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Math 8801: Functional Analysis
Homework #2

(1) Denote $\ell^2 = \{x = (x_1, x_2, \dots) : x_k \in \mathbb{C}, \sum_{k=1}^{\infty} |x_k|^2 < \infty\}$. We know that ℓ^2 is a Banach space with norm $\|x\| = \left(\sum_{k=1}^{\infty} |x_k|^2\right)^{1/2}$. Let (λ_n) be a bounded sequence of complex numbers. Define a map $T : \ell^2 \to \ell^2$,

$$T(x_1, x_2, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \dots)$$

First, we show that T is a linear continuous operator. Put $M = \sup_{k \in \mathbb{N}} |\lambda_k| < \infty$.

For $x = (x_1, x_2, \dots) \in \ell^2$,

$$\sum_{k=1}^{\infty} |\lambda_k x_k|^2 = \sum_{k=1}^{\infty} |\lambda_k|^2 |x_k|^2 \leq M^2 \sum_{k=1}^{\infty} |x_k|^2 < \infty \qquad (1)$$

Thus, $T(x) \in \ell^2$ and T is well-defined. For $x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots) \in \ell^2$ and $c \in \mathbb{C}$,

$$\begin{aligned} T(cx+y) &= T(cx_1 + y_1, cx_2 + y_2, \dots) \\ &= (\lambda_1(cx_1 + y_1), \lambda_2(cx_2 + y_2), \dots) \\ &= c(\lambda_1 x_1, \lambda_2 x_2, \dots) + (\lambda_1 y_1, \lambda_2 y_2, \dots) \\ &= cT(x) + T(y). \end{aligned}$$

Hence, T is a linear map. The inequality (1) implies

$$\|Tx\|^2 \leq M^2 \|x\|^2 \quad \forall x \in \ell^2.$$

Hence, T is continuous and $\|T\|_{\mathcal{L}(\ell^2)} \leq M$.

Next, we describe the spectrum of T. To do so, we first find all eigenvalues of T. Let $\lambda \in \mathbb{C}$ be an eigenvalue of T. The definition of eigenvalues says that there exists $x=(x_1,x_2,\dots)\in \ell^2\setminus\{0\}$ such that $Tx=\lambda x$. This equation is equivalent to

$$\lambda_k x_k = \lambda x_k \quad \forall k\in\mathbb{N}.$$

Since $x_j\neq 0$ for some $j\in\mathbb{N}$, $\lambda=\lambda_j$. This implies the eigenvalues of T are among $\lambda_1,\lambda_2,\lambda_3,\dots$ For each $j\in\mathbb{N}$, we put $x^j=(0,0,\dots,0,\underset{\stackrel{\uparrow}{j}}{1},0,\dots)\neq 0$. Then

$$Tx^j = T(0,\dots,0,\underset{\stackrel{\uparrow}{j}}{1},0,\dots) = (0,\dots,0,\underset{\stackrel{\uparrow}{j}}{\lambda_j},0,\dots) = \lambda_j x^j,$$

which implies λ_j is an eigenvalue of T. We conclude that the set of all eigenvalues of T is $\{\lambda_n : n\in\mathbb{N}\}$.

Denote by $\sigma(T)$ the spectrum of T. Because an eigenvalue is also a spectral value of T, $\{\lambda_n : n\in\mathbb{N}\}\subset\sigma(T)$. We also know that $\sigma(T)$ is closed in $\mathbb{C}$. Then

$$A := \overline{\{\lambda_n : n\in\mathbb{N}\}} \subset \sigma(T).$$

We show that the two sets are actually equal. It is to show that for each $\lambda\in\mathbb{C}\setminus A$, $T-\lambda\, Id_{\ell^2}$ has a linear continuous inverse. A is compact because it is bounded and closed in $\mathbb{C}$. For $\lambda\in\mathbb{C}\setminus A$, $dist(\lambda,A)=\delta>0$. Then

$$|\lambda_k-\lambda|\geq\delta \quad \forall k\in\mathbb{N}.$$

The sequence $\frac{1}{\lambda_1-\lambda},\frac{1}{\lambda_2-\lambda},\dots$ is bounded in $\mathbb{C}$. According to the first part of our proof, the map $\tilde{T}:\ell^2\to\ell^2$, $\tilde{T}(x_1,x_2,\dots)=\left(\frac{x_1}{\lambda_1-\lambda},\frac{x_2}{\lambda_2-\lambda},\dots\right)$ is well-defined, linear and continuous.

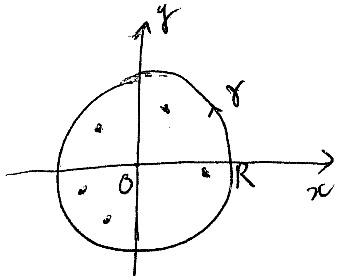
$$\begin{aligned}\tilde{T}\circ(T-\lambda Id_{\ell^2})(x) &= \tilde{T}(Tx-\lambda x) = \tilde{T}((\lambda_1-\lambda)x_1,(\lambda_2-\lambda)x_2,\dots)\\ &= (x_1,x_2,\dots)\\ &= x,\end{aligned}$$

$$(T-\lambda \,\mathrm{Id}_{\ell^2})\circ \underbrace{\tilde{T}(x)}_{=y} = Ty-\lambda y = ((\lambda_1-\lambda)y_1, (\lambda_2-\lambda)y_2, \dots)$$
$$= \left((\lambda_1-\lambda)\frac{x_1}{\lambda_1-\lambda}, (\lambda_2-\lambda)\frac{x_2}{\lambda_2-\lambda}, \dots\right)$$
$$= (x_1, x_2, \dots)$$
$$= x.$$

Therefore, $\tilde{T}$ is the inverse of $T-\lambda\,\mathrm{Id}_{\ell^2}$.

2 Let $A\in M_n(\mathbb{C})$. It can be viewed as a linear operator of $\mathbb{C}^n$ whose matrix representation in the standard basis $(e_1, e_2, \dots, e_n)$ is A. Because $\mathbb{C}^n$ is finite-dimensional, the spectral values of A are also the eigenvalues. Thus, $\sigma(A)$ is exactly the set of roots of the characteristic polynomial $P(\lambda)=\det(\lambda I_n - A)$, which is a finite set.

Let $R>\|A\|_{\mathcal{L}(\mathbb{C}^n)}$ and γ be the closed path $\gamma(t)=Re^{it}$, $0\le t\le 2\pi$. All eigenvalues of A have modulus less than or equal to $\|A\|_{\mathcal{L}(\mathbb{C}^n)}$, so they are enclosed in γ. We show that



$$A^k = \frac{1}{2\pi i}\int_\gamma \lambda^k(\lambda I_n - A)^{-1}\,d\lambda \quad \forall k=0,1,2,\dots$$

The first step is to show that for $\lambda\in\mathbb{C}$, $|\lambda|=R$,

$$(\lambda I_n - A)^{-1} = \frac{1}{\lambda}\left(I_n + \frac{A}{\lambda} + \frac{A^2}{\lambda^2} + \dots\right) \qquad (1)$$

where the series is taken with respect to $\mathcal{L}(\mathbb{C}^n)$-norm.

$$\left\|\frac{A^j}{\lambda^j}\right\| = \frac{1}{|\lambda|^j}\|\underbrace{A\circ A\circ\dots\circ A}_{j \text{ times}}\| \le \frac{1}{R^j}\|A\|^j = \left(\frac{\|A\|}{R}\right)^j.$$

Because $\sum_{j=0}^{\infty}\left(\frac{\|A\|}{R}\right)^j<\infty$, the series on the right hand side of (1) converges uniformly for $\lambda\in\gamma$. In particular, $\lim_{m\to\infty}\frac{A^m}{\lambda^m}=0$.

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Denote $B = RHS(1)$. We have

$$(\lambda I_n - A)\cdot B = \lim_{m\to\infty}\left(I_n - \frac{A}{\lambda}\right)\cdot\left(I_n + \frac{A}{\lambda} + \frac{A^2}{\lambda^2} + \dots + \frac{A^m}{\lambda^m}\right)$$
$$= \lim_{m\to\infty}\left(I_n - \frac{A^{m+1}}{\lambda^{m+1}}\right) = I_n,$$

$$B(\lambda I_n - A) = \lim_{m\to\infty}\left(I_n + \frac{A}{\lambda} + \frac{A^2}{\lambda^2} + \dots + \frac{A^m}{\lambda^m}\right)\left(I_n - \frac{A}{\lambda}\right)$$
$$= \lim_{m\to\infty}\left(I_n - \frac{A^{m+1}}{\lambda^{m+1}}\right) = I_n.$$

Therefore, B is the inverse of $\lambda I_n - A$. The equation (1) is justified.

Then $\lambda^k(\lambda I_n - A)^{-1} = \sum_{j=0}^{\infty} \lambda^{k-1-j} A^j$. Because this series converges uniformly for $\lambda \in \gamma$,

$$\int_\gamma \lambda^k(\lambda I_n - A)^{-1} d\lambda = \sum_{j=0}^{\infty} \int_\gamma \lambda^{k-1-j} A^j d\lambda = \sum_{j=0}^{\infty}\left(\int_\gamma \lambda^{k-1-j} d\lambda\right) A^j. \qquad (2)$$

We have
$$\int_\gamma \lambda^m d\lambda = \int_0^{2\pi} \gamma(t)^m \gamma'(t)\, dt = \int_0^{2\pi} (Re^{it})^m R i e^{it}\, dt$$
$$= R^{m+1} i \int_0^{2\pi} e^{i(m+1)t}\, dt.$$

If $m = -1$ then $\int_\gamma \lambda^{-1} d\lambda = i\int_0^{2\pi} dt = 2\pi i$.

If $m \neq -1$ then $\int_\gamma \lambda^m d\lambda = R^{m+1} i \left.\frac{e^{i(m+1)t}}{i(m+1)}\right|_{t=0}^{t=2\pi} = 0.$

Thus,
$$\int_\gamma \lambda^{k-1-j} d\lambda = \begin{cases} 2\pi i & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

By (2) we get $\int_\gamma \lambda^k(\lambda I_n - A)^{-1} d\lambda = 2\pi i A^k.$

Therefore, $A^k = \frac{1}{2\pi i}\int_\gamma \lambda^k(\lambda I_n - A)^{-1} d\lambda \quad \forall k = 0, 1, 2, \dots$

Next, we use these identities to prove Cayley-Hamilton Theorem, which says that $P(A)=0$. Write $P(\lambda)=\sum_{k=0}^{m} a_k\lambda^k$, $a_k\in\mathbb{C}$. Then

$$P(A)=\sum_{k=0}^{m} a_k A^k = \sum_{k=0}^{m} a_k \frac{1}{2\pi i}\int_\gamma \lambda^k(\lambda I_n - A)^{-1}d\lambda$$
$$=\frac{1}{2\pi i}\int_\gamma \underbrace{P(\lambda)(\lambda I_n - A)^{-1}}_{Q(\lambda)}d\lambda$$

Here $Q(\lambda)=\det(\lambda I_n - A)(\lambda I_n - A)^{-1}$ is equal to the adjugate matrix of $\lambda I_n - A$. Each entry of matrix $Q(\lambda)$ is thus a polynomial of λ. This implies Q is a holomorphic function on $\mathbb{C}$. Therefore,

$$P(A)=\frac{1}{2\pi i}\int_\gamma Q(\lambda)d\lambda = 0.$$

③ Let X be a complex Hilbert space and $T: X\to X$ be a quasi-hermitian operator. By definition, T is linear continuous and there exists a self-adjoint operator G with $(Gx,x)>0$ for all $x\in X\setminus\{0\}$ such that $GT=T^*G$.

