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Math 8800: Functional Analysis
Homework #1

① Let X be a normed space over $\mathbb{R}$ (respectively $\mathbb{C}$) such that every linear functional $f: X \to \mathbb{R}$ (respectively $\mathbb{C}$) is continuous. We show that X is finite-dimensional.

Suppose by contradiction that X is infinite dimensional. Then it contains a countably infinite set of linearly independent vectors $\{x_1, x_2, x_3, \dots\}$. By replacing x_i with $\frac{x_i}{\|x_i\|}$, we can assume $\|x_k\| = 1$ for all $k \in \mathbb{N}$. Let Y be the linear span of $\{x_1, x_2, x_3, \dots\}$. Define a linear map $T: Y \to \mathbb{R}$ (respectively $\mathbb{C}$) such that $Tx_k = 4^k$ for all $k \in \mathbb{N}$. Because Y is a vector subspace of X, T can extend to a linear map on X, which is continuous by our assumption. Thus, T is continuous on Y. This means there exists a number $C > 0$ such that $|Tx| \leq C\|x\|$ for all $x \in Y$. Because

$$|Tx - Ty| \leq C\|x - y\| \quad \forall x, y \in Y,$$

T maps a Cauchy sequence in Y to a Cauchy sequence in $\mathbb{R}$ (respectively $\mathbb{C}$). Put $y_k = 2^{-k} x_k \in Y$. Then

$$\|y_m - y_n\| = \|2^{-m} x_m - 2^{-n} x_n\| \leq 2^{-m}\|x_m\| + 2^{-n}\|x_n\| = 2^{-m} + 2^{-n} < 2 \cdot 2^{-m} \quad \forall n > m.$$

Thus, (y_k) is a Cauchy sequence in Y. On the other hand,

$$Ty_k = 2^{-k} Tx_k = 2^{-k} 4^k = 2^k,$$

which is not a Cauchy sequence in $\mathbb{R}$ (respectively $\mathbb{C}$). This is a contradiction.

② Let $X = \mathbb{R}[x]$ be the vector space (over $\mathbb{R}$) of polynomials in one variable with real coefficients. We show that there is no norm on X that could turn it into a Banach space.

Suppose otherwise: X together with some norm $\|.\|$ is a Banach space. For each $n = 0, 1, 2, \ldots$ we denote by x_n the monomial x^n. Then $\{x_1, x_2, x_3, \ldots\}$ is a basis for X. Put

$$s_n = \sum_{k=0}^{n} \frac{x_k}{\|x_k\|} 2^{-k} \qquad \forall n \in \mathbb{N}.$$

For $m, n \in \mathbb{N}$, $m < n$,

$$\|s_n - s_m\| = \left\| \sum_{k=m+1}^{n} \frac{x_k}{\|x_k\|} 2^{-k} \right\| \leq \sum_{k=m+1}^{n} \left\| \frac{x_k}{\|x_k\|} 2^{-k} \right\| = \sum_{k=m+1}^{n} 2^{-k} = 2^{-m} - 2^{-n} < 2^{-m}.$$

Thus, (s_n) is a Cauchy sequence in X. Since X is complete, (s_n) converges to some $y \in X$.

Each $x \in X$ is written uniquely as $x = \sum_{k=0}^{\infty} a_k x_k$ where $a_k \in \mathbb{R}$ and only finitely many of which are nonzero. Let $\pi_k : X \to \mathbb{R}$, $\pi(x) = a_k$ be the projection on the k'th component of x. Because $y \in X$, there exists $r \in \mathbb{N}$ such that $\pi_r(y) = 0$. Note that $\pi_r(s_n) = \dfrac{2^{-r}}{\|x_r\|}$ $\forall n \geq r$.

Thus, $$\lim_{n\to\infty} \pi_r(s_n) = \frac{2^{-r}}{\|x_r\|} > \pi_r(y).$$

This implies that a contradiction would be found if we could show π_r is continuous. For each $k \in \mathbb{N}$, let X_k be the linear span of $\{x_0, x_1, \ldots, x_k\}$. Let $T_k : X_k \to \mathbb{R}$ be the restriction of π_k on X_k. Since T_k is linear and $\dim X_k < \infty$, T_k is continuous.

