Name: Tuan Pham
ID: 4652218

Math 8801: Functional Analysis
Homework #3

① Denote $E = \{u \in C^2([\alpha,\beta]) : u(\alpha) = u(\beta) = 0\}$. Let $b \in C^1([\alpha,\beta])$ and let L be the differential operator on (α,β) given by $Lu = -u_{xx} + b(x)u_x$. We show that there exists a function $\rho \in C^2([\alpha,\beta])$, $\rho > 0$ on $[\alpha,\beta]$, such that

$$\int_\alpha^\beta (Lu)v\rho\,dx = \int_\alpha^\beta u(Lv)\rho\,dx \quad \forall u,v \in E. \qquad (1)$$

For any $u,v \in E$, $\rho \in C^2([\alpha,\beta])$,

$$\int_\alpha^\beta (Lu)v\rho\,dx = \int_\alpha^\beta (-u_{xx}+b(x)u_x)v\rho\,dx = -\int_\alpha^\beta u_{xx}v\rho\,dx + \int_\alpha^\beta b(x)u_x v\rho\,dx$$

$$= \underbrace{-(u_x v\rho)\Big|_\alpha^\beta}_{=0} + \int_\alpha^\beta u_x(v\rho)_x\,dx + \underbrace{(b(x)v\rho u)\Big|_\alpha^\beta}_{=0} - \int_\alpha^\beta (b(x)v\rho)_x u\,dx$$

$$= \underbrace{u(v\rho)_x\Big|_\alpha^\beta}_{=0} - \int_\alpha^\beta u(v\rho)_{xx}\,dx - \int_\alpha^\beta (b(x)v\rho)_x u\,dx$$

$$= \int_\alpha^\beta u\left[-(v\rho)_{xx} - (b(x)v\rho)_x\right]dx.$$

To get (1), it suffices to find $\rho \in C^2([\alpha,\beta])$, $\rho > 0$ on $[\alpha,\beta]$, such that

$$-(v\rho)_{xx} - (b(x)v\rho)_x = (Lv)\rho \quad \forall v \in E. \qquad (2)$$

We have

$$LHS(2) - RHS(2) = -(v_{xx}\rho + 2v_x\rho_x + v\rho_{xx}) - (b'(x)v\rho + b(x)v_x\rho + b(x)v\rho_x) - (-v_{xx} + b(x)v_x)\rho$$

$$= -v\left(\rho_{xx} + b'(x)\rho + b(x)\rho_x\right) - 2v_x\left(\rho_x + b(x)\rho\right)$$

$$= -v\left(\rho_x + b(x)\rho\right)_x - 2v_x\left(\rho_x + b(x)\rho\right).$$

To make this expression equal to 0 for all $v \in E$, it suffices that

$$\rho_x + b(x)\rho = 0 \quad \forall x \in [\alpha, \beta]. \qquad (3)$$

Put $B(x) = \int_0^x b(t)\,dt$. Since $b \in C^1([\alpha,\beta])$, $B \in C^2([\alpha,\beta])$. Multiplying both sides of (3) by $e^{B(x)}$, we get

$$\frac{d}{dx}\left[\rho(x)e^{B(x)}\right] = \left(\rho_x + b(x)\rho\right)e^{B(x)} = 0 \qquad \forall x \in [\alpha,\beta]$$

This gives $\rho(x)\,e^{B(x)} \equiv c$. We can choose $\rho(x) = e^{-B(x)}$. With this choice, $\rho \in C^2([\alpha,\beta])$ and $\rho > 0$ on $[\alpha,\beta]$.

Put $F = C([\alpha,\beta])$ and define $\|v\|_F = \left(\int_\alpha^\beta v(x)^2\,dx\right)^{1/2} \quad \forall v \in F.$

Then F is a normed space. Put $E = \{u \in C^2([\alpha,\beta]) : u(\alpha) = u(\beta) = 0\}$ and define $\|u\|_E = \left(\int_\alpha^\beta u'(x)^2\,dx\right)^{1/2}$. Then E is a normed space.

Let $f \in F$. We show that there exists unique $u \in E$ such that $Lu = f$. If $u \in E$ and $Lu = f$ then

$$-e^{-B(x)}f = e^{-B(x)}\left(-u_{xx} + b(x)u_x\right) = e^{-B(x)}u_{xx} - b(x)e^{-B(x)}u_x = \frac{d}{dx}\left[u_x e^{-B(x)}\right]. \qquad (4)$$

Integrating both sides over $[\alpha, x] \subset [\alpha,\beta]$, we get

$$u_x(x)e^{-B(x)} - \underbrace{u_x(\alpha)e^{-B(\alpha)}}_{c} = -\int_\alpha^x e^{-B(t)}f(t)\,dt$$

Thus, $u_x(x) = c\,e^{B(x)} - \int_\alpha^x e^{B(x)-B(t)}f(t)\,dt.$

Integrating both sides over $[\alpha,x] \subset [\alpha,\beta]$, we get

$$u(x) = c\int_\alpha^x e^{B(t)}dt - \int_\alpha^x\int_\alpha^t e^{B(t)-B(s)} f(s)\,ds\,dt \qquad (5)$$

For any choice of constant c, the function u given by this formula is in $C^2([\alpha,\beta])$ and satisfies $Lu=f$, $u(\alpha)=0$. The constraint $u(\beta)=0$ corresponds to a unique choice of c.

$$c = \frac{\int_\alpha^\beta\int_\alpha^t e^{B(t)-B(s)} f(s)\,ds\,dt}{\int_\alpha^\beta e^{B(t)}dt}.$$

Next, for $u\in E$, $f\in F$, $Lu=f$, we show that $\|u\|_E \le \theta\|f\|_F$ for some number $\theta>0$ independent of u and f. In other words, L, if viewed as a map from E to F, has a continuous inverse.