(i) Suppose T is also a compact operator. We show that all spectral values of T are real. Denote by I the identity map on X. If $a+bi$, where $a,b\in\mathbb{R}$, $b\neq 0$, is a spectral value of T then i is a spectral value of $\frac{1}{b}T-\frac{a}{b}I$. Indeed,

$$(a+bi)I - T\in\mathcal{L}^\times(X) \Leftrightarrow b\left[iI-\left(\tfrac{1}{b}T-\tfrac{a}{b}I\right)\right]\in\mathcal{L}^\times(X)$$
$$\Leftrightarrow iI-\left(\tfrac{1}{b}T-\tfrac{a}{b}I\right)\in\mathcal{L}^\times(X).$$

Thus, it suffices to show that i is not a spectral value of $\frac{1}{b}T-\frac{a}{b}I$ for any

$a, b \in \mathbb{R}, b \neq 0$. The operator $\frac{1}{b}T$ is quasi-hermitian (with the same map G) and compact. By considering $\frac{1}{b}T$ instead of T, we are to show that i is not a spectral value of $T - aI$ for every $a \in \mathbb{R}$. Fix $a \in \mathbb{R}$ and put $A = T - aI$. We show that $iI - A \in \mathcal{L}^\times(X)$. Suppose $I + A^2 \in \mathcal{L}^\times(X)$. Then

$$(iI - A)\underbrace{(-iI - A)(I + A^2)^{-1}}_{B} = (I + A^2)(I + A^2)^{-1} = I,$$

$$B(iI - A) = (I + A^2)^{-1}(-iI - A)(iI - A) = (I + A^2)^{-1}(I + A^2) = I.$$

Note that $(I+A^2)^{-1}$ and $(-iI - A)$ commute because $(I + A^2)$ and $(-iI - A)$ commute. Then B is the continuous inverse of $iI - A$. Therefore, the remaining task is to show $I + A^2 \in \mathcal{L}^\times(X)$. It is true for $X = \{0\}$. Consider the case $X \neq \{0\}$. We have

$$I + A^2 = I + (T - aI)^2 = (1 + a^2)I - 2aT + T^2$$

$$= (1 + a^2)\Big[I - \underbrace{\tfrac{1}{1 + a^2}(2aT - T^2)}_{\tilde{T}}\Big].$$

Put $S = I - \tilde{T}$. Then $I + A^2 = (1 + a^2)S$. Suppose there exists a number $c > 0$ such that

$$\|(I + A^2)x\| \geq c \qquad \forall x \in X, \|x\| = 1. \qquad (1)$$

Then $(I + A^2)$ is injective. Thus, S is also injective. Because T is a compact operator, so is $\tilde{T}$. Thus S is a Fredholm operator. One corollary in the lecture on 09/26/2014 says that $Sx = b$ is solvable for every $b \in X$ if and only if $Sx = 0$ has a unique solution $x = 0$. In our case, S is injective, so it is surjective by this corollary. Thus, $(I + A^2)$ is also surjective. Then $(I + A^2)$

has a linear inverse. Because of (1), $(I+A^2)^{-1}$ is continuous. Therefore, we only need to show that there is a number $c>0$ satisfying (1). Suppose by contradiction that there is no such c. Then for each $n \in \mathbb{N}$, there exists $x_n \in X$, $\|x_n\| = 1$ such that $\|(I+A^2)x_n\| \leq \frac{1}{n}$. Then

$$(I+A^2)x_n \longrightarrow 0 \quad \text{as } n \to \infty.$$

We have

$$(I+A^2)x_n = x_n + (T-aI)^2 x_n = (1+a^2)x_n - 2aTx_n + T^2 x_n.$$

Because T is compact, and (x_n) is bounded, (Tx_n) has a convergent subsequence (Tx_{n_k}). By replacing (x_n) with (x_{n_k}), we can assume (Tx_n) converges. Then

$$x_n = \frac{1}{1+a^2}\left((I+A^2)x_n + 2aTx_n + T(Tx_n)\right)$$

also converges as $n \to \infty$. Write $\lim x_n = x_0$. Because $\|x_n\| = 1 \ \forall n$, $\|x_0\| = 1$.

$$|((I+A^2)x_n, Gx_n)| \leq \|(I+A^2)x_n\| \, \|Gx_n\| \leq \|(I+A^2)x_n\| \, \|G\|.$$

Thus, $\lim((I+A^2)x_n, Gx_n) = 0$. We have

$$GA = G(T-aI) = GT - aG = T^*G - aG = (T^* - aI)G = A^*G. \qquad (2)$$

Hence,

$$((I+A^2)x_n, Gx_n) = (x_n, Gx_n) + (A^2x_n, Gx_n) = (Gx_n, x_n) + (Ax_n, A^*Gx_n)$$

$$\overset{(2)}{=\!=} (Gx_n, x_n) + \underbrace{(Ax_n, GAx_n)}_{\geq 0}.$$

This implies $((I+A^2)x_n, Gx_n) \geq (Gx_n, x_n)$. $\qquad$ (3)

We have

$$\cancel{Kx_n}\ |(Gx_n, x_n) - (Gx_0, x_0)| = |(Gx_n, x_n - x_0) + (Gx_n - Gx_0, x_0)|$$

$$\leq \|Gx_n\| \, \|x_n - x_0\| + \|G(x_n - x_0)\| \, \|x_0\|$$

$$\leq \|G\| \|x_n - x_0\| + \|G\| \|x_n - x_0\| \|x_0\|$$

$$\longrightarrow 0 \text{ as } n \to \infty.$$

Thus, $\lim (Gx_n, x_n) = (Gx_0, x_0)$. Taking $n \to \infty$ in (3), we get $0 \geq (Gx_0, x_0)$.
Thus $x_0 = 0$. This contradicts the fact that $\|x_0\| = 1$.

(ii) Suppose X is finite dimensional. We show that T is quasi-hermitian if and only if X has a basis in which T is represented by a real diagonal matrix.

First, assume T is quasi-linear. Define $((x, y)) := (Gx, y)$ for all $x, y \in X$.
Then

$$((cx_1 + x_2, y)) = G(cx_1 + x_2, y) = c\,G(x_1, y) + G(x_2, y)$$

$$= c((x_1, y)) + ((x_2, y)) \quad \forall x_1, x_2, y \in X, \ \forall c \in \mathbb{C}$$

$$((y, x)) = (Gy, x) = (y, Gx) = \overline{(Gx, y)} = \overline{((x, y))}.$$

$$((x, x)) = (Gx, x) \geq 0.$$

$$((x, x)) = 0 \Leftrightarrow (Gx, x) = 0 \Leftrightarrow x = 0.$$

Thus, $((\cdot,\cdot))$ is an inner product on X. Since X is finite dimensional, $(X, ((\cdot,\cdot)))$ is also a Hilbert space. We have

$$((Tx, y)) = (GTx, y) = (T^*Gx, y) = (Gx, Ty) = ((x, Ty)) \quad \forall x, y \in X.$$

Hence, T is self-adjoint as an operator on $(X, ((\cdot,\cdot)))$. Because X is finite dimensional, T is a compact operator. Moreover, the spectral values of T are also its eigenvalues. By the spectral Theorem for self-adjoint compact operators on a Hilbert space, we conclude that X has a basis consisting of eigenvectors of T. In this basis, T is represented by a diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Each λ_k is an eigenvalue of T, so it is real by Part (i).

Next, let $T: X \to X$ be a linear operator that is represented by matrix $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ in some basis of X. We show that T is quasi-hermitian. Let $\mathcal{B}$ be an orthonormal basis of X. Denote by $[f]_{\mathcal{B}}$ the matrix representation of a linear operator f in basis $\mathcal{B}$. There exists a matrix $P \in GL(n, \mathbb{C})$ such that $[T]_{\mathcal{B}} = P^{-1} D P$. We also know that $[T^*]_{\mathcal{B}} = [T]_{\mathcal{B}}^*$, where A^* denotes the complex conjugate of the transpose of matrix A.

Because D is a real diagonal matrix, $D^* = D$. Then

$$[T^*]_{\mathcal{B}} = (P^{-1} D P)^* = P^* D (P^{-1})^* = P^* P (P^{-1} D P) P^{-1} (P^{-1})^* = P^* P [T]_{\mathcal{B}} (P^* P)^{-1} \quad (4)$$

Let $F: X \to X$ be the linear operator whose matrix representation in $\mathcal{B}$ is P. Put $G = F^* F$. Then $[G]_{\mathcal{B}} = [F^* F]_{\mathcal{B}} = [F]_{\mathcal{B}}^* [F]_{\mathcal{B}} = P^* P$. Substituting $[G]_G$ into (4), we get

$$[T^*]_{\mathcal{B}} = [G]_{\mathcal{B}} [T]_{\mathcal{B}} [G]_{\mathcal{B}}^{-1} = [G T G^{-1}]_{\mathcal{B}}.$$

Hence, $T^* = G T G^{-1}$ which gives $GT = T^* G$. Because $G^* = (F^* F)^* = F^* F = G$, G is self-adjoint.

$$(Gx, x) = (F^* F x, x) = (Fx, Fx) \geq 0 \quad \forall x \in X,$$

$$(Gx, x) = 0 \iff Fx = 0.$$

F is invertible because $[F]_{\mathcal{B}} = P$ is an invertible matrix. Thus, $Fx = 0$ if and only if $x=0$. We get

$$(Gx,x) = 0 \Longleftrightarrow x = 0.$$

Therefore, T is quasi-hermitian.

④ Let X be a Hilbert space and $\mathcal{L}(X)$ be the space of linear continuous operators on X.

(i) Let $A \in \mathcal{L}(X)$. Suppose A has a left inverse B_1 and a right inverse B_2. We show that $B_1 = B_2$. By definition, $B_1 A = A B_2 = I$ where I denotes the identity map on X. Using the associativity of map composition, we get

$$B_1 = B_1 I = B_1(AB_2) = (B_1 A) B_2 = I B_2 = B_2.$$

Consequently, B_1 is also a right inverse of A. This implies A is invertible.

(ii) Let $A \in \mathcal{L}(X)$. Suppose A has a left inverse B. First, we show that A is injective. That is to show $\ker A = \{0\}$. For $x \in \ker A$,

$$x = (BA)x = B(Ax) = B(0) = 0.$$

Next, we show that $A(X)$ is closed in X. Let (y_n) be a sequence in $A(X)$ that converges to some $y_0 \in X$. Because B is continuous,

$$\lim By_n = By_0. \qquad (1)$$

Write $y_n = Ax_n$ for some $x_n \in X$. Because $By_n = BAx_n = x_n$, (1) says that (x_n) converges to $x_0 = By_0$. Then $\lim y_n = \lim Ax_n = Ax_0$. Thus, $y_0 = Ax_0 \in A(X)$.

Next, we show that there exists a number $c > 0$ such that

$$\|Ax\| \geq c\|x\| \quad \forall x \in X. \qquad (2)$$

If $X = \{0\}$, (2) is true because both sides are zero. If $X \neq \{0\}$ then (2) is equivalent to

$$\|Ax\| \geq c \qquad \forall x \in X, \|x\| = 1.$$

Suppose by contradiction that there is no such c. Then for every $n \in \mathbb{N}$, there exists $x_n \in X$, $\|x_n\| = 1$ such that $\|Ax_n\| < \frac{1}{n}$. Because $Ax_n \to 0$ as $n \to \infty$,

$$x_n = BAx_n = B(Ax_n) \longrightarrow B(0) = 0 \quad \text{as } n \to \infty.$$

This is a contradiction because $\|x_n\| = 1$ for every $n \in \mathbb{N}$.

(iii) Let $A \in \mathcal{L}(X)$. We show that A has a left inverse if and only if A^* has a right inverse. For every $B \in \mathcal{L}(X)$,

$$(BAx, y) = (Ax, B^*y) = (x, A^*B^*y),$$

$$(B^*x, y) = \overline{(y, B^*x)} = \overline{(By, x)} = (x, By) \qquad \forall x, y \in X.$$

Thus, $(BA)^* = A^*B^*$ and $(B^*)^* = B$.

If A has a left inverse B then $A^*B^* = (BA)^* = I^* = I$; thus, A^* has a right inverse B^*. If A^* has a right inverse B then $B^*A = B^*(A^*)^* = (A^*B)^* = I^* = I$; thus, A has a left inverse B^*.

Next, suppose that A is self-adjoint and has a left inverse B. We show that A is invertible. As showed earlier, B^* is a right inverse of A^*. Since $A^* = A$, A has a right inverse. Because A has a left inverse and a right inverse, they must be the same according to Part (i). Therefore, A is invertible.

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(iv) Suppose X is finite dimensional. Let $A \in \mathcal{L}(X)$. We show that A is invertible if and only if A has a left inverse.

If A is invertible, it has a left inverse by definition. Now suppose A has a left inverse. We know by Part (ii) that A is injective. Thus $\ker A = \{0\}$.
We have $\dim X = \underbrace{\dim \ker A}_{=0} + \dim A(X) = \dim A(X)$.
Because $A(X)$ is a vector subspace of X and $\dim A(X) = \dim X < \infty$, $A(X) = X$.
Thus, A is surjective. Then A has a linear inverse A^{-1}. Because X is finite dimensional, A^{-1} is also continuous. Therefore, A is invertible in $\mathcal{L}(X)$.

5 Let X be a complex Banach space and $T: X \to X$ be a linear continuous operator. The definitions of Riesz points and Riesz operators are given in the statement of the problem. In the following, we write "$\lambda - T$" for the map $\lambda\, id_X - T$.