By Hahn-Banach theorem, T_k can extend to a linear continuous functional on X, which we continue to denote by T_k. Because $\bigcup_{k=1}^{\infty} X_k = X$,

$$\lim_{n\to\infty} T_k x = \pi_r(x) \qquad \forall x \in X.$$

By Banach-Steinhaus theorem, the sequence (T_k) is bounded in X^*. That is, there exists a number $M > 0$ such that

$$|T_k x| \leq M\|x\| \quad \forall x \in X, \forall k \in \mathbb{N}.$$

Thus, $|\pi_r(x)| = \lim_{k\to\infty} |T_k x| \leq M\|x\|$ for all $x \in X$. Together with the linearity of π_r, we conclude that π_r is continuous.

③ Let X be a Banach space and $T: X \to X$ be a compact linear operator. Suppose that $S = Id_X - T$ is invertible from X to $Y = S(X)$. We show that $S^{-1} = Id_Y - Q$ where $Q: Y \to X$ is a compact operator.

It is clear that $Q = Id_Y - S^{-1}$ is a linear map. To show that Q is continuous, is equivalent to to show that S^{-1} is continuous. It suffices to show S^{-1} is continuous at $0 \in Y$. Let (y_n) be a sequence in Y which converges to 0. Put $x_n = S^{-1}(y_n)$. Then $S(x_n) = y_n \to 0$. We need to show (x_n) converges to 0.

Consider two cases.

• (x_n) is bounded.

Suppose by contradiction that $x_n \not\to 0$. Then there exists a subsequence (x_{n_k}) and a number $\varepsilon > 0$ such that $\|x_{n_k}\| \geq \varepsilon$ for all $k \in \mathbb{N}$. By replacing (x_n) by this subsequence, we can assume $\|x_n\| \geq \varepsilon \quad \forall n \in \mathbb{N}$. (1)

Because (x_n) is bounded and T is compact, the sequence (Tx_n) has a convergent subsequence (Tx_{n_k}). Because $x_{n_k} = Tx_{n_k} + Sx_{n_k}$, the sequence (x_{n_k}) also converges. Let $a = \lim x_{n_k}$. By the continuity of S, $Sx_{n_k} \to Sa$. Then $Sa = 0$. Since S is injective, $a = 0$. Thus, $x_{n_k} \to 0$ as $k \to \infty$. This contradicts (1).

- (x_n) is unbounded.

We show that this case actually does not happen. By replacing (x_n) by a subsequence if necessary, we can assume $\|x_n\| \to \infty$ as $n \to \infty$. Let $x_n' = \frac{x_n}{\|x_n\|}$. Then $\|x_n'\| = 1$ and $Sx_n' = \frac{Sx_n}{\|x_n\|} = y_n' \to 0$. On the other hand, (Tx_n') has a convergent subsequence because (x_n') is bounded. Say $Tx_{n_k}' \to a$. Then

$$x_{n_k}' = Tx_{n_k}' + Sx_{n_k}' \to a + 0 = a \quad \text{as } k \to \infty.$$

By the continuity of the norm and of S, we have

$$\|a\| = \lim_{k\to\infty} \|x_{n_k}'\| = 1,$$

$$Sa = \lim_{k\to\infty} Sx_{n_k}' = 0.$$

The second identity implies $a = 0$ because S is injective. But this contradicts the first identity. We have showed that Q is continuous.