Since $u(\alpha)=u(\beta)=0$, there exists $\gamma\in(\alpha,\beta)$ such that $u_x(\gamma)=0$. Taking the integral of both sides of (4) from γ to x, we get

$$u_x(x)e^{-B(x)} = -\int_\gamma^x e^{-B(t)}f(t)\,dt.$$

Thus, $$u_x(x) = -\int_\gamma^x e^{B(x)-B(t)}f(t)\,dt. \qquad (6)$$

Note that $$B(x)-B(t) = \int_t^x b(s)\,ds \le (\beta-\alpha)M,$$

where $M=\sup_{s\in[\alpha,\beta]}|b(s)|$. Then (6) implies

$$|u_x(x)| \le \int_\alpha^\beta (\beta-\alpha)M|f(t)|\,dt \le M(\beta-\alpha)(\beta-\alpha)^{1/2}\left(\int_\alpha^\beta |f(t)|^2dt\right)^{1/2} = M(\beta-\alpha)^{3/2}\|f\|_F.$$

Thus, $\|u_x\|_F = \left(\int_\alpha^\beta u_x(x)^2 dx \right)^{1/2} \leq \left(\int_\alpha^\beta \left[M(\beta-\alpha)^{3/2} \|f\|_F \right]^2 dx \right)^{1/2}$

$$= M(\beta-\alpha)^2 \|f\|_F .$$

Then

$$\|u\|_E = \|u_x\|_F \leq \underbrace{M(\beta-\alpha)^2}_{\theta} \|f\|_F . \qquad (7)$$

Next, we show that L is invertible in sense that for every $f \in L^2(\alpha,\beta)$, there exists unique $u \in H_0^1(\alpha,\beta)$ such that

$$\int_\alpha^\beta (u_x v_x + b(x) u_x v)\, dx = \int_\alpha^\beta f v\, dx \quad \forall v \in H_0^1(\alpha,\beta). \qquad (8)$$

We will also see that the inverse map $L^{-1}: L^2(\alpha,\beta) \to H_0^1(\alpha,\beta)$ is continuous, i.e.

$$\|u\|_{H_0^1} = \left(\int_\alpha^\beta |u_x|^2 dx \right)^{1/2} \leq \theta \|f\|_{L^2}$$

for $\theta = M(\beta-\alpha)^2$.

Let $f \in L^2(\alpha,\beta)$. Because F is dense in $L^2(\alpha,\beta)$, there exists a sequence (f_n) in F such that $f_n \to f$. We showed earlier that there exists $u_n \in E$ such that $Lu_n = f_n$. Multiplying both sides of $-u_n'' + b(x) u_n' = f_n$ by $v \in \mathcal{D}(\alpha,\beta)$ and taking the integral over $[\alpha,\beta]$, we get

$$\int_\alpha^\beta (u_n' v' + b(x) u_n' v)\, dx = \int_\alpha^\beta f_n v\, dx \qquad (9)$$

Because $\mathcal{D}(\alpha,\beta)$ is dense in $H_0^1(\alpha,\beta)$, (9) holds for all $v \in H_0^1(\alpha,\beta)$. Since L is linear, $L(u_n - u_m) = f_n - f_m$. By the estimate (7),

$$\|u_n - u_m\|_{H_0^1} \leq \theta \|f_n - f_m\|_{L^2} .$$

Thus, (u_n) is a Cauchy sequence in $H_0^1(\alpha,\beta)$. Because $H_0^1(\alpha,\beta)$ is a Banach space, (u_n) converges to some $u \in H_0^1(\alpha,\beta)$. Letting $n\to\infty$ in both sides of (9), we get

$$\int_\alpha^\beta (u'v' + b(x)u'v)\,dx = \int_\alpha^\beta fv\,dx \quad \forall v \in H_0^1(\alpha,\beta).$$

Thus, u satisfies (8). Because $Lu_n = f_n$, $\|u_n\|_{H_0^1} \le \theta \|f_n\|_{L^2}$ for all $n \in \mathbb{N}$. Letting $n\to\infty$, we get $\|u\|_{H_0^1} \le \theta \|f\|_{L^2}$.

We now show the uniqueness of $u \in H_0^1(\alpha,\beta)$ satisfying (8). It suffices to show that if

$$\int_\alpha^\beta (u_x v_x + b(x)u_x v)\,dx = 0 \qquad \forall v \in H_0^1(\alpha,\beta) \qquad (10)$$

then $u = 0$. Let $\rho \in C^2([\alpha,\beta])$, $\rho(x) = e^{-B(x)}$ for all $x \in [\alpha,\beta]$. We know that $\rho_x + b(x)\rho = 0$. Since $u \in H_0^1(\alpha,\beta)$, $v = \rho u \in H_0^1(\alpha,\beta)$. Then

$$v_x + b(x)v = (\rho_x + b(x)\rho)u + \rho u_x = \rho u_x.$$

Substituting $v = \rho u$ into (10), we get

$$0 = \int_\alpha^\beta u_x(v_x + b(x)v)\,dx = \int_\alpha^\beta \rho u_x^2\,dx.$$

Because $\rho(x) > 0$ for all $x \in [\alpha,\beta]$ and is continuous, $\min_{x\in[\alpha,\beta]} \rho(x) = c > 0$. Then

$$\|u\|_{H_0^1}^2 = \int_\alpha^\beta u_x^2\,dx \le c^{-1}\int_\alpha^\beta \rho u_x^2\,dx = 0.$$

Thus, $u = 0$.

