \geq ||Tx|| - 1.$$

Then

$$4||x + u_1|| \geq 3||x + u_1|| + ||x + u_1|| \geq \frac{3}{2} + (||Tx|| - 1) = ||Tx|| + \frac{1}{2} \geq \left\|Tx + \frac{1}{2}v_1\right\|. \tag{3.9}$$

Define a map $\tilde{T} : X_1 \oplus \mathbb{R}u_1 \to Y_1 \oplus \mathbb{R}v_1$,

$$\tilde{T}(x + cu_1) = Tx + \frac{c}{2}v_1 \quad \forall x \in X_1, \forall c \in \mathbb{R}.$$

Then $\tilde{T}$ is linear and bijective. We show that $||\tilde{T}\tilde{x}|| \leq 4||\tilde{x}||$ for all $\tilde{x} \in X_1 \oplus \mathbb{R}u_1$. Take $\tilde{x} \in X_1 \oplus \mathbb{R}u_1$. If $\tilde{x} \in X_1$ then $||\tilde{T}\tilde{x}|| = ||T\tilde{x}|| \leq ||\tilde{x}|| \leq 4||\tilde{x}||$.

Consider the case $\tilde{x} \notin X_1$. Then $\tilde{x} = x + cu_1$ for some $x \in X_1$, $c \in \mathbb{R}\backslash\{0\}$. Put $y = c^{-1}x \in X_1$. Then $\tilde{x} = c(y + u_1)$.

$$||\tilde{T}\tilde{x}|| = |c| \left\|\tilde{T}(y + u_1)\right\| = |c| \left\|Ty + \frac{1}{2}v_1\right\| \overset{(3.9)}{\leq} |c|4\,\|y + u_1\| = 4\,\|c(y + u_1)\| = 4||\tilde{x}||.$$

We have showed that $\tilde{T}$ is continuous. Put $X_2 = X_1 \oplus \mathbb{R}u_1$ and $Y_2 = Y_1 \oplus \mathbb{R}v_1$. Because X_1 is closed in X, X_2 is also closed in X. Thus, X_2 is a Banach space. Similarly, Y_2 is a Banach space. By the Open Mapping theorem, $\tilde{T}$ has a continuous inverse. Thus, $\tilde{T}$ is an isomorphism. □

4 Remark 4

We show that there is no Fredholm operator from $L^r(\Omega)$ to $L^s(\Omega)$, where $r, s \in (1, \infty), r \neq s$. Another isomorphism invariance of a Banach space beside separability and reflexivity is its "types". Some background about this notion is needed for our problem.

Definition 4.1. Let X be a Banach space and $p \in [1, 2]$. Then X is said to be of *type p* if there exists a number $C > 0$ (could depend on p) such that

$$\frac{1}{2^n} \sum_{\text{all signs}} \left\| \sum_{i=1}^{n} \pm x_i \right\|^p \leq C \sum_{i=1}^{n} \|x_i\|^p \quad \forall n \in \mathbb{N}, \forall x_1, ..., x_n \in X.$$

The above definition can be stated as follows. Let $\epsilon_1, \epsilon_2, \epsilon_3, ...$ be an independent sequence of identically distributed random variables, each satisfying $P(\epsilon_i = 1) = P(\epsilon_i = -1) = \frac{1}{2}$. The space X is said to be of type p if there exists a number $C > 0$ (could depend on p) such that

$$\mathbb{E} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|^p \leq C \sum_{i=1}^{n} \|x_i\|^p \quad \forall n \in \mathbb{N}, \forall x_1, ..., x_n \in X.$$

Proposition 4.2. *Let X be a Banach space of type p. We have the following statements.*

(i) Every Banach space isomorphic to X is also of type p.

(ii) X is of type r for every $r \in [1, p)$.

The largest type of X, if exists, is called the *best type* of X. Thanks to the parallelogram identity, every Hilbert space is of type 2. Then by Part (i), every finite dimensional space is of type 2, which is its best type.