To show Q is compact, we take an arbitrary bounded sequence (y_n) in Y and show that the sequence (x_n), $x_n = Qy_n$, has a convergent subsequence in X. We have

$$Sx_n = S(Qy_n) = S(y_n - S^{-1}y_n) = Sy_n - y_n = -Ty_n. \qquad (2)$$

Because (y_n) is bounded and T is compact, there exists a subsequence (y_{n_k}) such that (Ty_{n_k}) converges. By replacing (y_n) by (y_{n_k}), we can assume (Ty_n) converges. Since (y_n) is bounded and Q is continuous, $x_n = Qy_n$ is a bounded sequence. Thus, there exists a subsequence (x_{n_k}) such that (Tx_{n_k}) converges. Thus,

$$x_{n_k} = Sx_{n_k} + Tx_{n_k} \overset{(2)}{=\!=} -Ty_{n_k} + Tx_{n_k}$$

converges in X.

④ Let $X = C([0,1])$ be the norm space with $\|f\| = \sup_{x\in[0,1]} |f(x)|$. Let $k \in C([0,1]\times[0,1])$ and define $K: X \to X$, $Kf(x) = \int_0^1 k(x,y)f(y)\,dy \quad \forall f\in X, \forall x\in[0,1]$

Let $A: X\to X$ be a linear continuous map, and put $B = A\circ K$. We show that there exists $b \in C([0,1]\times[0,1])$ such that

$$Bf(x) = \int_0^1 b(x,y)f(y)\,dy \quad \forall f\in X, \forall x\in[0,1].$$

For each $y\in[0,1]$, denote $k_y(x) = k(x,y)$. Then $k_y\in X$. Define a function $b: [0,1]\times[0,1]\to \mathbb{R}$, $b(x,y) = (Ak_y)(x)$ for $x,y\in[0,1]$. First we show that b is continuous.

$$b(x,y) - b(x,z) = (Ak_y - Ak_z)(x) = (A(k_y - k_z))(x)$$

$$\leq \|A(k_y - k_z)\|_X \leq \|A\|_{\mathcal{L}(X,X)} \|k_y - k_x\|_X. \qquad (1)$$

Because k is continuous on the compact set $[0,1]\times[0,1]$, it is uniformly continuous. For each $\varepsilon>0$, there exists $\delta = \delta(\varepsilon)>0$ such that

$$|k(x',y') - k(x,y)| < \varepsilon \quad \forall x',y',x,y\in[0,1],\ |x'-x|<\delta,\ |y'-y|<\delta.$$

Thus, $\|k_y - k_z\|_X = \sup_{x\in[0,1]} |k(x,y) - k(x,z)| < \varepsilon \quad \forall y,z\in[0,1],\ |y-z|<\delta.$

From (1) we get

$$|b(x,y)-b(x,z)| \leq \varepsilon\|A\| \quad \forall x,y,z\in[0,1],\ |y-z|<\delta. \qquad (2)$$

Let $\{(x_n,y_n)\}_n$ be a sequence in $[0,1]\times[0,1]$ that converges to (x_0,y_0). There is $N_1=N_1(\varepsilon)\in\mathbb{N}$ such that $|y_n-y_0|<\delta$ for all $n>N_1$. Then

$$|b(x_n,y_n)-b(x_0,y_0)| \leq |b(x_n,y_n)-b(x_n,y)|+|b(x_n,y)-b(x,y)|$$
$$\overset{(2)}{\leq} \varepsilon\|A\| + |Ak_{y_0}(x_n)-Ak_{y_0}(x)|.$$

Because $Ak_{y_0}\in X$, $\lim|Ak_{y_0}(x_n)-Ak_{y_0}(x)|=0$. There exists $N_2=N_2(\varepsilon)\in\mathbb{N}$ such that $|Ak_{y_0}(x_n)-Ak_{y_0}(x)|<\varepsilon$ for all $n>N_2$. Thus,

$$|b(x_n,y_n)-b(x_0,y_0)| < \varepsilon\|A\|+\varepsilon=(\|A\|+1)\varepsilon \quad \forall n>\max\{N_1,N_2\}.$$

Hence, $\lim b(x_n,y_n)=b(x_0,y_0)$, and b is continuous on $[0,1]\times[0,1]$. Put $\tilde{B}:X\to X$,

$$\tilde{B}f(x)=\int_0^1 b(x,y)f(y)\,dy \quad \forall f\in X,\ \forall x\in[0,1].$$

We show that $\tilde{B}=B$. In other words, for a fixed $f\in X$, we need to show $\tilde{B}f=A(Kf)$.