(2) Let X be a Hilbert space and $T \in \mathcal{L}(X)$ be self-adjoint. We show that T is compact if and only if some power of T is compact.

Suppose T is compact. We know that the composition of a compact operator and a continuous operator is a compact operator. Thus, for every $k \geq 2$, $T^k = T \circ T^{k-1}$ is compact.

Next, suppose T^m is compact for some $m \geq 2$. We show that T is also compact. Recall that for any $A \in \mathcal{L}(X)$, $(A^k)^* = (A^*)^k$. Indeed, for any $x, y \in X$,

$$(A^k x, y) = (A^{k-1}x, A^* y) = (A^{k-2}x, (A^*)^2 y) = \cdots = (Ax, (A^*)^{k-1} y) = (x, (A^*)^k y).$$

Because $2^m \geq m$, $T^{2^m} = T^m \circ T^{2^m - m}$ is a compact operator. We prove by backward induction in $k \in \{0, 1, \ldots, m\}$ that T^{2^k} is compact. If this is done, $T = T^{2^0}$ is compact.

We know T^{2^m} is compact. Suppose T^{2^k} is compact for some $1 \leq k \leq m$. We show that $T^{2^{k-1}}$ is also compact. Put $A = T^{2^{k-1}} \in \mathcal{L}(X)$. Then

$$A^* = (T^{2^{k-1}})^* = (T^*)^{2^{k-1}} = T^{2^{k-1}} = A.$$

Thus, $A^* A = A^2 = T^{2^k}$ is compact. By Problem 5, Part (b), of Homework#1 we conclude that A is compact.

3 Let $a : (\alpha, \beta) \to \mathbb{R}$ be a measurable function with $0 < c \leq a(x) \leq C < \infty$ almost everywhere in (α, β). Consider the eigenvalue problem

$$\begin{cases} -a(x) u'' = \lambda u \\ u(\alpha) = u(\beta) = 0 \end{cases}$$

Put $b(x) = \frac{1}{a(x)}$. Then $b : (\alpha, \beta) \to \mathbb{R}$ is a measurable function with $0 < C^{-1} \leq b(x) \leq c^{-1} < \infty$. The eigenvalue problem is equivalent to

$$\begin{cases} -u'' = \lambda b(x) u, \\ u(\alpha) = u(\beta) = 0. \end{cases} \qquad (I)$$

We interpret (I) as the problem of finding $u \in H_0^1(\alpha,\beta)$ satisfying

$$\int_\alpha^\beta u'v'\,dx = \lambda \int_\alpha^\beta b(x)\,uv\,dx \quad \forall v \in H_0^1(\alpha,\beta). \qquad (1)$$

First, we show that there are only countably many $\lambda \in \mathbb{R}$ such that (1) has a nontrivial solution. Further properties of λ and u will be showed after we finish this step.

Consider the problem
$$\begin{cases} -u'' = b(x) f \\ u(\alpha) = u(\beta) = 0, \end{cases} \qquad (\mathrm{II})$$

where $f \in L^2(\alpha,\beta)$. We interpret (II) as the problem of finding $u \in H_0^1(\alpha,\beta)$ satisfying
$$\int_\alpha^\beta u'v'\,dx = \int_\alpha^\beta b(x) f v\,dx \quad \forall v \in H_0^1(\alpha,\beta). \qquad (2)$$

Recall that $H_0^1(\alpha,\beta)$ is a Hilbert space with inner product $(u,v) = \int_\alpha^\beta u'v'\,dx$.

For each $f \in L^2(\alpha,\beta)$, the map $v \in H_0^1(\alpha,\beta) \mapsto \int_\alpha^\beta b(x) f v\,dx \in \mathbb{R}$ is linear continuous. By Riesz's representation theorem, there exists unique $u \in H_0^1(\alpha,\beta)$ such that
$$(u,v) = \int_\alpha^\beta b(x) f v \quad \forall v \in H_0^1(\alpha,\beta).$$

Thus, (2) has a unique solution. Define the solution operator $G: L^2(\alpha,\beta) \to H_0^1(\alpha,\beta)$, $Gf = u$. In other words,
$$\int_\alpha^\beta (Gf)'v'\,dx = \int_\alpha^\beta b(x) f v\,dx \quad \forall v \in H_0^1(\alpha,\beta)\ \forall f \in L^2(\alpha,\beta). \qquad (3)$$

It follows from (3) that G is linear. Moreover,

$$\|u\|_{H_0^1}^2 = (u,u) = \int_\alpha^\beta b(x) f u \le c^{-1}\|f\|_{L^2}\|u\|_{L^2} \le c^{-1}\theta\|f\|_{L^2}\|u\|_{H_0^1},$$

where θ depends only on α and β. Then $\|u\|_{H_0^1} \le c^{-1}\theta\|f\|_{L^2}$. This implies G is continuous.

We know that the injection map $i: H_0^1(\alpha,\beta) \hookrightarrow L^2(\alpha,\beta)$ is a compact operator. View G as the map $i\circ G: L^2(\alpha,\beta)\to L^2(\alpha,\beta)$. Then G is is a compact operator.