(a) Let λ and μ be two distinct Riesz points of T. By the definition of Riesz points,

$$X = Y_\lambda \oplus Z_\lambda = Y_\mu \oplus Z_\mu$$

where
$$\begin{cases} Y_\lambda, Y_\mu, Z_\lambda, Z_\mu \text{ are } T\text{-invariant subspaces of } X, \\ Z_\lambda \text{ and } Z_\mu \text{ are finite dimensional}, \\ (\mu - T)|_{Z_\mu} \text{ and } (\lambda - T)|_{Z_\lambda} \text{ are nilpotent operators}, \\ (\mu - T)|_{Y_\mu} \text{ and } (\lambda - T)|_{Y_\lambda} \text{ are } \cancel{\text{linear}} \text{ isomorphism onto } Y_\mu \text{ and } Y_\lambda \\ \qquad\qquad\qquad\qquad\qquad \text{respectively.} \end{cases}$$

We show that $Z_\mu \subset Y_\lambda$.

Suppose by contradiction that there exists $x \in Z_\mu \setminus Y_\lambda$. Because $X = Y_\lambda \oplus Z_\lambda$, $x = y + z$ for some $y \in Y_\lambda$, $z \in Z_\lambda$, $z \neq 0$. We know that there are $m, n \in \mathbb{N}$ such that $(\lambda - T)^m|_{Z_\lambda} = 0$ and $(\mu - T)^n|_{Z_\mu} = 0$. Since $x \in Z_\mu$, $(\mu - T)^n \cancel{x} = 0$. Thus,

$$(\mu - T)^n y + (\mu - T)^n z = 0 \qquad (1)$$

Because Y_λ is invariant under T, it is also invariant under $\mu - T$. The same is true for Z_λ. Thus, $(\mu - T)^n y \in Y_\lambda$ and $(\mu - T)^n z \in Z_\lambda$. Then (1) implies $(\mu - T)^n z \in Z_\lambda \cap Y_\lambda = \{0\}$. Thus, $(\mu - T)^n z = 0$.

On the other hand, $(\lambda - T)^m z = 0$ because $z \in Z_\lambda$. We know that $\mathbb{C}[t]$ is a principal entire ring. In this ring, $(\mu - t)^n$ and $(\lambda - t)^m$ are relatively prime because they have no common root. By a property of principal entire rings (Lang "Algebra", Proposition 5.1, page 112), there exists $f(t), g(t) \in \mathbb{C}[t]$ such that $f(t)(\mu - t)^n + g(t)(\lambda - t)^m = 1$. Then

$$\begin{aligned} z &= [f(T)(\mu - T)^n + g(T)(\lambda - T)^m] z \\ &= f(T)(\mu - T)^n z + g(T)(\lambda - T)^m z \\ &= 0. \end{aligned}$$

This is a contradiction.

Next, we show $X = Z_\lambda \oplus Z_\mu \oplus (Y_\lambda \cap Y_\mu)$. It suffices to show that $Y_\lambda = Z_\mu \oplus (Y_\lambda \cap Y_\mu)$. Because $Z_\mu \subset Y_\lambda$, $Z_\mu + (Y_\lambda \cap Y_\mu) \subset Y_\lambda$. For each $x \in Y_\lambda \subset X$, $x = y + z$ for some $y \in Y_\mu$ and $z \in Z_\mu$. Since $Z_\mu \subset Y_\lambda$, $z \in Y_\lambda$. Thus,

$y = x - z \in Y_\lambda$. Then $y \in Y_\lambda \cap Y_\mu$. Then

$$x = \underbrace{y}_{\in Y_\lambda \cap Y_\mu} + \underbrace{z}_{\in Z_\mu} \in (Y_\lambda \cap Y_\mu) + Z_\mu .$$

Hence, $Y_\lambda = Z_\mu + (Y_\lambda \cap Y_\mu)$. In addition, $Z_\mu \cap (Y_\lambda \cap Y_\mu) \subset Z_\mu \cap Y_\mu = \{0\}$.

Thus, the sum $Z_\mu + (Y_\lambda \cap Y_\mu)$ is a direct sum.

(b) Suppose T^m is a compact operator for some $m \in \mathbb{N}$. We show that T is a Riesz operator.

A theorem about the spectrum of a compact operator on a complex Banach space was mentioned in class on 10/10/2014. It implies that a compact operator is a Riesz operator. Thus, T^m is a Riesz operator. We first show that

$$\lambda^m \in \sigma(T^m) \quad \forall \lambda \in \sigma(T). \qquad (2)$$

Suppose this is not true, i.e. there exists $\lambda \in \sigma(T)$ such that $\lambda^m - T^m \in \mathcal{L}^{\times}(X)$.

Then

$$(\lambda - T)\underbrace{(\lambda^{m-1} + \lambda^{m-2}T + \cdots + \lambda T^{m-2} + T^{m-1})}_{B}(\lambda^m - T^m)^{-1} = (\lambda^m - T^m)(\lambda^m - T^m)^{-1} = id_X,$$

$$B(\lambda^m - T^m)^{-1}(\lambda - T) = (\lambda^m - T^m)^{-1} B (\lambda - T) = (\lambda^m - T^m)^{-1}(\lambda^m - T^m) = id_X .$$

Note that B and $(\lambda^m - T^m)^{-1}$ commute because B and $(\lambda^m - T^m)$ commute.

Thus, $B(\lambda^m - T^m)^{-1}$ is a continuous inverse of $\lambda - T$. This is a contradiction because $\lambda \in \sigma(T)$. We have proved (2).

Next, let $\lambda \in \sigma(T) \setminus \{0\}$. We show that λ is an isolated point of $\sigma(T)$. Suppose otherwise. Then there is a sequence (λ_n) in $\sigma(T) \setminus \{\lambda\}$ that converges to λ. Thanks to (2), $\lambda^m \in \sigma(T^m)$ and $\lambda_n^m \in \sigma(T^m)$ for all $n \in \mathbb{N}$. Because $\lambda^m \neq 0$, it is an isolated point of $\sigma(T^m)$. Because $\lim_{n\to\infty} \lambda_n^m = \lambda^m$, the sequence $(\lambda_n^m)_{n\in\mathbb{N}}$ has to be constant after some index. Write

$$\lambda_n^m = C \quad \forall n > N.$$

Then λ_n belongs to the set of m'th roots of C, which is a finite set. There is at least one m'th root of C that is equal to λ_n for infinitely many $n \in \mathbb{N}$. Thus, (λ_n) has a constant subsequence, say (λ_{n_k}). Because (λ_n) converges to λ, $\lambda_{n_k} = \lambda$ for all $k \in \mathbb{N}$. This is a contradiction because $\lambda_n \neq \lambda$ for all $n \in \mathbb{N}$.

Next, we find two subspaces Y and Z of X such that

$$\begin{cases} X = Y \oplus Z, & (3) \\ Y \text{ is closed in } X,\ Z \text{ is finite dimensional}, & (4) \\ Y \text{ and } Z \text{ are invariant under } T, & (5) \\ (\lambda - T)|_Y \in \mathcal{L}^\times(Y), & (6) \\ (\lambda - T)|_Z \text{ is nilpotent}. & (7) \end{cases}$$

Condition (5) is equivalent to that Y and Z are invariant under $\lambda - T$. For each polynomial $f(t) = \sum_{k=0}^{n} a_k t^k$ with complex coefficients, we denote $f(T) := \sum_{k=0}^{n} a_k T^k \in \mathcal{L}(X)$ with the convention $T^0 = id_X$. We observe that

$$\underbrace{f(T)\,g(T)}_{\substack{\text{composition of maps} \\ \text{in } \mathcal{L}(X)}} = \underbrace{(fg)}_{\substack{\text{multiplication} \\ \text{of polynomials}}}(T) = g(T)\,f(T) \qquad \forall f, g \in \mathbb{C}[t].$$

Put $f(t) = \lambda - t$, $g(t) = \lambda^m - t^m$, $h(t) = \lambda^{m-1} + \lambda^{m-2}t + \cdots + \lambda t^{m-2} + t^{m-1}$.

Then $g(t) = f(t)h(t)$.

Because λ^m is a spectral value of the compact operator T^m, there exists $n \in \mathbb{N}$ such that for $\mathcal{U} = \operatorname{Im} g^n(T)$, $V = \ker g^n(T)$,

$$\begin{cases} X = \mathcal{U} \oplus V, \\ \mathcal{U} \text{ is closed in } X,\ V \text{ is finite dimensional}, \\ \mathcal{U} \text{ and } V \text{ are invariant under } g(T), \\ g(T)|_{\mathcal{U}} \in \mathcal{L}^{\times}(\mathcal{U}). \end{cases}$$

We show that $\mathcal{U}$ and V are invariant under $f(T)$. Once this is proved, $\mathcal{U}$ and V are also invariant under T. For $u \in \mathcal{U}$, $u = g^n(T)u'$ for some $u' \in X$. Then

$$f(T)u = f(T)g^n(T)u' = g^n(T)f(T)u' \in \mathcal{U}.$$

Thus, $\mathcal{U}$ is invariant under $f(T)$. For $v \in V$, $g^n(T)v = 0$. Then

$$g^n(T)f(T)v = f(T)g^n(T)v = f(T)0 = 0.$$

This means $f(T)v \in \ker g^n(T) = V$. Thus, V is invariant under $f(T)$.

We see that $g(t)$ has no multiple root because $g(t)$ and $g'(t)$ has no common root. Let $\lambda_1, \lambda_2, \dots, \lambda_{m-1}$ be the roots of $g(t)$ beside λ. Because $g^n(T) = (-1)^{(m+1)n}(\lambda - T)^n(\lambda_1 - T)^n \cdots (\lambda_{m-1} - T)^n$ vanishes V, V is decomposed into T-invariant subspaces.

$$V = \underbrace{\ker[(\lambda - T)|_V^n]}_{Z} \oplus \underbrace{\ker[(\lambda_1 - T)|_V^n] \oplus \cdots \oplus \ker[(\lambda_{m-1} - T)|_V^n]}_{Z'}. \quad (8)$$

Then $X = \mathcal{U} \oplus V = \underbrace{\mathcal{U} \oplus Z'}_{Y} \oplus Z$.

Since V is finite dimensional, so are Z and Z'. Because $\mathcal{U}$ is closed in X and Z' is a finite dimensional subspace of X, $Y = \mathcal{U} \oplus Z'$ is closed in X. This property was taught in class on 9/8/2014. Because $\mathcal{U}$ and Z' are invariant under $f(T)$, so is Y. With the choice of Y and Z above, we get (3), (4), (5). By the definition of Z,

$$(\lambda - T)|_Z^n z = (\lambda - T)|_V^n z = 0 \quad \forall z \in Z.$$

Thus, $(\lambda - T)|_Z$ is nilpotent. We get (7). To get (6), we first assume $f(T)|_{\mathcal{U}} \in \mathcal{L}^\times(\mathcal{U})$ and $f(T)|_{Z'} \in \mathcal{L}^\times(Z')$. Define a map $\tilde{T} : Y \to Y$,

$$\tilde{T}x = f(T)|_{\mathcal{U}}^{-1} x_1 + f(T)|_{Z'}^{-1} x_2$$

where $x = x_1 + x_2$, $x_1 \in \mathcal{U}$, $x_2 \in Z'$. In a shorter notation, $\tilde{T} = f(T)|_{\mathcal{U}}^{-1} \oplus f(T)|_{Z'}^{-1}$.

As a linear map, $\tilde{T}$ is the inverse of $f(T)|_Y$. To show $\tilde{T}$ is continuous, we take a sequence (y_n) in Y that converges to 0 and show that $(\tilde{T}y_n)$ converges to 0. Write $y_n = u_n + z_n'$ with $u_n \in \mathcal{U}$ and $z_n' \in Z'$. Then

$$\tilde{T}y_n = \tilde{T}u_n + \tilde{T}z_n' = f(T)|_{\mathcal{U}}^{-1} u_n + f(T)|_{Z'}^{-1} z_n' . \qquad (9)$$

The direct sum $Y = \mathcal{U} \oplus Z'$ gives us a linear projection $P : Y \to Z'$. Its range is finite dimensional and its kernel is $\mathcal{U}$, which is closed in Y. By a result in class on 9/10/2014, P is continuous. Thus,

$$\lim z_n' = \lim P(y_n) = 0,$$

$$\lim u_n = \lim (y_n - z_n') = 0 - 0 = 0.$$

By (9) we get $\lim \tilde{T} y_n = 0$. Therefore, it remains to show $f(T)|_{\mathcal{U}} \in \mathcal{L}^{\times}(\mathcal{U})$ and $f(T)|_{Z'} \in \mathcal{L}^{\times}(Z')$.