For each $n\in\mathbb{N}$, we divide the interval $[0,1]$ into n equal subintervals $[y_{n,i},y_{n,i+1}]$ with $y_{n,i}=\frac{i}{n}$ $\forall\, 0\leq i\leq n$. Put

$$g_n(x)=\frac{1}{n}\sum_{i=0}^{n} b(x,y_{n,i})f(y_{n,i}). \qquad (3)$$

Then $g_n\in X$ because each summand is continuous in $x\in[0,1]$. We show that $g_n\to\tilde{B}f$ in X.

$$|g_n(x)-\tilde{B}f(x)| = \left|\sum_{i=0}^{n}\int_{y_{n,i}}^{y_{n,i+1}} b(x,y_{n,i})f(y_{n,i})\,dy-\int_0^1 b(x,y)f(y)\,dy\right|$$
$$=\left|\sum_{i=0}^{n}\int_{y_{n,i}}^{y_{n,i+1}}\big(b(x,y_{n,i})f(y_{n,i})-b(x,y)f(y)\big)\,dy\right|$$

$$\leq \sum_{i=0}^{n} \int_{y_{n,i}}^{y_{n,i+1}} \underbrace{|b(x, y_{n,i}) f(y_{n,i}) - b(x,y) f(y)|}_{\{1\}} dy \qquad (4)$$

The function $(x,y) \mapsto b(x,y)f(y)$ is continuous on the compact set $[0,1] \times [0,1]$. Thus, it is uniformly continuous. For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|b(x',y')f(y') - b(x,y)f(y)| < \varepsilon \quad \forall x', y', x, y \in [0,1], \ |x'-x| < \delta, \ |y'-y| < \delta.$$

For $n > \delta^{-1}$, $|y_{n,i+1} - y_{n,i}| = n^{-1} < \delta$. Thus, $\{1\} < \varepsilon$ for all $x \in [0,1]$. Then we imply from (4) that

$$|g_n(x) - \tilde{B}_f(x)| \leq \sum_{i=0}^{n} \int_{y_{n,i}}^{y_{n,i+1}} \varepsilon \, dy = \int_0^1 \varepsilon \, dy = \varepsilon \quad \forall x \in [0,1], \ \forall n > \delta^{-1}.$$

Thus, $\|g_n - \tilde{B}_f\|_X \leq \varepsilon$ for all $n > \delta^{-1}$. We thus proved that $g_n \to \tilde{B}_f$ in X.

Next, we show that $g_n \to A(Kf)$ in X. By (3),

$$g_n(x) = \frac{1}{n} \sum_{i=0}^{n} A k_{y_{n,i}}(x) f(y_{n,i}) = A\left(\frac{1}{n} \sum_{i=0}^{n} k_{y_{n,i}}(x) f(y_{n,i}) \right).$$

Put $h_n(x) = \frac{1}{n} \sum_{i=0}^{n} k(x, y_{n,i}) f(y_{n,i})$. Then $h_n \in X$ (because each summand is continuous in $x \in [0,1]$) and $g_n = A h_n$. Thanks to the continuity of A, to show $g_n \to A(Kf)$ in X, we only need to show $h_n \to Kf$ in X. This can be done by completely the same procedure as we proved $g_n \to \tilde{B}_f$ in X, with b being replaced by k.

(5) Let X be a Hilbert space and T be a linear continuous operator on X.

(a) We construct an example showing that $T^2 = T \circ T$ is a compact operator but T is not.