Define
$$((f,g)) = \int_\alpha^\beta b(x) f(x) g(x)\,dx \quad \forall f, g \in L^2(\alpha,\beta).$$
This is a well-defined bilinear form on $L^2(\alpha,\beta)$ because $C^{-1}\le b(x)\le c^{-1}$ almost everywhere in (α,β). This form is also symmetric.

$$((f,f)) = \int_\alpha^\beta b(x) f(x)^2\,dx \begin{cases} \le c^{-1}\int_\alpha^\beta f(x)^2\,dx \\ \ge C^{-1}\int_\alpha^\beta f(x)^2\,dx. \end{cases}$$

Thus,
$$C^{-1}\|f\|_{L^2}^2 \le ((f,f)) \le c^{-1}\|f\|_{L^2}^2 \quad \forall f\in L^2(\alpha,\beta) \qquad (4)$$
This shows that $((\cdot,\cdot))$ is positive definite. Thus, $((\cdot,\cdot))$ is an inner product on $L^2(\alpha,\beta)$. The estimate (4) also implies that the norm on $L^2(\alpha,\beta)$ given by $((\cdot,\cdot))$ is equivalent to the norm given by the usual inner product $(\cdot,\cdot)_{L^2}$. Since the latter norm space is complete, so is the former one. Thus, $(L^2(\alpha,\beta), ((\cdot,\cdot)))$ is a Hilbert space. Another consequence of the equivalence of the usual norm and new norm on $L^2(\alpha,\beta)$ is that G is a compact operator on $(L^2(\alpha,\beta), ((\cdot,\cdot)))$.

Denote by X the Hilbert space $(L^2(\alpha,\beta),((.,.)))$. The inner product on X is thus

$$(f,g)_X = \int_\alpha^\beta b(x) f g \, dx \quad \forall f,g \in X.$$

We show that G is a self-adjoint operator on X. Let $f,g \in X$.

$$(Gf,g)_X = \int_\alpha^\beta b(x)(Gf)g\,dx \overset{(3)}{=} \int_\alpha^\beta (Gf)'(Gg)'\,dx$$

$$= \int_\alpha^\beta (Gg)'(Gf)'\,dx \overset{(3)}{=} \int_\alpha^\beta b(x)(Gg)f\,dx = (f,Gg)_X.$$

Thus, G is self-adjoint.

By the spectral theorem for compact self-adjoint operators on Hilbert spaces, we have the following statements.

(i) G has countably many spectral values, i.e. $\sigma(G)$ is countably infinite or finite.

(ii) Each spectral value is a real eigenvalue.

(iii) The only possible accumulation point of $\sigma(G)$ is 0.

Next, we show that $(Gf,f)_X > 0$ for all $f \in X\setminus\{0\}$. Let $f \in X$.

$$(Gf,f)_X = \int_\alpha^\beta b(x)(Gf)f \overset{(3)}{=} \int_\alpha^\beta (Gf)'(Gf)' = \int_\alpha^\beta [(Gf)']^2\,dx \geq 0. \quad (5)$$

Suppose $(Gf,f)=0$. Then (5) implies $(Gf)'=0$ almost everywhere in (α,β). Thus, Gf is equal to a constant function almost everywhere in (α,β). Then (3) implies

$$\int_\alpha^\beta b(x) f v\,dx = 0 \qquad \forall v \in H_0^1(\alpha,\beta).$$

In other words, $(f,v)_X = 0$ for all $v \in H_0^1(\alpha,\beta)$. We know that $H_0^1(\alpha,\beta)$ is dense in $L^2(\alpha,\beta)$ with the usual norm. Because of the equivalence of the usual norm and new norm on $L^2(\alpha,\beta)$, $H_0^1(\alpha,\beta)$ is also dense in X. Thus, $(f,v)_X = 0$ for all $v \in X$. By taking $v=f$, we conclude that $f=0$. We have showed $(Gf,f) > 0$ for all $f \in X\setminus\{0\}$. This implies all eigenvalues of G are positive numbers.

Next, we show that the eigenspace $E(\lambda')$ of each eigenvalue λ' of G is one-dimensional. Let $u \in E(\lambda')$. We first show that $u \in H^2(\alpha,\beta)$ and satisfies the equation $-u'' = \lambda b(x) u$, with $\lambda = \frac{1}{\lambda'}$, almost everywhere in (α,β).
Because $Gu = \lambda' u$, by replacing f with u in (3), we get

$$\int_\alpha^\beta u'v'\,dx = \int_\alpha^\beta \lambda b(x) u v\,dx \qquad \forall v \in H_0^1(\alpha,\beta). \qquad (6)$$

Put $w(x) = \int_\alpha^x \lambda b(t) u(t)\,dt$. Because $\lambda C^{-1} \le \lambda b(x) \le \lambda c^{-1}$ for almost every $t \in [\alpha,\beta]$ and $u \in L^2(\alpha,\beta) \subset L^1(\alpha,\beta)$, the integrand $\lambda b(t) u(t) \in L^2(\alpha,\beta) \subset L^1(\alpha,\beta)$. Thus, w is continuous on $[\alpha,\beta]$. In particular, $w \in L^2(\alpha,\beta)$. Moreover, w has generalized derivative $w'(x) = \lambda b(x) u(x) \in L^2(\alpha,\beta)$. Thus, $w \in H^1(\alpha,\beta)$.

$$\int_\alpha^\beta u'v'\,dx = RHS(6) = \int_\alpha^\beta w'v\,dx = wv\Big|_\alpha^\beta - \int_\alpha^\beta wv'\,dx = -\int_\alpha^\beta wv'\,dx \qquad \forall v \in H_0^1(\alpha,\beta).$$