By the decomposition (8), all eigenvalues of $f(T)|_{Z'}$ are $\lambda_1 - \lambda, \lambda_2 - \lambda, \dots, \lambda_{m-1} - \lambda$. None of them is zero. Thus, $\ker f(T)|_{Z'} = \{0\}$. Because Z' is finite dimensional, $f(T)|_{Z'}$ is a linear isomorphism. By the same reason, $f(T)|_{Z'}^{-1}$ is continuous. Thus, $f(T)|_{Z'} \in \mathcal{L}^{\times}(Z')$.

Next, we show $f(T)|_{\mathcal{U}} \in \mathcal{L}^{\times}(\mathcal{U})$. Because $f(t)$ is a polynomial of degree 1, $h(t)$ is a polynomial of $f(t)$. We proved earlier that $\mathcal{U}$ is invariant under $f(T)$, Then it is also invariant under $h(T)$. Because $g(T)|_{\mathcal{U}}$ and $h(T)|_{\mathcal{U}}$ commute, $g(T)|_{\mathcal{U}}^{-1}$ and $h(T)|_{\mathcal{U}}^{*}$ also commute. Then

$$f(T)|_{\mathcal{U}}\, h(T)|_{\mathcal{U}}\, g(T)|_{\mathcal{U}}^{-1} = g(T)|_{\mathcal{U}}\, g(T)|_{\mathcal{U}}^{-1} = id_{\mathcal{U}},$$

$$h(T)\, g(T)|_{\mathcal{U}}^{-1}\, f(T) = g(T)|_{\mathcal{U}}^{-1}\, h(T)|_{\mathcal{U}} f(T)|_{\mathcal{U}} = g(T)|_{\mathcal{U}}^{-1}\, g(T)|_{\mathcal{U}} = id_{\mathcal{U}}.$$

Therefore, $f(T)|_{\mathcal{U}}$ has an inverse in $\mathcal{L}(\mathcal{U})$.

(c) Let $\mathcal{K}(X) \subset \mathcal{L}(X)$ be the space of all compact operators on X. We show that

$$T \text{ is a Riesz operator} \iff \operatorname*{dist}_{n\to\infty} \{\operatorname{dist}(T^n, \mathcal{K}(X))\}^{1/n} = 0,$$

where the distance is taken in the normed space $\mathcal{L}(X)$. First, we prove the following lemma.

[Let $T \in \mathcal{L}(X)$ and $\lambda_1, \lambda_2, \dots, \lambda_m$ be distinct nonzero Riesz points of $\sigma(T)$. Then

$$X = Z_{\lambda_1} \oplus \cdots \oplus Z_{\lambda_m} \oplus \underbrace{(Y_{\lambda_1} \cap \dots \cap Y_{\lambda_m})}_{Y}. \qquad (10)$$

Let $P: X \to X$ be the linear projection on Y. Then

$$\sigma(T \circ P) \setminus \{0\} \subset \sigma(T) \setminus \{\lambda_1, \dots, \lambda_m\}$$]

We show (10) by induction on $m \in \mathbb{N}$. For $m = 1$, (10) is true by the definition of Riesz points. For $m=2$, (10) is true by Part (a). Suppose (10) is true for some $m \in \mathbb{N}$, $m \geq 2$. Consider $m+1$ distinct nonzero Riesz points $\lambda_1, \dots, \lambda_m, \lambda_{m+1}$ of $\sigma(T)$. By the induction hypothesis, $X = Z_{\lambda_1} \oplus \dots \oplus Z_{\lambda_m} \oplus Y$ where $Y = Y_{\lambda_1} \cap \dots \cap Y_{\lambda_m}$. It suffices to show $Y = Z_{\lambda_{m+1}} \oplus (Y \cap Y_{\lambda_{m+1}})$.

For each $1 \leq i \leq m$, we apply Part (a) for $\lambda = \lambda_i$ and $\mu = \lambda_{m+1}$. Then $Z_{\lambda_{m+1}} \subset Y_{\lambda_i}$. Thus, $Z_{\lambda_{m+1}} \subset Y_{\lambda_1} \cap \dots \cap Y_{\lambda_m} = Y$. This also gives us $Z_{\lambda_{m+1}} + (Y \cap Y_{\lambda_{m+1}}) \subset Y$. For each $y \in Y$, $y = z + y'$ for some $z \in Z_{\lambda_{m+1}}$ and $y' \in Y \cap Y_{\lambda_{m+1}}$. Since $Z_{\lambda_{m+1}} \subset Y$, $z \in Y$. Then $y' = y - z \in Y$. Then $y' \in Y \cap Y_{\lambda_{m+1}}$. Then $y \in Z_{\lambda_{m+1}} + (Y \cap Y_{\lambda_{m+1}})$. We get

$$Y = Z_{\lambda_{m+1}} + (Y \cap Y_{\lambda_{m+1}}).$$

Moreover, $Z_{\lambda_{m+1}} \cap (Y \cap Y_{\lambda_{m+1}}) \subset Z_{\lambda_{m+1}} \cap Y_{\lambda_{m+1}} = \{0\}$. Thus, the sum is a direct sum.

We show (11) directly, without using induction on m. Let $\lambda \in \sigma(T \circ P) \setminus \{0\}$.

$$T\circ P(x) = \begin{cases} 0 & \text{if } x \in Z, \\ Tx & \text{if } x \in Y, \end{cases}$$

where $Z = Z_{\lambda_1} \oplus \cdots \oplus Z_{\lambda_m}$. Then

$$(\lambda - T\circ P)(x) = \begin{cases} \lambda x & \text{if } x \in Z, \\ \lambda - Tx & \text{if } x \in Y. \end{cases} \qquad (12)$$

Suppose $\lambda \notin \sigma(T)$. Then $\lambda - T$ is invertible in $\mathcal{L}(X)$. Then $\lambda - T\circ P$ has a linear inverse

$$\tilde{T}x := (\lambda - T\circ P)^{-1}(x) = \begin{cases} \lambda^{-1}x & \text{if } x \in Z, \\ (\lambda - T)^{-1}x & \text{if } x \in Y. \end{cases}$$

Let $Q: X \to X$ be the linear projection on Z. Then $id_X = Q + P$, and

$$\tilde{T} = \lambda^{-1}Q + (\lambda - T)^{-1} \circ (id_X - Q).$$

The linear projection $Q_1 : X \to Z$ has finite dimensional range and closed kernel, which is Y. Thus, Q_1 is continuous. The inclusion map $Q_2 : Z \hookrightarrow X$ is also continuous. Thus, $Q = Q_2 \circ Q_1$ is continuous. Therefore, $\tilde{T}$ is continuous. This means $\lambda - T\circ P$ has a continuous inverse. This is a contradiction because $\lambda \in \sigma(T\circ P)$. We have showed $\lambda \in \sigma(T)$.

Suppose by contradiction that $\lambda \in \{\lambda_1, \ldots, \lambda_m\}$. We can assume $\lambda = \lambda_1$. Because λ_1 is a Riesz point of $\sigma(T)$, $R := (\lambda - T)|_{Y_1} \in \mathcal{L}^\times(Y_1)$. We know $Y \subset Y_1$. Then according to (12), $\lambda - T\circ P$ has a linear inverse

$$\tilde{T}x = \begin{cases} \lambda^{-1}x & \text{if } x \in Z, \\ R^{-1}x & \text{if } x \in Y. \end{cases}$$

The linear projection $P_1 : X \to Y$ is continuous because $P_1 x = Px = x - Qx$

for all $x \in X$. The inclusion map $P_2 : Y_1 \hookrightarrow X$ is also continuous. We have

$$\tilde{T} = \lambda^{-1} Q + P_2 \circ R^{-1} \circ P_1,$$

which is a continuous function. This is a contradiction because $\lambda \in \sigma(T \circ P)$.

We have proved the lemma. ∎

Return to the problem.

($\Longrightarrow$) Suppose T is a Riesz operator. Put $\delta_n = \{\mathrm{dist}(T^n, \mathcal{K}(X))\}^{1/n}$.

We want to show $\lim \delta_n = 0$. Consider three following cases.

- $\underline{\sigma(T) = \{0\}}$

The spectral radius of T is $\rho(T) = 0$. By Gelfand's formula,

$$0 = \rho(T) = \lim_{n \to \infty} \|T^n\|^{1/n}.$$

In addition,

$$0 \leq \mathrm{dist}(T^n, \mathcal{K}(X)) \leq \|T^n - 0\| = \|T^n\| \quad \forall n \in \mathbb{N},$$

which implies $0 \leq \delta_n \leq \|T^n\|^{1/n}$. Thus, $\lim \delta_n = 0$.

- $\underline{\sigma(T) \neq \{0\} \text{ and is finite}}$

Write $\sigma(T) \setminus \{0\} = \{\lambda_1, \dots, \lambda_m\}$. Because T is a Riesz operator, each λ_i is a Riesz point of $\sigma(T)$. By the lemma,

$$X = \underbrace{Z_{\lambda_1} \oplus \dots \oplus Z_{\lambda_m}}_{Z} \oplus \underbrace{(Y_{\lambda_1} \cap \dots \cap Y_{\lambda_m})}_{Y}.$$

Let $P : X \to X$ be the linear projection on Y. Then

$$\sigma(T \circ P) \setminus \{0\} \subset \sigma(T) \setminus \{\lambda_1, \dots, \lambda_m\} \subset \{0\}.$$

Thus, $\sigma(T\circ P)=\{0\}$ and $\rho(T\circ P)=\{0\}$. Let $Q_1: X\to Z$ be the projection on Z, $Q_2: Z\hookrightarrow X$ be the inclusion map and $Q=Q_2\circ Q_1=id_X-P$. Since Y is the intersection of closed sets of X, it is also closed. Because $\dim Z<\infty$ and $\ker Q_1=Y$ is closed in X, Q_1 is continuous. Because Q_1 is finite-rank, it is compact. Thus, Q is compact. Put $A=T\circ Q=T-T\circ P$. Then A is a compact operator.

We show by induction on $n\in\mathbb{N}$ that $(T-A)^n=T^n-A_n$, where $A_n\in\mathcal{K}(X)$. For $n=1$, $A_1=A\in\mathcal{K}(X)$. Suppose the claim is true for some $n\in\mathbb{N}$. Then

$$(T-A)^{n+1}=(T-A)^n(T-A)=(T^n-A_n)(T-A)$$
$$=T^{n+1}\underbrace{-A_nT-T^nA+A_nA}_{\in\mathcal{K}(X)}.$$

Thus, the claim is also true for $n+1$. Then

$$\operatorname{dist}(T^n,\mathcal{K}(X))\leq\|T^n-A_n\|=\|(T-A)^n\|=\|(T\circ P)^n\|.$$

This implies $\delta_n\leq\|(T\circ P)^n\|^{1/n}$. Letting $n\to\infty$ and using Gelfand's formula we get

$$\lim_{n\to\infty}\delta_n\leq\lim_{n\to\infty}\|(T\circ P)^n\|^{1/n}=\rho(T\circ P)=0.$$

Therefore, $\lim\delta_n=0$.

- <u>$\sigma(T)$ is infinite</u>

Because $\sigma(T)$ is bounded, it has at least one accumulation point. Because each point in $\sigma(T)\setminus\{0\}$ is an isolated point of $\sigma(T)$, $\sigma(T)$ is countable.

Because $\sigma(T)$ is closed, the accumulation point of $\sigma(T)$ belongs to $\sigma(T)$. This point could only be 0. Hence, we can write $\sigma(T) = \{0, \lambda_1, \lambda_2, \lambda_3, \dots\}$ where (λ_n) is a sequence of ~~distic~~ distinct nonzero complex numbers that converges to 0. For each $m \in \mathbb{N}$, by the lemma we have

$$X = \underbrace{Z_{\lambda_1} \oplus \cdots \oplus Z_{\lambda_m}}_{Z_m} \oplus \underbrace{(Y_{\lambda_1} \cap \cdots \cap Y_{\lambda_m})}_{Y_m}.$$

Let $P_m : X \to X$ be the linear projection on Y_m and $Q_m : X \to X$ be the linear projection on Z_m. By the lemma,

$$\sigma(T \circ P_m) \setminus \{0\} \subset \sigma(T) \setminus \{\lambda_1, \dots, \lambda_m\} = \{0, \lambda_{m+1}, \lambda_{m+2}, \dots\}$$

Thus, $\sigma(T \circ P_m) \subset \{\lambda_{m+1}, \lambda_{m+2}, \dots\}$. By the previous case, $T \circ Q_n = T - T \circ P_m$ is a compact operator. Also,

$$(T - T \circ Q_m)^n = T^n - A_{m,n}$$

where $A_{m,n} \in \mathcal{K}(X)$. Hence,

$$\operatorname{dist}(T^n, \mathcal{K}(X)) \le \| T^n - A_{m,n} \| = \| (T - T \circ Q_m)^n \| = \| (T \circ P_m)^n \|.$$

This implies $\delta_n \le \| (T \circ P_m)^n \|^{1/n}$. Letting $n \to \infty$ and using Gelfand's formula we get

$$\lim_{n\to\infty} \delta_n \le \lim_{n \to \infty} \| (T \circ P_m)^n \|^{1/n} = \rho(T \circ P_m) = \sup_{k > m} |\lambda_k|.$$

Because this is true for all $m \in \mathbb{N}$, we can take the limit of both sides as $m \to \infty$.

$$\lim_{n \to \infty} \delta_n \le \limsup_{k \to \infty} |\lambda_k| = 0$$

since $\lim \lambda_k = 0$. Therefore, $\lim \delta_n = 0$.