Proof of Proposition 4.2. (i) Let $T : X \to Y$ be an isomorphism. There exists a number $C_1 > 0$ such that

$$C_1^{-1} \|x\| \leq \|Tx\| \leq C_1 \|x\| \quad \forall x \in X.$$

Let $n \in \mathbb{N}$ and $y_1, ..., y_n \in Y$. Put $x_i = T^{-1}(y_i)$. Then

$$C_1^{-1} \|x_i\| \leq \|y_i\| \leq C_1 \|x_i\|,$$

$$C_1^{-1} \left\| \sum_{i=1}^{n} \pm x_i \right\| \leq \left\| \sum_{i=1}^{n} \pm y_i \right\| \leq C_1 \left\| \sum_{i=1}^{n} \pm x_i \right\|.$$

Then

$$\frac{1}{2^n} \sum_{\text{all signs}} \left\| \sum_{i=1}^{n} \pm y_i \right\|^p \leq \frac{C_1^p}{2^n} \sum_{\text{all signs}} \left\| \sum_{i=1}^{n} \pm x_i \right\|^p \leq C_1^p C \sum_{i=1}^{n} \|x_i\|^p \leq C_1^{-2p} C \sum_{i=1}^{n} \|y_i\|^p.$$

Thus, Y is of type p.

(ii) Because X is of type p,

$$\mathbb{E}\left\|\sum_{i=1}^{n}\varepsilon_i x_i\right\|^p \leq C\sum_{i=1}^{n}\|x_i\|^p \quad \forall n\in\mathbb{N}, \forall x_1, ..., x_n \in X. \tag{4.1}$$

By Hölder's inequality, $(\mathbb{E}|f|^r)^{1/r} \leq (\mathbb{E}|f|^p)^{1/p}$ for every random variable f. Taking $f = \left\|\sum\limits_{i=1}^{n}\varepsilon_i x_i\right\|$, we get

$$\left(\mathbb{E}\left\|\sum_{i=1}^{n}\varepsilon_i x_i\right\|^r\right)^{\frac{1}{r}} \leq \left(\mathbb{E}\left\|\sum_{i=1}^{n}\varepsilon_i x_i\right\|^p\right)^{\frac{1}{p}}. \tag{4.2}$$

For any $s > 1$ and nonnegative numbers $a_1, a_2, ..., a_n$, we have

$$a_1^s + ... + a_n^s \leq (a_1 + ... + a_n)^s.$$

Take $a_i = ||x_i||^r$ and $s = \frac{p}{r}$. Then

$$\left(\sum_{i=1}^{n}\|x_i\|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n}\|x_i\|^r\right)^{\frac{1}{r}} \tag{4.3}$$

Substituting (4.2) and (4.3) into (4.1), we get

$$\mathbb{E}\left\|\sum_{i=1}^{n}\varepsilon_i x_i\right\|^r \leq C^{\frac{r}{p}}\sum_{i=1}^{n}\|x_i\|^r.$$

Thus, X is of type r. ☐

Proposition 4.3. *Let X be a Banach space of best type p, and Y be a closed subspace with finite codimension. Then Y is also of best type p.*

Proof. Since X is of type p, Y is also of type p. Suppose by contradiction that Y is of type $q \in (p, 2]$. Then there exists a number $C > 0$ such that

$$\left(\mathbb{E}\left\|\sum_{i=1}^{n}\varepsilon_i y_i\right\|^q\right)^{\frac{1}{q}} \leq C\left(\sum_{i=1}^{n}\|y_i\|^q\right)^{\frac{1}{q}} \quad \forall n\in\mathbb{N}, \forall y_1, ..., y_n \in Y.$$

Write $X = Y \oplus Z$ where Z is a finite dimensional subspace of X. The projection maps $\pi_Y : X \to Y$ and $\pi_Z : X \to Z$ are continuous. Thus, there is a number $C_2 > 0$ such that

$$\|\pi_Y x\| \leq C_2\|x\|, \quad \|\pi_Z x\| \leq C_2\|x\| \quad \forall x \in X.$$