Then $$\int_\alpha^\beta (u'+w)v'\,dx = 0 \qquad \forall v \in H_0^1(\alpha,\beta).$$

According to a lemma in the lecture on 11/07/2014, $u'+w$ is equal to a constant function almost everywhere in (α,β). Since $w \in H^1(\alpha,\beta)$, $u' \in H^1(\alpha,\beta)$ and $u'' = -w' = -\lambda b(x) u(x)$ almost everywhere in (α,β). We have showed that $u \in H^2(\alpha,\beta) \cap H_0^1(\alpha,\beta)$ and

$$-u'' = \lambda b(x)\, u(x) \qquad \text{a.e. in } (\alpha,\beta). \qquad (7)$$

The second step in showing $\dim E(\lambda') = 1$ is to show that if $u \in E(\lambda')$ and $u(\gamma) = u'(\gamma) = 0$ for some $\gamma \in [\alpha,\beta]$ then $u = 0$ in $[\alpha,\beta]$.

Let $u \in E(\lambda')$ such that $u(\gamma) = u'(\gamma) = 0$ for some $\gamma \in [\alpha,\beta]$. Integrating (7) from γ to x, we get

$$-u'(x) = \int_\gamma^x \lambda b(t)\, u(t)\, dt.$$

Integrating both sides from γ to x, we get

$$-u(x) = \int_\gamma^x \int_\gamma^t \lambda b(s)\, u(s)\, ds\, dt \overset{\text{Fubini}}{=\!=} \int_\gamma^x \int_s^x \lambda b(s)\, u(s)\, dt\, ds = \int_\gamma^x (x-s)\lambda b(s) u(s) ds.$$

Thus, $|u(x)| = \left| \int_\gamma^x (x-s)\, \lambda b(s)\, u(s) ds \right| \leq \left| \int_\gamma^x |x-s|\, \lambda c^{-1}\, |u(s)|\, ds \right|$

$$\leq (\beta-\alpha)\lambda c^{-1} \left| \int_\gamma^x |u(s)|\, ds \right| \quad \forall x \in [\alpha,\beta].$$

By Gronwall's inequality (Lemma 6.1, Amann "Ordinary Differential Equations" 1990, page 89), we get $|u(x)| = 0$ for every $x \in [\alpha,\beta]$. Thus, $u = 0$.

A consequence of the above result is that if $u \in E(\lambda') \setminus \{0\}$ then it has only finitely many zeros in $[\alpha,\beta]$. Indeed, suppose otherwise. Then the set of zeros of u has an accumulation point x_0. There is a sequence (x_n) in $(\alpha,\beta)\setminus\{x_0\}$ of zeros of u that converges to x_0. Because $u \in H^1(\alpha,\beta) \hookrightarrow C([\alpha,\beta])$,

$$u(x_0) = \lim_{n\to\infty} u(x_n) = 0,$$

$$u'(x_0) = \lim_{n\to\infty} \frac{u(x_n) - u(x_0)}{x_n - x_0} = 0.$$

This implies $u = 0$ on $[\alpha,\beta]$, which is a contradiction.

Next, let $u_1, u_2 \in E(\lambda')\setminus\{0\}$. We show that u_1 and u_2 are scalar multiples of each other.

$$-u_1'' = \lambda b(x)\, u_1 \qquad \text{a.e. in } (\alpha,\beta)$$

$$-u_2'' = \lambda b(x)\, u_2 \qquad \text{a.e. in } (\alpha,\beta)$$

Thus, $u_1''u_2 - u_1u_2'' = -\lambda b(x)u_1u_2 - (-\lambda b(x)u_1u_2) = 0$ almost everywhere in (α,β). For every $\varphi \in \mathcal{D}(\alpha,\beta)$,

$$\begin{aligned}
\int_\alpha^\beta (u_1'u_2 - u_1u_2')\varphi'\,dx &= \int_\alpha^\beta u_1'(u_2\varphi' + u_2'\varphi)\,dx - \int_\alpha^\beta u_2'(u_1\varphi' + u_1'\varphi)\,dx \\
&= \int_\alpha^\beta u_1'(u_2\varphi)'\,dx - \int_\alpha^\beta u_2'(u_1\varphi)'\,dx \\
&= -\int_\alpha^\beta u_1'' u_2\varphi\,dx + \int_\alpha^\beta u_2'' u_1\varphi\,dx \\
&= -\int_\alpha^\beta \underbrace{(u_1''u_2 - u_2''u_1)}_{=0}\varphi\,dx \quad = 0.
\end{aligned}$$

Thus, $u_1'u_2 - u_1u_2'$ is equal to a constant function almost everywhere in (α,β). It is, in fact, a constant function because it is continuous. Since $u_1(\alpha) = u_2(\alpha) = 0$, $u_1'u_2 - u_1u_2' = 0$ in $[\alpha,\beta]$.

We know that u_1 has finitely many zeros in $[\alpha,\beta]$, say $\alpha = x_1 < x_2 < \dots < x_n = \beta$. On the interval (x_1,x_2), $u_1 \neq 0$. Then

$$0 = \frac{u_1'u_2 - u_1u_2'}{u_1^2} = -\left(\frac{u_2}{u_1}\right)'.$$

Thus, $\frac{u_2}{u_1}$ is a constant function c_0 on (x_1,x_2). Then $u = u_2 - c_0u_1$ is a member of $E(\lambda')$ that vanishes in (x_1, x_2). Since nonzero members of $E(\lambda')$ has only finitely many zeros in $[\alpha,\beta]$, we conclude that $u=0$ in $[\alpha,\beta]$. Thus, $u_2 = c_0 u_1$ in $[\alpha,\beta]$. We have showed that $\dim E(\lambda') = 1$.