($\Longleftarrow$) First we prove the following lemma.

<u>Lemma 2</u> Let $A, B \in \mathcal{L}(X)$, $S = I - A - B$, where I denotes the identity map on X. Suppose A is compact and $\|B\| < \frac{1}{2}$. Then there are subspaces Y and Z of X such that

$$\begin{cases} X = Y \oplus Z, \\ Y \text{ is closed in } X,\ Z \text{ is finite dimensional}, \\ Y \text{ and } Z \text{ are invariant under } S, \\ S|_Y \in \mathcal{L}^{\times}(Y), \\ S|_Z \text{ is nilpotent.} \end{cases}$$

Our proof requires several steps as follows.

(i) $\ker S$ is finite dimensional.

(ii) $S(X)$ is closed and finite co-dimensional.

(iii) If $\ker S = \{0\}$ then $S : X \to S(X)$ has a continuous inverse.

(iv) There exists $n_0 \in \mathbb{N}$ such that $\ker S^{n_0} = \ker S^{n_0+1}$ and $S^{n_0}(X) = S^{n_0+1}(X)$.

(v) Put $Z = \ker S^{n_0}$ and $Y = S^{n_0}(X)$ where n_0 is the number found in step (iv). Then Y and Z satisfy the claim of the lemma.

<u>Proof of (i)</u>

As a vector subspace of X, $\ker S$ is a normed space. It is finite dimensional if and only if the closed unit ball $B_1(0) = \{x \in \ker S : \|x\| \le 1\}$ is compact. Let (x_n) be a sequence in $B_1(0)$. We show that (x_n) has a convergent subsequence. Because $0 = Sx_n = x_n - Ax_n - Bx_n$, $Ax_n = x_n - Bx_n$. Because (x_n) is bounded and A is compact, (Ax_n) has a convergent subsequence (Ax_{n_k}). By replacing (x_n) with (x_{n_k}), we can assume (Ax_n) converges. For each $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that $\|Ax_n - Ax_m\| < \varepsilon$ for all $m, n > N(\varepsilon)$. Thus,

$$\begin{aligned} \|x_n - x_m\| = \|Ax_n - Ax_m + B(x_n - x_m)\| &\leq \|Ax_n - Ax_m\| + \|B(x_n - x_m)\| \\ &< \varepsilon + \|B\| \, \|x_n - x_m\| \\ &\leq \varepsilon + \frac{1}{2}\|x_n - x_m\| \quad \forall m, n > N(\varepsilon). \end{aligned}$$

Hence, $\|x_n - x_m\| < 2\varepsilon$ for all $m, n > N(\varepsilon)$. This implies (x_n) is a Cauchy sequence. Because $\ker S = S^{-1}(\{0\})$ and S is continuous, $\ker S$ is closed in X. Because X is a Banach space, so is $\ker S$. Thus, the Cauchy sequence (x_n) converges in $\ker S$.

<u>Proof of (ii)</u>

Let (y_n) be a sequence ~~tha~~ in $S(X)$ that converges to $y_0 \in X$. We show $y_0 \in S(X)$. Consider two following cases.

- There is a bounded sequence (x_n) such that $y_n = Sx_n$.

Then (Ax_n) has a convergent subsequence (Ax_{n_k}). By replacing (x_n) with (x_{n_k}), we can assume (Ax_n) converges. Then by the same argument as in the proof of (i), we can show that (x_n) converges in X. Write $\lim x_n = x_0$. Because S is continuous, $\lim y_n = \lim Sx_n = Sx_0$. Thus, (y_n) converges to $y_0 = Sx_0 \in S(X)$.

- Every sequence (x_n) in X such that $y_n = Sx_n$ is unbounded.

We show that this case cannot happen. Suppose otherwise. Let (x_n) be such a sequence. Put $V = \ker S$. For each $n \in \mathbb{N}$, we can take $\tilde{x}_n = x_n - v_n$ for any $v_n \in V$ and get a sequence $(\tilde{x}_n)$ satisfying $y_n = S\tilde{x}_n$. The fact that (x_n) cannot be replaced by a bounded sequence implies that the sequence $\mathrm{dist}(x_n, V)$ is unbounded. By replacing (x_n) with a suitable subsequence, we can assume $\mathrm{dist}(x_n, V) \to \infty$ as $n \to \infty$.

Put $x_n' = \dfrac{x_n}{\mathrm{dist}(x_n, V)}$. Then $\mathrm{dist}(x_n', V) = \mathrm{dist}\left(\dfrac{x_n}{\mathrm{dist}(x_n, V)}, V\right) = \dfrac{\mathrm{dist}(x_n, V)}{\mathrm{dist}(x_n, V)} = 1.$

Put $y_n' = \dfrac{y_n}{\mathrm{dist}(x_n, V)}$. Because $y_n \to y_0$ and $\mathrm{dist}(x_n, V) \to \infty$, $y_n' \to 0$.

Because $\mathrm{dist}(x_n', V)$ is a bounded sequence, we can find a bounded sequence (x_n'') in X such that $Sx_n'' = Sx_n'$. Note that $x_n'' = x_n' - v_n$ for some $v_n \in V$. Hence, $\mathrm{dist}(x_n'', V) = \mathrm{dist}(x_n', V) = 1$. Then $y_n' = Sx_n'' = x_n'' - Ax_n'' - Bx_n''$.

Since (x_n'') is bounded, (Ax_n'') has a convergent subsequence (Ax_{n_k}''). By replacing (x_n'') with (x_{n_k}''), we can assume (Ax_n'') converges. Thus, $x_n'' - Bx_n'' = y_n' + Ax_n''$ converges.

$$\|x_n'' - x_m''\| \leq \|(x_n'' - Bx_n'') - (x_m'' - Bx_m'')\| + \underbrace{\|B(x_n'' - x_m'')\|}_{\leq \frac{1}{2}\|x_n'' - x_m''\|}$$

Hence, $\|x_n'' - x_m''\| \leq 2\|(x_n'' - Bx_n'') - (x_m'' - Bx_m'')\|$. This implies (x_n'') is a Cauchy sequence in X. Thus, (x_n'') converges to some $x_0 \in X$. Then

$Sx_0 = \lim Sx_n'' = \lim y_n' = 0$. Thus, $x_0 \in V$. On the other hand,

$$\mathrm{dist}(x_0, V) = \lim_{n\to\infty} \mathrm{dist}(x_n, V) = 1,$$

which implies $x_0 \notin V$. This is a contradiction. We have showed that $S(X)$ is closed in X.

Next, we show that $S(X)$ is finite-codimensional in X. Suppose otherwise. Then there are one-dimensional subspaces $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \dots$ of X such that we have a direct sum $S(X) \oplus \left(\bigoplus_{k=1}^{\infty} \mathcal{U}_k \right) \subset X$. Put $V_0 = S(X)$ and $V_n = V_{n-1} \oplus \mathcal{U}_n$ for all $n \geq 1$. Because $S(X)$ is closed and has finite codimension in V_n, V_n is also closed in X. This property was mentioned in class on 9/8/2014.

For each $n \in \mathbb{N}$ and $\alpha \in (0,1)$, we show that there exists $x_{n,\alpha} \in V_n$ such that $\|x_{n,\alpha}\| = 1$ and $\mathrm{dist}(x_{n,\alpha}, V_{n-1}) \geq \alpha$. Suppose this is not true for some $n \in \mathbb{N}, \alpha \in (0,1)$. Let $u \in \mathcal{U}_n$ be a unit vector. Each $x \in V_n$ is written as $x = v + cu$ where $v \in V_{n-1}$, $c \in \mathbb{C}$. Because V_{n-1} is closed in X and $u \notin V_{n-1}$, $\mathrm{dist}(u, V_{n-1}) > 0$. Take

$$c = \frac{\alpha}{\mathrm{dist}(u, V_{n-1})} > 0.$$

Then for every $v \in V_{n-1}$,

$$\mathrm{dist}(v+cu, V_{n-1}) = \mathrm{dist}(cu, V_{n-1}) = c\,\mathrm{dist}(u, V_{n-1}) = \alpha.$$

Thus, $\|v+cu\| \neq 1$ for all $v \in V_{n-1}$. The map $v \in V_{n-1} \mapsto \|v+cu\| \in \mathbb{R}$

is continuous. Hence, $\|v+cu\|>1$ for all $v\in V_{n-1}$, or $\|v+cu\|<1$ for all $v\in V_{n-1}$. Because

$$\|v+cu\| \geq \|v\|-\|cu\| \to \infty \text{ as } v\to\infty,$$

we have $\|v+cu\|>1$ for all $v\in V_{n-1}$. Replacing v by cv, we get

$$c\|v+u\|>1 \quad \forall v\in V_{n-1}.$$

Thus, $1\leq c\inf_{v\in V_{n-1}}\|v+u\| = c\,\mathrm{dist}(u,V_{n-1}) = \alpha$. This is a contradiction.

We have showed that for each $n\in\mathbb{N}$ and $\alpha\in(0,1)$, there exists $x_{n,\alpha}\in V_n$ such that $\|x_{n,\alpha}\|=1$ and $\mathrm{dist}(x_{n,\alpha},V_{n-1})\geq\alpha$.

Now take $\alpha=\|B\|+\frac{1}{2}\in(0,1)$ and denote $x_{n,\alpha}$ by x_n. Because (x_n) is a bounded sequence, (Ax_n) has a convergent subsequence (Ax_{n_k}). By replacing (x_n) with (x_{n_k}), (V_n) with (V_{n_k}), we can assume (Ax_n) converges. Note that this time V_{n-1} is still contained in V_n but may not have codimension 1 in V_n. We have

$$Ax_n+Bx_n = x_n - Sx_n.$$

Thus, $$Ax_{n+1}-Ax_n+B(x_{n+1}-x_n) = \underbrace{x_{n+1}}_{\in V_{n+1}} - \underbrace{x_n}_{\in V_n} - \underbrace{S(x_{n+1}-x_n)}_{\in S(X)\subset V_n}.$$

Hence, $\|Ax_{n+1}-Ax_n+B(x_{n+1}-x_n)\| \geq \mathrm{dist}(x_{n+1},V_n)\geq\alpha$.

Then $$\|Ax_{n+1}-Ax_n\| \geq \alpha-\|B(x_{n+1}-x_n)\| \geq \alpha-\|B\|\underbrace{\|x_{n+1}-x_n\|}_{\leq 2}$$
$$\geq \alpha-2\|B\|$$
$$=\frac{1}{2}-\|B\|>0.$$

This contradicts the fact that $\lim (Ax_{n+1} - Ax_n) = 0$.

Proof of (iii)

Suppose $\ker S = \{0\}$. Let (y_n) be a sequence in $S(X)$ that converges to $y_0 \in S(X)$. We show that $x_n = S^{-1}y_n$ converges to $x_0 = S^{-1}y_0$. First we show that (x_n) is bounded. Suppose otherwise. By replacing (x_n) with a suitable subsequence, we can assume $\|x_n\| \to \infty$ as $n \to \infty$. Put $x_n' = \frac{x_n}{\|x_n\|}$. Then (x_n') is a bounded sequence in X. Thus, (Ax_n') has a convergent subsequence (Ax_{n_k}'). By replacing (x_n') with (x_{n_k}'), we can assume (Ax_n') converges. We have

$$x_n' - Bx_n' = Sx_n' + Ax_n' = \underbrace{\frac{y_n}{\|x_n\|}}_{y_n'} + Ax_n'.$$

Because $y_n \to y_0$ and $\|x_n\| \to \infty$, $y_n' \to 0$. Thus, $x_n' - Bx_n'$ converges.

$$\begin{aligned} \|x_n' - x_m'\| &= \|(x_n' - Bx_n') - (x_m' - Bx_m') + B(x_n' - x_m')\| \\ &\leq \|(x_n' - Bx_n') - (x_m' - Bx_m')\| + \underbrace{\|B\|}_{<\frac{1}{2}} \|x_n' - x_m'\|. \end{aligned}$$

Hence, $\|x_n' - x_m'\| \leq 2\|(x_n' - Bx_n') - (x_m' - Bx_m')\|$. This implies (x_n') is a Cauchy sequence. Thus, (x_n') converges to $x_0' \in X$. Then

$$Sx_0' = \lim Sx_n' = \lim y_n' = 0.$$

Since S is injective, $x_0' = 0$. On the other hand, $\|x_0'\| = \lim \|x_n'\| = 1$. This is a contradiction. We have showed that (x_n) is bounded.