To do so, we have to work with an infinite-dimensional space. The simplest of such spaces is a Hilbert space with a countable orthonormal basis $(e_1, e_2, \dots)$

$$X = \left\{ x = \sum_{k=1}^{\infty} c_k e_k : c_k \in \mathbb{R}, \sum_{k=1}^{\infty} c_k^2 < \infty \right\}, \quad \|x\| = \left(\sum_{k=1}^{\infty} c_k^2 \right)^{1/2}.$$

We want to define $T \in \mathcal{L}(X,X)$ such that $T^2 = T \circ T = 0$. Recall that a linear operator on $\mathbb{R}^2$ is represented by a 2x2 matrix. An operator L represented by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ satisfies $L^2 = 0$. If we view $\mathbb{R}^2$ as the linear space spanned by $\{e_1, e_2\}$ then $Le_1 = 0$, $Le_2 = e_1$. This is a hint for us to construct T satisfying $T^2 = 0$. Let Y be the linear span of $\{e_1, e_2, e_3, \dots\}$. Define a linear map $T : Y \to \mathbb{R}$ satisfying $Te_{2k-1} = 0$, $Te_{2k} = e_{2k-1}$ for all $k \in \mathbb{N}$. Then

$$T\left(\sum_{i=1}^{n} c_i e_i\right) = \sum_{i=1}^{n} c_i Te_i = \sum_{k=1}^{[\frac{n}{2}]} c_{2k} e_{2k-1} \qquad \forall c_1, \dots, c_m \in \mathbb{R}.$$

Thus,
$$\left\| T\left(\sum_{i=1}^{n} c_i e_i\right)\right\| = \left(\sum_{k=1}^{[\frac{n}{2}]} c_{2k}^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} c_i^2\right) = \left\|\sum_{i=1}^{n} c_i e_i\right\|.$$

This implies T is continuous and $\|T\|_{Y^*} \leq 1$. Because Y is dense in X, T extends uniquely to a linear continuous map on X, which we continue to denote by T. We have

$$T^2 e_{2k} = T(Te_{2k}) = Te_{2k-1} = 0,$$
$$T^2 e_{2k-1} = T(Te_{2k-1}) = T(0) = 0 \qquad \forall k \in \mathbb{N}.$$

Thus, $T^2 = 0$ on Y. Because T^2 is continuous on X, and Y is dense in X, $T^2 = 0$ on X.

Now we show that T is not compact. Put $x_k = e_{2k}$ for all $k \in \mathbb{N}$.

Then (x_k) is a bounded sequence in X.

$$\|Tx_k - Tx_i\| = \|Te_{2k} - Te_{2i}\| = \|e_{2k-1} - e_{2i-1}\| = \sqrt{2} \quad \forall k, i \in \mathbb{N},\ k \neq i.$$

Thus, the sequence (Tx_k) doesn't have any convergent subsequence. Therefore, T is not a compact operator.

(b) Suppose that T^*T is a compact operator on X. We show that T is also compact. Let (x_n) be a sequence with $\|x_n\| \leq 1$ for all $n \in \mathbb{N}$. We show that (Tx_n) has a convergent subsequence. Because T^*T is compact, (T^*Tx_n) has a convergent subsequence. By replacing (x_n) by a subsequence if necessary, we can assume (T^*Tx_n) converges. Thus, (T^*Tx_n) is a Cauchy sequence. For each $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\|T^*Tx_n - T^*Tx_m\| < \varepsilon \quad \forall n, m > N.$$

By the definition of T^*, $(Tx, y) = (x, T^*y)$ for all $x, y \in X$. Replacing y by Tx, we get $\|Tx\|^2 = (x, T^*Tx)$. Then substituting x by $x_n - x_m$, we get

$$\begin{aligned}\|Tx_n - Tx_m\|^2 &= (x_n - x_m, T^*Tx_n - T^*Tx_m) \\ &\overset{\text{Schwarz}}{\leq} \underbrace{\|x_n - x_m\|}_{\leq 2}\ \underbrace{\|T^*Tx_n - T^*Tx_m\|}_{< \varepsilon} \\ &< 2\varepsilon \quad \forall n, m > N.\end{aligned}$$

Thus, (Tx_n) is a Cauchy sequence. Since X is complete, (Tx_n) converges.