By the spectral theorem for compact self-adjoint operators on Hilbert spaces, $X = L^2(\alpha,\beta)$ is the Hilbert direct sum of the eigenspaces of eigenvalues of G. Since each ~~eigenvalue~~ eigenspace is one-dimensional and X is infinite dimensional, there are infinitely many eigenspaces. Thus, the eigenvalues of G is a sequence of distinct positive numbers that converges to zero. We call them $\lambda_1', \lambda_2', \lambda_3', \dots$ with $\lambda_1' > \lambda_2' > \lambda_3' > \cdots > 0$. Put $\lambda_j = \frac{1}{\lambda_j'}$. Then $0 < \lambda_1 < \lambda_2 < \cdots$ and $\lim_{j\to\infty} \lambda_j = \infty$. Take $u_j \in E(\lambda_j')$ with $\|u_j\|_X = 1$. Then

$$X = \bigoplus_{j=1}^{\infty} E(\lambda_j') \qquad \text{(Hilbert direct sum)}$$
$$= \bigoplus_{j=1}^{\infty} \mathbb{R} u_j \qquad \text{(Hilbert direct sum)}.$$

4 Consider a bilinear form $B : H^1(\alpha,\beta) \times H^1(\alpha,\beta) \to \mathbb{R}$, $B(u,v) = \int_\alpha^\beta u'v'\,dx$. For $a, b \in \mathbb{R}$, let $l : H^1(\alpha,\beta) \to \mathbb{R}$ be a linear map given by $l(v) = a\,v(\alpha) + b\,v(\beta)$. We show that the problem of finding $u \in H^1(\alpha,\beta)$ satisfying

$$B(u,v) = l(v) \qquad \forall v \in H^1(\alpha,\beta) \qquad (1)$$

is solvable if and only if $a+b=0$.

Suppose u solves Problem (1), i.e.

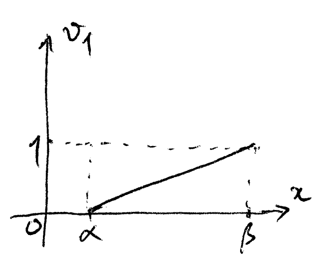
$$\int_\alpha^\beta u'v'\,dx = av(\alpha)+bv(\beta) \quad \forall v\in H^1(\alpha,\beta). \qquad (2)$$

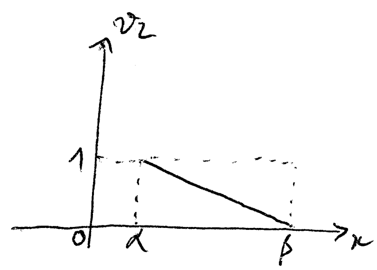
Then $$\int_\alpha^\beta u'v'\,dx = 0 \quad \forall v\in \mathcal{D}(\alpha,\beta). \qquad (3)$$

Since $u\in H^1(\alpha,\beta)$, $u'\in L^2(\alpha,\beta)\subset L^1_{loc}(\alpha,\beta)$. By a lemma in the lecture on 11/07/2014, we conclude from (3) that u' is equal to a constant function c almost everywhere in (α,β). Then (2) becomes

$$av(\alpha)+bv(\beta) = \int_\alpha^\beta cv'\,dx = c(v(\beta)-v(\alpha)) \qquad \forall v\in H^1(\alpha,\beta).$$

Then $$(a+c)\,v(\alpha)+(b-c)v(\beta) \quad \forall v\in H^1(\alpha,\beta). \qquad (4)$$





Let v_1 and v_2 be affine functions on $[\alpha,\beta]$ with $v_1(\alpha)=0$, $v_1(\beta)=1$ and $v_2(\alpha)=1$, $v_2(\beta)=0$. We see that $v_1, v_2\in H^1(\alpha,\beta)$ because they are smooth functions on $[\alpha,\beta]$. Replacing v with v_1 in (4), we get $b-c=0$. Replacing v with v_2 in (4), we get $a+c=0$. Thus,

$$a+b = (a+c)+(b-c) = 0.$$

Now suppose $a+b=0$. We determine all solutions to Problem (1). We showed earlier that every solution $u\in H^1(\alpha,\beta)$ has a constant derivative, i.e. $u'=c=-a$ almost everywhere in (α,β). Then

$$u(x) = \int_\alpha^x u(t)\,dt = \int_\alpha^x -a\,dt = -ax+a\alpha \quad \forall x\in[\alpha,\beta]. \qquad (5)$$

Conversely, the function u given by (5) solves Problem 1. Indeed, for every $v \in H^1(\alpha,\beta)$,

$$\int_\alpha^\beta u'v'\,dx = \int_\alpha^\beta -av'\,dx = -a(v(\beta)-v(\alpha)) = av(\alpha) - av(\beta)$$
$$= av(\alpha) + bv(\beta).$$

We see that once Problem (1) has a solution, it has a unique solution.

5 Let X be an infinite-dimensional Hilbert space and $\{e_n : n \in \mathbb{N}\}$ be an orthonormal subset. Put $S = \{\sqrt{n}\, e_n : n \in \mathbb{N}\}$. Let V be a neighborhood of 0 in the weak topology (X,τ). It is also called a weak neighborhood of 0. We show that V contains infinitely many elements of S.