Suppose by contradiction that (x_n) does not converge to x_0. Then there exist a number $\varepsilon > 0$ and a subsequence (x_{n_k}) such that $\|x_{n_k} - x_0\| > \varepsilon$ for all $k \in \mathbb{N}$. Put $z_k = x_{n_k}$. Then $Sz_k = Sx_{n_k} = y_{n_k} \to y_0 = Sx_0$ as $k \to \infty$.

Because (z_k) is bounded, (Az_k) has a convergent subsequence (Az_{k_m}).

By replacing (z_k) with (z_{k_m}), we can assume (Az_k) converges.

$$z_k - Bz_k = \underbrace{Sz_k}_{\text{Converges}} + \underbrace{Az_k}_{\text{Converges}}.$$

Thus, $z_k - Bz_k$ converges.

$$\begin{aligned}\|z_k - z_m\| &= \|(z_k - Bz_k) - (z_m - Bz_m) + B(z_k - z_m)\| \\ &\leq \|(z_k - Bz_k) - (z_m - Bz_m)\| + \underbrace{\|B\|}_{\leq \frac{1}{2}} \|z_k - z_m\|.\end{aligned}$$

Thus, $\|z_k - z_m\| \leq 2\|(z_k - Bz_k) - (z_m - Bz_m)\|$. This implies (z_k) is a Cauchy sequence. Hence, (z_k) converges to some $z_0 \in X$. Then $Sz_0 = \lim Sz_k = y_0 = Sx_0$.

Thus, $z_0 = x_0$. We get $\lim z_k = x_0$. This contradicts the fact that $\|z_k - x_0\| > \varepsilon$ for all $k \in \mathbb{N}$.

Proof of (iv)

Both parts of the statement can be proved by using the following lemma.

Lemma 3 Let F be a normed space and E, $E \neq \{0\}$, $E \neq F$, be a closed subspace of F with finite codimension. For each $\alpha \in (0,1)$, there exists $x \in F$, $\|x\| = 1$ such that $\operatorname{dist}(x, E) \geq \alpha$.

The proof follows by the same method which we used in the bottom half of page 27 and the top half of page 28. Indeed, suppose Lemma 3 is not true for some $\alpha \in (0,1)$. Take $y \in F \setminus E$, $\|y\| = 1$. Because E is closed in F, $\mathrm{dist}(y, E) > 0$. Put $E' = E \oplus \mathbb{C}y$ and

$$c = \frac{\alpha}{\mathrm{dist}(y, E)} > 0.$$

For each $z \in E$, $\mathrm{dist}(z + cy, E) = \mathrm{dist}(cy, E) = c\,\mathrm{dist}(y, E) = \alpha$. Then $\|z + cy\| \neq 1$ for all $z \in E$. The map $z \in E \mapsto \|z + cy\| \in \mathbb{R}$ is continuous. Thus, either $\|z + cy\| > 1$ for all $z \in E$ or $\|z + cy\| < 1$ for all $z \in E$. We have

$$\|z + cy\| \geq \|z\| - \|cy\| = \|z\| - c \to \infty \quad \text{as } z \to \infty.$$

Thus, $\|z + cy\| > 1$ for all $z \in E$. Replacing z with cz, we get $c\|z + y\| > 1$ for all $z \in E$. Hence,

$$1 \leq c \inf_{z \in E} \|z + y\| = c\,\mathrm{dist}(y, E) = \alpha.$$

This is a contradiction. We have proved Lemma 3.

Return to the proof of (iv). For each $n \in \mathbb{N}$, we put $Z_n = \ker S^n$ and $Y_n = S^n(X)$. Then $Z_1 \subset Z_2 \subset Z_3 \subset \dots \subset X$ and $Y_1 \supset Y_2 \supset Y_3 \supset \dots$

First, we show by inducting on $n \in \mathbb{N}$ that each Z_n is finite dimensional. By Part (i), Z_1 is finite dimensional. Suppose $Z_n = V_0$ is finite dimensional for some $n \in \mathbb{N}$. Suppose by contradiction that $Z_{n+1} = W$ is infinite dimensional. Then there exist nonzero finite dimensional subspaces $U_1, U_2, U_3, \dots$ of W such that we have a direct sum $V_0 \oplus (\bigoplus_{k=1}^{\infty} U_k) \subset W$.

Put $V_k = V_{k-1} \oplus U_k$ for every $k \in \mathbb{N}$.

Because V_{k-1} is finite dimensional, it is closed in V_k. Put $\alpha = \frac{\|B\| + 1}{2} \in (0,1)$.

By Lemma 3, there exists $x_k \in V_k$, $\|x_k\| = 1$, $\operatorname{dist}(x_k, V_{k-1}) \geq \alpha$. Because (x_k) is a bounded sequence, (Ax_k) has a convergent subsequence. By replacing (x_k) with a suitable subspace, we can assume (Ax_k) converges.

$$Ax_k + Bx_k = x_k - Sx_k.$$

Thus, $Ax_{k+1} - Ax_k + Bx_{k+1} - Bx_k = x_{k+1} - x_k - S(x_{k+1} - x_k)$.

Because $x_{k+1} - x_k \in V_k \subset W = \ker S^{n+1}$, we have $S(x_{k+1} - x_k) \in \ker S^n = V_0 \subset V_k$.

Thus,

$$\| Ax_{k+1} - Ax_k + Bx_{k+1} - Bx_k \| \geq \operatorname{dist}(x_{k+1}, V_k) \geq \alpha.$$

Then

$$\begin{aligned} \|Ax_{k+1} - Ax_k\| &\geq \alpha - \|B(x_{k+1} - x_k)\| \geq \alpha - \|B\| \underbrace{\|x_{k+1} - x_k\|}_{\leq 2} \\ &\geq \alpha - 2\|B\| \\ &= \frac{1}{2} - \|B\| > 0 \qquad \forall k \in \mathbb{N}. \end{aligned}$$

This contradicts the fact that $\lim (Ax_{k+1} - Ax_k) = 0$. We have showed that each Z_n is finite dimensional.

Next, we show that there exists $n_1 \in \mathbb{N}$ such that $Z_{n_1} = Z_{n_1+1}$. Suppose by contradiction that there is no such n_1. Then $Z_1 \subsetneq Z_2 \subsetneq Z_3 \subsetneq \ldots$ Because each Z_{n-1} is finite dimensional, it is closed in Z_n. By Lemma 3, there exists $x_n \in Z_n$, $\|x_n\| = 1$, $\operatorname{dist}(x_n, V_{n-1}) \geq \alpha = \|B\| + \frac{1}{2}$. Because (x_n) is bounded, (Ax_n) has a convergent subsequence. By replacing (x_n) with a suitable subsequence, we can assume (Ax_n) converges.

$$Ax_n + Bx_n = x_n - Sx_n.$$

Thus,
$$Ax_{n+1} - Ax_n + Bx_{n+1} - Bx_n = x_{n+1} - x_n - S(x_{n+1} - x_n).$$

Because $x_{n+1} - x_n \in Z_n = \ker S^n$, we have $S(x_{n+1} - x_n) \in \ker S^n = Z_n$. Thus,

$$\| Ax_{n+1} - Ax_n + Bx_{n+1} - Bx_n \| \geqslant \operatorname{dist}(x_{n+1}, V_n) \geqslant \alpha.$$

Then
$$\begin{aligned} \| Ax_{n+1} - Ax_n \| &\geqslant \alpha - \| B(x_{n+1} - x_n) \| \geqslant \alpha - \|B\| \underbrace{\| x_{n+1} - x_n \|}_{\leq 2} \\ &\geqslant \alpha - 2\|B\| \\ &= \frac{1}{2} - \|B\| > 0 \quad \forall n \in \mathbb{N}. \end{aligned}$$

This contradicts the fact that $\lim (Ax_{n+1} - Ax_n) = 0$.

Next, we show by induction on $n \in \mathbb{N}$ that Y_n is closed and has finite codimension in X. By Part (ii), the claim is true for $n=1$. Suppose Y_n is closed and has finite codimension in X for some $n \in \mathbb{N}$. To show Y_{n+1} is closed and has finite codimension in X, it suffices to show Y_{n+1} is closed and has finite codimension in Y_n. The proof in the following is similar to the proof of Part (ii).

Let (y_k) be a sequence in Y_{n+1} that converges to $y_0 \in Y_n$. We show $y_0 \in Y_{n+1}$. Consider two following cases.

- There is a bounded sequence (x_k) in Y_n such that $y_k = Sx_k$.

Then (Ax_k) has a convergent subsequence. By replacing (x_k) with a suitable subsequence, we can assume (Ax_k) converges.

$$x_k = Sx_k + Ax_k + Bx_k = y_k + Ax_k + Bx_k.$$

Thus, $\|x_k - x_m\| \leq \|y_k - y_m\| + \|Ax_k - Ax_m\| + \underbrace{\|B(x_k - x_m)\|}_{\leq \frac{1}{2}\|x_k - x_m\|}$

Then $\|x_k - x_m\| \leq 2\|y_k - y_m\| + 2\|Ax_k - Ax_m\|$. This implies (x_k) is a Cauchy sequence in Y_n. Note that Y_n is a Banach space because it is closed in X.

Then (x_k) converges to some $x_0 \in Y_n$. Then

$$y_0 = \lim y_k = \lim Sx_k = Sx_0 \in Y_{n+1}.$$

■ Every sequence (x_k) in Y_n such that $y_k = Sx_k$ is unbounded.

We show that this case cannot happen. Let (x_k) be such a sequence. Put

$$\tilde{Z}_n = \ker(S|_{Y_n}) = \{x \in Y_n : Sx = 0\}.$$

For each $k \in \mathbb{N}$, we can take $\tilde{x}_k = x_k - z_k$ for any $z_k \in \tilde{Z}_n$, and get a sequence $(\tilde{x}_k)$ satisfying $y_k = S\tilde{x}_k$. The fact that (x_k) cannot be replaced by a bounded sequence implies that the sequence $\{\mathrm{dist}(x_k, \tilde{Z}_n)\}_{k\in\mathbb{N}}$ is unbounded. By replacing (x_k) with a suitable subsequence, we can assume $\mathrm{dist}(x_k, \tilde{Z}_n) \to \infty$ as $k \to \infty$. Put

$$x_k' = \frac{x_k}{\mathrm{dist}(x_k, \tilde{Z}_n)} \quad \text{and} \quad y_k' = \frac{y_k}{\mathrm{dist}(x_k, \tilde{Z}_n)}.$$

Then $\mathrm{dist}(x_k', \tilde{Z}_n) = \dfrac{\mathrm{dist}(x_k, \tilde{Z}_n)}{\mathrm{dist}(x_k, \tilde{Z}_n)} = 1$. Since $y_k \to y_0$ and $\mathrm{dist}(x_k, \tilde{Z}_n) \to \infty$, $y_k' \to 0$ as $k \to \infty$.

Because $\{\mathrm{dist}(x_k', \tilde{Z}_n)\}_{k\in\mathbb{N}}$ is a bounded sequence, we can find a bounded sequence (x_k'') in Y_n such that $Sx_k'' = Sx_k'$. Note that $x_k'' = x_k' - z_k$ for some $z_k \in \widetilde{Z}_n$. Hence, $\operatorname{dist}(x_k'', \widetilde{Z}_n) = \operatorname{dist}(x_k', \widetilde{Z}_n) = 1$. We have

$$y_k' = Sx_k'' = x_k'' - Ax_k'' - Bx_k''.$$

Since (x_k'') is bounded, (Ax_k'') has a convergent subsequence. By replacing (x_k'') with a suitable subsequence, we can assume (Ax_k'') converges. Thus,

$x_k'' - Bx_k' = y_k' + Ax_k''$ converges.

$$\|x_k'' - x_m''\| = \|(x_k'' - Bx_k'') - (x_m'' - Bx_m'')\| + \underbrace{\|B(x_k'' - x_m'')\|}_{\le \frac{1}{2}\|x_k'' - x_m''\|}$$

Hence, $\|x_k'' - x_m''\| \le 2\|(x_k'' - Bx_k'') - (x_m'' - Bx_m'')\|$. This implies (x_k'') is a Cauchy sequence in Y_n. Thus, (x_k'') converges to some $x_0 \in Y_n$. Then

$$Sx_0 = \lim Sx_k'' = \lim y_k' = 0.$$

Thus, $x_0 \in \widetilde{Z}_n$. On the other hand, $\operatorname{dist}(x_0, \widetilde{Z}_n) = \lim_{k\to\infty} \operatorname{dist}(x_k'', \widetilde{Z}_n) = 1$, which implies $x_0 \notin \widetilde{Z}_n$. This is a contradiction. We have showed that Y_{n+1} is closed in Y_n.