Because Z is finite dimensional, it is of type 2. By Proposition 4.2, Part (ii), Z is also of type q. Take $x_1, ..., x_n \in X$ and write $x_i = y_i + z_i$ for $y_i \in Y$, $z_i \in Z$. Then $\|y_i\|, \|z_i\| \le C_2 \|x_i\|$.

$$
\begin{aligned}
\left(\mathbb{E}\left\|\sum_{i=1}^{n}\varepsilon_i x_i\right\|^q\right)^{\frac{1}{q}} &= \left(\mathbb{E}\left\|\sum_{i=1}^{n}\varepsilon_i y_i + \sum_{i=1}^{n}\varepsilon_i z_i\right\|^q\right)^{\frac{1}{q}} \\
&\le \left(\mathbb{E}\left\|\sum_{i=1}^{n}\varepsilon_i y_i\right\|^q\right)^{\frac{1}{q}} + \left(\mathbb{E}\left\|\sum_{i=1}^{n}\varepsilon_i z_i\right\|^q\right)^{\frac{1}{q}} \\
&\le C\left(\sum_{i=1}^{n}\|y_i\|^q\right)^{\frac{1}{q}} + C_1\left(\sum_{i=1}^{n}\|z_i\|^q\right)^{\frac{1}{q}} \\
&\le CC_2\left(\sum_{i=1}^{n}\|x_i\|^q\right)^{\frac{1}{q}} + C_1C_2\left(\sum_{i=1}^{n}\|x_i\|^q\right)^{\frac{1}{q}} \\
&= (CC_2 + C_1C_2)\left(\sum_{i=1}^{n}\|x_i\|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

Hence, X is of type q. Because p is the best type of X, $q \le p$. This is a contradiction. □

An important result that is needed for our problem is that the best type of $L^p(\Omega)$ is known. Theorem 6.2.14 in [AK06, p.140] states that:

> Let μ be a probability measure. Then the best type of $L^p(\mu)$ is equal to p if $1 \le p \le 2$, and is equal to 2 if $2 < p < \infty$.

We now have enough tools to prove the following result.

Proposition 4.4. Let $r, s \in (1,\infty), r \neq s$, and $\Omega \subset \mathbb{R}^n$ be a subset of positive finite measure. Then there is no Fredholm operator from $L^r(\Omega)$ to $L^s(\Omega)$.

Proof. Suppose by contradiction that there is a Fredholm operator from $T : L^r(\Omega) \to L^s(\Omega)$. Then there is a closed, finite-codimensional subspace X (repestively Y) of $L^r(\Omega)$ (respectively $L^s(\Omega)$) such that X and Y are isomorphic. The dual map $T^* : L^{s'}(\Omega) \to L^{r'}(\Omega)$ is also a Fredholm operator. Thus, there is a closed, finite-codimensional subspace $\tilde{X}$ (repestively $\tilde{Y}$) of $L^{r'}(\Omega)$ (respectively $L^{s'}(\Omega)$) such that $\tilde{X}$ and $\tilde{Y}$ are isomorphic.

By Proposition 4.3, the best type of X (respectively $Y, \tilde{X}, \tilde{Y}$) is equal to the best type of $L^r(\Omega)$ (respectively $L^s(\Omega), L^{r'}(\Omega), L^{s'}(\Omega)$). We have the following chart.

	Best type of			
	X	Y	$\tilde{X}$	$\tilde{Y}$
$r < 2, s < 2$	r	s		
$r < 2, s \geq 2$	r	2		
$r = 2, s < 2$	2	s		
$r = 2, s > 2$			2	s'
$r > 2, s < 2$	2	s		
$r > 2, s = 2$			r'	2
$r > 2, s > 2$			r'	s'

We see that either the best type of X is not equal to the best type of Y, or the best type of $\tilde{X}$ is not equal to the best type of $\tilde{Y}$. This is a contradiction. □

References

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[LT77] J. Lindenstrauss and L. Tzafriri: *Classical Banach spaces. I. Sequence spaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92. Springer-Verlag, Berlin-New York, 1977.

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