We know that 0 has a basis of weak neighborhoods consisting of the sets

$$V_{f_1,\dots,f_m;\varepsilon_1,\dots,\varepsilon_m} = \{x \in X : |f_k(x)| < \varepsilon_k \ \forall 1 \le k \le m\},$$

for $f_1,\dots,f_m \in X^*$ and $\varepsilon_1,\dots,\varepsilon_m > 0$. Thus, V contains such a set. By shrinking V if necessary, we can assume $V = V_{f_1,\dots,f_m;\varepsilon_1,\dots,\varepsilon_m}$ for some $f_1,\dots,f_m \in X^*$ and $\varepsilon_1,\dots,\varepsilon_m > 0$.

Because $f_k \in X^*$, by Riesz's representation theorem, there exists $x_k \in X$ such that $f_k(x) = (x, x_k)$ for all $x \in X$. Because $\{e_n : n \in \mathbb{N}\}$ is an orthonormal subset of X, we have the Parseval's inequality

$$\sum_{n=1}^\infty |(e_n, x_k)|^2 \le \|x_k\|^2.$$

Thus, $$\sum_{n=1}^\infty \underbrace{\sum_{k=1}^m |(e_n,x_k)|^2}_{a_n} = \sum_{k=1}^m \sum_{n=1}^\infty |(e_n,x_k)|^2 \le \sum_{k=1}^m \|x_k\|^2 < \infty. \qquad (1)$$

Put $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m\}$. We show that there are infinitely many $n \in \mathbb{N}$ such that $a_n < \dfrac{\varepsilon^2}{n \log(n+2)}$.

Suppose otherwise. Then there exists $N \in \mathbb{N}$ such that $a_n \geq \dfrac{\varepsilon^2}{n \log(n+2)}$ for all $n \geq N$. Then
$$\sum_{n=1}^{\infty} a_n \geq \sum_{n=N}^{\infty} a_n \geq \varepsilon^2 \sum_{n=N}^{\infty} \frac{1}{n \log(n+2)}. \qquad (2)$$

The function $f: [1,\infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{x \log(x+2)}$ is nonnegative and decreasing.
$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{dx}{x \log(x+2)} \geq \int_2^{\infty} \frac{dx}{(x+2)\log(x+2)}$$
$$\underset{y=\log(x+2)}{=} \int_{\log 4}^{\infty} \frac{dy}{y} = (\log y)\Big|_{\log 4}^{\infty} = \infty.$$

By the integral test for convergence, the series $\displaystyle\sum_{n=N}^{\infty} \frac{1}{n \log(n+2)} = \sum_{n=N}^{\infty} f(n)$ diverges. Then (2) gives $\displaystyle\sum_{n=1}^{\infty} a_n = \infty$. This contradicts (1).

Therefore, the set $\left\{n \in \mathbb{N} : a_n < \dfrac{\varepsilon^2}{n \log(n+2)}\right\}$ is infinite. We can order this set as a (strictly) increasing sequence (n_i) in $\mathbb{N}$. For every $1 \leq k \leq m$,
$$|(e_{n_i}, x_k)|^2 \leq a_{n_i} < \frac{\varepsilon^2}{n_i \log(n_i+2)} \qquad \forall i \in \mathbb{N}$$

Thus,
$$|(\sqrt{n_i}\, e_{n_i}, x_k)| \leq \frac{\varepsilon}{\sqrt{\log(n_i+2)}} \leq \frac{\varepsilon}{\sqrt{\log 3}} < \varepsilon \leq \varepsilon_k \qquad \forall i \in \mathbb{N}.$$

We get
$$|f_k(\sqrt{n_i}\, e_{n_i})| < \varepsilon_k \quad \forall i \in \mathbb{N},\ \forall 1 \leq k \leq m.$$

Thus, $\sqrt{n_i}\, e_{n_i} \in V$ for all $i \in \mathbb{N}$. We have showed that V contains infinitely many elements of S.

Next, we use this result to show that (X, τ) is not metrizable. Suppose otherwise. Then there is a metric $d: X \times X \to \mathbb{R}$ such that the topology generated by the metric space (X, d) coincides τ. For each $n \in \mathbb{N}$, put

$$B_n = \{x \in X : d(x, 0) < \frac{1}{n}\}.$$

Then B_n is an open neighborhood of 0 in (X, τ). We proved earlier that $B_1 \cap S$ is an infinite set. Thus, there exists $n_1 \in \mathbb{N}$, $n_1 > 2$ such that $\sqrt{n_1}\, e_{n_1} \in B_1$. Suppose $n_1, n_2, \ldots, n_k \in \mathbb{N}$ are defined such that

$$\begin{cases} n_1 < n_2 < \ldots < n_k, \\ n_i > 2^i \quad \forall 1 \leq i \leq k, \\ \sqrt{n_i}\, e_{n_i} \in B_i \quad \forall 1 \leq i \leq k. \end{cases}$$

We showed earlier that $B_{k+1} \cap S$ is an infinite set. Thus, there exists $n_{k+1} \in \mathbb{N}$, $n_{k+1} > \max\{n_k, 2^{k+1}\}$ such that $\sqrt{n_{k+1}}\, e_{n_{k+1}} \in B_{k+1}$. The process keeps going. Then we get a strictly increasing sequence (n_k) in $\mathbb{N}$ such that $\sqrt{n_k}\, e_{n_k} \in B_k$ for all $k \in \mathbb{N}$. Thus, $(\sqrt{n_k}\, e_{n_k})$ converges to 0 in (X, d). Since the topologies on (X, d) and (X, τ) are the same, $(\sqrt{n_k}\, e_{n_k})$ converges to 0 in (X, d). By the definition of weak convergence,

$$\lim_{k \to \infty} f(\sqrt{n_k}\, e_{n_k}) = 0 \qquad \forall f \in X^*. \qquad (3)$$

Let Y be the vector subspace of X spanned by $\{e_{n_1}, e_{n_2}, \ldots\}$. Define a linear map $g : Y \to \mathbb{R}$, $g(e_{n_k}) = \frac{1}{\sqrt{n_k}}$ for all $k \in \mathbb{N}$. We show that $g \in Y^*$.