Next, we show that Y_{n+1} has finite codimension in Y_n. Suppose otherwise. Then there are nonzero finite-dimensional subspaces $U_1, U_2, U_3, \dots$ of Y_n such that we have a direct sum $Y_{n+1} \oplus \left(\bigoplus_{k=1}^{\infty} U_k\right) \subset Y_n$. Put $V_0 = Y_{n+1}$ and $V_k = V_{k-1} \oplus U_k$ for all $k \in \mathbb{N}$. Because V_0 is closed in X and has finite codimension in V_k, V_k is also closed in X. Thus, V_{k-1} is closed in V_k. By Lemma 3, there exists $x_k \in V_k$, $\|x_k\| = 1$, $\operatorname{dist}(x_k, V_{k-1}) \ge \alpha = \|B\| + \frac{1}{2}$.

Because (x_k) is bounded, (Ax_k) has a convergent subsequence. By replacing (x_k) with a suitable subsequence, we can assume (Ax_k) converges.

$$Ax_k + Bx_k = x_k - Sx_k$$

Thus, $Ax_{k+1} - Ax_k + B(x_{k+1} - x_k) = x_{k+1} - \underbrace{x_k}_{\in V_k} - \underbrace{S(x_{k+1} - x_k)}_{\cancel{\in S(X) = Y_1 \subset Y_n = V.}\ \in S(Y_n) = Y_{n+1} = V_0 \subset V_k}$

Thus, $\|Ax_{k+1} - Ax_k + B(x_{k+1} - x_k)\| \geq \operatorname{dist}(x_{k+1}, V_k) \geq \alpha.$

Then $\|Ax_{k+1} - Ax_k\| \geq \alpha - \|B(x_{k+1} - x_k)\| \geq \alpha - \|B\| \underbrace{\|x_{k+1} - x_k\|}_{\leq 2}$

$$\geq \alpha - 2\|B\|$$

$$= \frac{1}{2} - \|B\| > 0 \quad \forall k \in \mathbb{N}.$$

This contradicts the fact that $\lim (Ax_{k+1} - Ax_k) = 0$.

We have showed that Y_{n+1} has finite codimension in Y_n. Thereby, we have showed that Y_n is closed and has finite codimension in X for every $n \in \mathbb{N}$.

Now we show that there exists $n_2 \in \mathbb{N}$ such that $Y_{n_2} = Y_{n_2+1}$. Suppose by contradiction that there is no such n_2. Then $Y_1 \supsetneq Y_2 \supsetneq Y_3 \supsetneq \cdots$. Because each Y_{n+1} is closed in Y_n, by Lemma 3 there exists $x_n \in Y_n$, $\|x_n\| = 1$, $\operatorname{dist}(x_n, Y_{n+1}) \geq \alpha = \|B\| + \frac{1}{2}$. Because (x_n) is bounded, (Ax_n) has a convergent subsequence. By replacing (x_n) by a suitable subsequence, we can assume (Ax_n) converges.

$$Ax_n + Bx_n = x_n - Sx_n.$$

Thus, $Ax_{n+1} - Ax_n + B(x_{n+1} - x_n) = \underbrace{x_{n+1}}_{\in Y_{n+1}} - x_n - \underbrace{S(x_{n+1} - x_n)}_{\in S(Y_n) = Y_{n+1}}$.

Then $\| Ax_{n+1} - Ax_n + B(x_{n+1} - x_n) \| \geqslant dist(x_n, Y_{n+1}) \geqslant \alpha$.

Thus, $\| Ax_{n+1} - Ax_n \| \geqslant \alpha - \| B(x_{n+1} - x_n) \| \geqslant \alpha - \|B\| \underbrace{\|x_{n+1} - x_n\|}_{\leq 2}$

$$\geq \alpha - 2\|B\|$$

$$= \frac{1}{2} - \|B\| > 0.$$

This contradicts the fact that $\lim (Ax_{n+1} - Ax_n) = 0$.

Therefore, we get $n_1, n_2 \in \mathbb{N}$ such that $Z_{n_1+1} = Z_{n_1}$ and $Y_{n_2+1} = Y_{n_2}$. Then $Z_n = Z_{n_1}$ for all $n \geqslant n_1$, and $Y_n = Y_{n_2}$ for all $n \geqslant n_2$ by the definition of Z_n and Y_n. Take $n_0 = \max\{n_1, n_2\}$. We get

$$Z_n = Z_{n_0}, \quad Y_n = Y_{n_0} \quad \forall n \geqslant n_0.$$

Proof of (v)

Put $Z = Z_{n_0} = \ker S^{n_0}$ and $Y = Y_{n_0} = S^{n_0}(X)$. According to the proof of Part (iv), Z is finite dimensional and Y is closed in X with finite codimension. We show that Z and Y are invariant under S. For $z \in Z = \ker S^{n_0+1}$, $S^{n_0+1} z = 0$. Thus, $S^{n_0}(Sz) = 0$. This implies $Sz \in \ker S^{n_0} = Z$. Thus, $S(Z) \subset Z$. For $y \in Y = S^{n_0}(X)$, there is $x \in X$ such that $y = S^{n_0} x$. Then $Sy = S(S^{n_0} x) = S^{n_0}(Sx) \in S^{n_0}(X) = Y$. Thus, $S(Y) \subset Y$.

Next, we show $X = Y \oplus Z$. For each $x \in X$, $S^{n_0} x \in S^{n_0}(X) = Y = S^{2n_0}(X)$. Thus, there exists $x' \in X$ such that $S^{n_0} x = S^{2n_0} x'$. Then $S^{n_0}(x - S^{n_0} x') = 0$.

Put $y = S^{n_0}x' \in Y$ and $z = x - y$. We get $S^{k_0}z = 0$, i.e. $z \in \ker S^{n_0} = Z$. Hence, $X = Y + Z$. Take any $x \in Y \cap Z$. Since $x \in Y = S^{n_0}(X)$, $x = S^{n_0}x'$ for some $x' \in X$. Because $x \in Z$, $S^{n_0}x = 0$. Thus, $S^{2n_0}x' = 0$. Thus, $x' \in \ker S^{2n_0} = Z = \ker S^{n_0}$. Then $S^{n_0}x' = 0$, i.e. $x = 0$. Hence, $Y \cap Z = \{0\}$ and we get a direct sum $X = Y \oplus Z$.

Because $Z = \ker S^{n_0}$, $S^{n_0}(Z) = \{0\}$. Therefore, $S|_Z$ is a nilpotent operator. We now show that $S|_Y \in \mathcal{L}^\times(Y)$. First, we show $S|_Y$ is injective. Let $x \in Y$ be such that $Sx = 0$. Since $x \in Y = S^{n_0}(X)$, there exists $x' \in X$ such that $x = S^{n_0}x'$. Then $0 = Sx = S^{n_0}(Sx')$. This implies $Sx' \in Z$. Then $x = S^{n_0-1}(Sx') \in S^{n_0-1}(Z) \subset Z$. Thus $x \in Y \cap Z = \{0\}$. Next, we show $S|_Y : Y \to Y$ is surjective. For each $y \in Y = S^{n_0+1}(X)$, there exists $x \in X$ such that $y = S^{n_0+1}x$. Put $x' = S^{n_0}x \in Y$. Then $Sx' = y$.

Next, we show that $S|_Y$ has a continuous inverse. Let (y_n) be a sequence in Y that converges to 0. We show that $x_n = (S|_Y)^{-1}y_n$ converges to 0. First, we show that (x_n) is bounded. Suppose by contradiction that (x_n) is unbounded. By replacing (x_n) with a suitable subsequence, we can assume $\|x_n\| \to \infty$ as $n \to \infty$. Put $x_n' = \frac{x_n}{\|x_n\|}$. Since (x_n') is a bounded sequence, (Ax_n') has a convergent subsequence. By replacing (x_n') with a suitable subsequence, we can assume (Ax_n') converges. Put $y_n' = \frac{y_n}{\|x_n\|}$. Then $y_n' \to 0$ because $y_n \to 0$ and $\|x_n\| \to \infty$.

We have $x_n' - Bx_n' = Sx_n' + Ax_n' = y_n' + Ax_n'$. Thus, $x_n' - Bx_n'$ converges.

$$\|x_n' - x_m'\| \leq \|(x_n' - Bx_n') - (x_m' - Bx_m')\| + \underbrace{\|B(x_n' - x_m')\|}_{\leq \frac{1}{2}\|x_n' - x_m'\|}.$$

Hence, $\|x_n' - x_m'\| \leq 2\|(x_n' - Bx_n') - (x_m' - Bx_m')\|$.

This implies (x_n') is a Cauchy sequence in Y. Thus, (x_n') converges to $x_0' \in Y$.

Then $Sx_0' = \lim Sx_n' = \lim y_n' = 0$. Because $S|_Y$ is injective, $x_0' = 0$.

On the other hand, $\|x_0'\| = \lim \|x_n'\| = 1$. This is a contradiction. We have showed that (x_n) is bounded.

Suppose by contradiction that (x_n) does not converge to 0. Then there exist a number $\varepsilon > 0$ and a subsequence (x_{n_k}) such that $\|x_{n_k}\| > \varepsilon$ for all $k \in \mathbb{N}$. Put $z_k = x_{n_k}$. Then $Sz_k = Sx_{n_k} = y_{n_k} \to 0$ as $k \to \infty$. Because (z_k) is bounded, (Az_k) has a convergent subsequence. By replacing (z_k) with a suitable subsequence, we can assume (Az_k) converges. Thus, $z_k - Bz_k = $

$= Sz_k + Az_k$ converges as $k \to \infty$.

$$\|z_k - z_m\| = \|(z_k - Bz_k) - (z_m - Bz_m) + B(z_k - z_m)\|$$
$$\leq \|(z_k - Bz_k) - (z_m - Bz_m)\| + \underbrace{\|B(z_k - z_m)\|}_{\leq \frac{1}{2}\|z_k - z_m\|}$$

Thus, $\|z_k - z_m\| \leq 2\|(z_k - Bz_k) - (z_m - Bz_m)\|$. This implies (z_k) is a Cauchy sequence in Y. Hence, (z_k) converges to some $z_0 \in Y$. Then $Sz_0 = \lim Sz_k = 0$. Since $S|_Y$ is injective, $z_0 = 0$. We get $\lim z_k = 0$. This contradicts the fact that $\|z_k\| > \varepsilon$ for all $k \in \mathbb{N}$. Thus, we finish the proof of Lemma 2. $\blacksquare$

Return to the proof of ($\Longleftarrow$) in Part (c). Suppose $T \in \mathcal{L}(X)$ satisfies $\lim_{n\to\infty} \{\mathrm{dist}(T^n, K(X))\}^{\frac{1}{n}} = 0$. We show that T is a Riesz operator. Let $\lambda \in \sigma(T) \setminus \{0\}$. We find two subspaces Y and Z of X such that

$$\begin{cases} X = Y \oplus Z, \\ Y \text{ is closed in } X, \text{ and } Z \text{ is finite dimensional}, \\ Y \text{ and } Z \text{ are invariant under } T, \\ (\lambda - T)|_Y \in \mathcal{L}^{\times}(Y), \\ (\lambda - T)|_Z \text{ is nilpotent.} \end{cases} \qquad (*)$$

There exists $m \in \mathbb{N}$ such that $\{\mathrm{dist}(T^m, K(X))\}^{\frac{1}{m}} < \frac{|\lambda|}{2}$. Then there exists $A' \in K(X)$ such that $\|T^m - A'\|^{\frac{1}{m}} < \frac{|\lambda|}{2}$. Put $B' = T^m - A'$. Then $T^m = A' + B'$ and $\|B'\| < \left(\frac{|\lambda|}{2}\right)^m$.

$$\lambda^m - T^m = \lambda^m (I - \lambda^{-m} T^m) = \lambda^m (I - \lambda^{-m} A' - \lambda^{-m} B').$$

Put $A = \lambda^{-m} A'$, $B = \lambda^{-m} B'$ and $S = I - A - B$. Then A is a compact operator and

$$\|B\| = |\lambda|^{-m} \|B'\| < |\lambda|^{-m} \left(\frac{|\lambda|}{2}\right)^m = \frac{1}{2^m} \leq \frac{1}{2}.$$

By Lemma 2, there exist two subspaces $\mathcal{U}$ and V of X such that

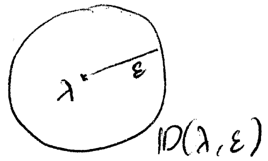
$$\begin{cases} X = \mathcal{U} \oplus V, \\ \mathcal{U} \text{ is closed in } X, \text{ and } V \text{ is finite dimensional}, \\ \mathcal{U} \text{ and } V \text{ are invariant under } S, \\ S|_{\mathcal{U}} \in \mathcal{L}^{\times}(\mathcal{U}), \\ S|_V \text{ is nilpotent.} \end{cases}$$

Put $f, g \in \mathbb{C}[t]$, $f(t) = \lambda - t$, $g(t) = \lambda^m - t^m$. Then $g(T) = \lambda^m S$. Then the above properties of $\mathcal{U}$ and V can be written as

$$\begin{cases} X = \mathcal{U} \oplus V, \\ \mathcal{U} \text{ is closed in } X, \text{ and } V \text{ is finite dimensional}, \\ \mathcal{U} \text{ and } V \text{ are invariant under } g(T), \\ g(T)|_{\mathcal{U}} \in \mathcal{L}^{\times}(\mathcal{U}), \\ g(T)|_V \text{ is nilpotent}. \end{cases} \qquad (**)$$

From now, we can get Y and Z satisfying $(*)$ by exactly the same arguments as in Part (b), pages 16–18. In Part (b), we get $\mathcal{U}$ and V satisfying $(**)$ by using the compactness of T^m. In this situation, we get $\mathcal{U}$ and V satisfying $(**)$ from Lemma 2. Once we get such $\mathcal{U}$ and V, the arguments for obtaining Y and Z sastisfying $(*)$ in both situations coincide. For a brief overview, the space Z is obtained by suitably decomposing $V = Z' \oplus Z$; then Y is chosen to be $\mathcal{U} \oplus Z'$. More specifically, V is invariant under T and $Z = \ker[(\lambda - T)|_V^n]$ for some $n \in \mathbb{N}$.

Next, we show that λ is an isolated point in $\sigma(T) \setminus \{0\}$. Put $T_1 = T|_Y \in \mathcal{L}(Y)$. Then $\lambda \notin \sigma(T_1)$. Because $\sigma(T_1)$ is closed in $\mathbb{C}$, there exists a number $\varepsilon > 0$ such that the disk $\mathbb{D}(\lambda, \varepsilon) \subset \mathbb{C}$ is disjoint with $\sigma(T_1)$.



Thus, $(\tilde{\lambda}-T)|_Y \in \mathcal{L}^\times(Y)$ for all $\tilde{\lambda} \in \mathbb{D}(\lambda,\varepsilon)$.

Put $T_2 = (\lambda - T)|_Z$. Because T_2 is a nilpotent operator on a finite dimensional space Z, all of its complex eigenvalues are equal to 0. Thus, if $a \in \mathbb{C}\setminus\{0\}$ then all eigenvalues of $a\cdot id_Z - T_2$ are equal to $a \neq 0$. This implies $a - T_2 \in \mathcal{L}^\times(Z)$. Hence $(\tilde{\lambda}-T)|_Z \in \mathcal{L}^\times(Z)$ for all $\tilde{\lambda} \neq \lambda$.

We get
$$\begin{cases} (\tilde{\lambda}-T)|_Y \in \mathcal{L}^\times(Y), \\ (\tilde{\lambda}-T)|_Z \in \mathcal{L}^\times(Z). \end{cases} \qquad \forall \tilde{\lambda} \in \mathbb{D}(\lambda,\varepsilon)\setminus\{\lambda\}.$$

Denote $P_1 : X \to Y$ and $Q_1 : X \to Z$ be the linear projection induced from the direct sum $X = Y \oplus Z$. Denote $P_2 : Y \hookrightarrow X$ and $Q_2 : Z \hookrightarrow X$ be the inclusion maps. Then $(\tilde{\lambda}-T)$ has a linear inverse.

$$(\tilde{\lambda}-T)^{-1} = P_2 \circ (\tilde{\lambda}-T)|_Y^{-1} \circ P_1 + Q_2 \circ (\tilde{\lambda}-T)|_Z^{-1} \circ Q_1.$$

By definition, P_2 and Q_2 are continuous. Because Q_1 has finite range and $\ker Q_1 = Y$ is closed in X, Q_1 is continuous. Since $P_1 = id_X - Q_1$, P_1 is also continuous. Thus, $(\tilde{\lambda}-T)^{-1}$ is continuous on X. This implies $\tilde{\lambda} - T \in \mathcal{L}^\times(X)$ for all $\tilde{\lambda} \in \mathbb{D}(\lambda,\varepsilon)\setminus\{\lambda\}$. Therefore, $(\mathbb{D}(\lambda,\varepsilon)\setminus\{\lambda\}) \cap \sigma(T) = \phi$.

6 Let X be a separable Hilbert space with Hilbert basis $\{e_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, we write $n = 2^\ell m$, where m is odd, and set $\alpha_n = \exp(-\ell)$.

~~Define a linear operator $A : X \to X$, $Ae_n = \alpha_n e_{n+1}$ for all $n \in \mathbb{N}$. Because $\alpha_n \in [0,1]$ for all $n \in \mathbb{N}$, A is continuous and $\|A\| \leq 1$. For each $k \in \mathbb{N}$, we define~~

Denote by Y the linear span of $\{e_1, e_2, e_3, \ldots\}$. We know that $X = \overline{Y}$.

Define a linear operator $A : Y \to Y$, $Ae_n = \alpha_n e_{n+1}$. For each $x \in Y$, we can write $x = \sum_{i=1}^{m} a_i e_i$. Thus,

$$\|Ax\|^2 = \left\| \sum_{i=1}^{m} a_i A e_i \right\|^2 = \left\| \sum_{i=1}^{m} a_i \alpha_i e_{i+1} \right\|^2 = \sum_{i=1}^{m} |a_i|^2 |\alpha_i|^2 \leq \sum_{i=1}^{m} |a_i|^2 \leq \|x\|^2.$$

Thus, A is continuous on Y. We can extend A to a unique linear continuous operator on X.

For each $k \in \mathbb{N}$, we define a linear operator $A_k : Y \to Y$,

$$A_k e_n = \begin{cases} 0 & \text{if } n = 2^k m \text{ with } m \text{ odd}, \\ Ae_n = \alpha_n e_{n+1} & \text{otherwise.} \end{cases}$$

We have
$$(A_k e_n, A_k e_m) = \begin{cases} 0 & \text{if } m \neq n \\ \leq |\alpha_n|^2 \leq 1 & \text{if } m = n. \end{cases}$$

For each $x = \sum_{i=1}^{m} a_i e_i \in Y$,

$$\|A_k x\|^2 = \left\| \sum_{i=1}^{m} a_i A_k e_i \right\|^2 = \sum_{i=1}^{m} |a_i|^2 \|A_k e_i\|^2 \leq \sum_{i=1}^{m} |a_i|^2 = \|x\|^2.$$

Thus, A_k is continuous on Y. We can extend A_k to a unique linear continuous operator on X.

(i) We show that $\rho(A) > 0$.

$$A^2 e_1 = A(\alpha_1 e_2) = \alpha_1 \alpha_2 e_3,$$

$$A^3 e_1 = A(\alpha_1 \alpha_2 e_3) = \alpha_1 \alpha_2 \alpha_3 e_4,$$

$$\cdots\cdots$$

$$A^n e_1 = \alpha_1 \alpha_2 \ldots \alpha_n e_{n+1},$$

$$\cdots\cdots$$

Thus, $\|A^n e_1\| = \alpha_1 \alpha_2 \cdots \alpha_n$. Take $n = 2^k - 1$. Then

$$\alpha_1, \alpha_2, \ldots, \alpha_n \in \{\exp(-j) : 0 \le j \le k\}.$$

We want to count the frequency that $\exp(-j)$ occurs in $\alpha_1, \alpha_2, \ldots, \alpha_n$. This is equal to the number of odd integers s such that $1 \le 2^j s < 2^k$. Such s lies in the set $\{1, 3, 5, \ldots, 2^{k-j} - 1\}$. This set has 2^{k-j-1} elements. Hence, $\exp(-j)$ occurs 2^{k-j-1} times in $\alpha_1, \alpha_2, \ldots, \alpha_n$. Thus,

$$\alpha_1 \alpha_2 \cdots \alpha_n = \prod_{j=0}^{k-1} (\exp(-j))^{2^{k-j-1}} = \exp\left(-\sum_{j=0}^{k-1} j 2^{k-j-1}\right) = \exp\left(-2^k \sum_{j=0}^{k-1} \frac{j}{2^{j+1}}\right).$$

Then $$\|A^n\|^{\frac{1}{n}} \geq \|A^n e_1\|^{\frac{1}{n}} = (\alpha_1 \alpha_2 \cdots \alpha_n)^{\frac{1}{n}} = \exp\left(-\frac{2^k}{2^k - 1} \sum_{j=0}^{k-1} \frac{j}{2^{j+1}}\right).$$

Letting $k \to \infty$ and using Gelfand's formula, we get

$$\rho(A) \geq \exp\left(-\sum_{j=0}^{\infty} \frac{j}{2^{j+1}}\right) > 0.$$

(ii) We show that each A_k is nilpotent and $\rho(A_k) = 0$.

By the definition of A_k,

$$A_k^j e_n = \begin{cases} 0 & \text{if none of } n, n+1, \ldots, n+j-1 \text{ is an odd number multiplied by } 2^k, \\ \alpha_n \alpha_{n+1} \cdots \alpha_{n+j-1} e_{n+j} & \text{otherwise.} \end{cases}$$

An integer of the form $2^k m$, where m is odd, occurs among 2^{k+1} adjacent integers. Thus, $A_k^{2^{k+1}} e_n = 0$ for every $n \in \mathbb{N}$. Thus, $A_k^{2^{k+1}} = 0$. We now show that every nilpotent operator has spectrum equal to $\{0\}$. Let $B \in \mathcal{L}(X)$ be a nilpotent operator. That is, $B^n = 0$ for some $n \in \mathbb{N}$. For each $\lambda \in \mathbb{C} \setminus \{0\}$,

$$(\lambda - B)(\lambda^{n-1} + \lambda^{n-2}B + \dots + \lambda B^{n-2} + B^{n-1}) = \lambda^n - B^n = \lambda^n I,$$

$$\underbrace{(\lambda^{n-1} + \lambda^{n-2}B + \dots + \lambda B^{n-2} + B^{n-1})}_{\tilde{B}}(\lambda - B) = \lambda^n - B^n = \lambda^n I.$$

Thus, $(\lambda - B)^{-1} = \lambda^{-n}\tilde{B} \in \mathcal{L}(X)$. Then $\lambda \in \mathbb{C} \setminus \sigma(B)$. This shows $\mathbb{C}\setminus\{0\} \subset \mathbb{C}\setminus\sigma(B)$. Thus, $\sigma(B) \subset \{0\}$. Because $\sigma(B) \neq \emptyset$, $\sigma(B) = \{0\}$. We have showed that $\rho(A_k) = 0$ for all $k \in \mathbb{N}$.

(iii) We show that $A_k \to A$ in $\mathcal{L}(X)$ as $k \to \infty$. By the definition of A_k,

$$(A - A_k)e_n = \begin{cases} Ae_n = \exp(-k)e_{n+1} & \text{if } n = 2^k s \text{ and } s \text{ is odd}, \\ 0 & \text{otherwise.} \end{cases}$$

Then
$$((A - A_k)e_n, (A - A_k)e_m) = \begin{cases} 0 & \text{if } m \neq n \\ \leq (\exp(-k))^2 & \text{if } m = n. \end{cases}$$

For each $x = \sum_{i=1}^{\infty} a_i e_i \in X$,

$$\|(A - A_k)x\|^2 = \left\|\sum_{i=1}^{\infty} a_i (A - A_k)e_i\right\|^2 = \sum_{i=1}^{\infty} |a_i|^2 \|(A - A_k)e_i\|^2$$
$$\leq (\exp(-k))^2 \sum_{i=1}^{\infty} |a_i|^2 = (\exp(-k))^2 \|x\|^2.$$

Thus, $\|A - A_k\| \leq \exp(-k)$. Therefore, $A_k \to A$ in $\mathcal{L}(X)$.