For each $x \in Y$, write $x = c_1 e_{n_1} + c_2 e_{n_2} + \dots + c_m e_{n_m}$. Then

$$\|x\| = \sqrt{c_1^2 + c_2^2 + \dots + c_m^2},$$

$$g(x) = g\left(\sum_{i=1}^m c_i e_{n_i}\right) = \sum_{i=1}^m c_i g(e_{n_i}) = \sum_{i=1}^m \frac{c_i}{\sqrt{n_i}} \le \left(\sum_{i=1}^m c_i^2\right)^{1/2} \left(\sum_{i=1}^m \frac{1}{n_i}\right)^{1/2}$$

$$\le \|x\| \left(\sum_{i=1}^m \frac{1}{2^i}\right)^{1/2} \le \|x\|.$$

Thus, g is continuous on Y. By Hahn-Banach theorem, g can extend to a linear continuous functional $\tilde{g} : X \to \mathbb{R}$. By (3), we have

$$\lim_{k\to\infty} \tilde{g}(\sqrt{n_k}\, e_{n_k}) = 0.$$

However, $\tilde{g}(\sqrt{n_k}\, e_{n_k}) = \sqrt{n_k}\, \tilde{g}(e_{n_k}) = \sqrt{n_k}\, g(e_{n_k}) = 1$ for all $k \in \mathbb{N}$. This is a contradiction.

(6) Let X be a normed vector space such that the dual space X^* is separable. We show that X is also separable.

If $X = \{0\}$, it is separable. Consider the case $X \neq \{0\}$. Because X^* is separable, it has a countable dense subset $\{f_n : n \in \mathbb{N}\}$. We know that $\|f_n\| = \sup_{\|x\|=1} f_n(x)$. Thus, there exists $x_n \in X$, $\|x_n\| = 1$ such that $f_n(x_n) \geq \frac{\|f_n\|}{2}$. Let Y be the linear span of $\{x_n : n \in \mathbb{N}\}$ over $\mathbb{Q}$, i.e.

$$Y = \left\{ \sum_{i=1}^n r_i x_i : n \in \mathbb{N}, r_i \in \mathbb{Q}\ \forall 1 \le i \le n \right\}.$$

We show that Y is a countable dense subset of X. For each $n \in \mathbb{N}$, put

$$A_n = \left\{ \sum_{i=1}^n r_i x_i : r_i \in \mathbb{Q}\ \forall 1 \le i \le n \right\}.$$

Then A_n is countable because the map $\mathbb{Q}^n \to A_n, (r_1, \dots, r_n) \mapsto \sum_{i=1}^n r_i x_i$ is surjective. Then $Y = \bigcup_{n=1}^\infty A_n$ is also countable. Now we show that Y is dense in X.

Let Y_1 be the linear span of $\{x_n : n \in \mathbb{N}\}$ over $\mathbb{R}$, i.e.

$$Y_1 = \left\{ \sum_{i=1}^n c_i x_i : n \in \mathbb{N},\ c_i \in \mathbb{R},\ \forall 1 \leq i \leq n \right\}.$$

Then Y_1 is a vector subspace of X and $Y \subset Y_1 \subset X$. It suffices to show that Y is dense in Y_1 and Y_1 is dense in X.

Let $x \in Y_1$ and write $x = \sum_{i=1}^n c_i x_i$ for $c_1, \dots, c_n \in \mathbb{R}$. Because $\mathbb{Q}$ is dense in $\mathbb{R}$, there are sequences $(c_1^{(k)}), (c_2^{(k)}), \dots, (c_n^{(k)})$ in $\mathbb{Q}$ such that $c_i^{(k)} \to c_i$ as $k \to \infty$. Because Y_1 is a normed vector space, the addition map $Y_1 \times Y_1 \to Y_1$, $(x', y') \mapsto x' + y'$, and the scalar-multiplication map $\mathbb{R} \times Y_1 \to Y_1$, $(c, x') \mapsto cx'$, are continuous. Thus,

$$\sum_{i=1}^n c_i^{(k)} x_i \longrightarrow \sum_{i=1}^n c_i x_i = x \quad \text{as } k \to \infty.$$

This implies x is the limit of a sequence in Y. Thus, Y is dense in Y_1.

Now we show that Y_1 is dense in X. Suppose otherwise. Then there exists $x \in X \setminus \overline{Y_1}$. Since Y_1 is a vector space, it is convex. For $y, z \in \overline{Y_1}$ and $c \in [0,1]$, there are sequences (y_n) and (z_n) in Y_1 such that $y_n \to y$ and $z_n \to z$. Then

$$\underbrace{(1-c)y_n + cz_n}_{\in Y_1} \longrightarrow (1-c)y + cz \quad \text{as } n \to \infty.$$

Then $(1-c)y + cz \in \overline{Y_1}$. It implies $\overline{Y_1}$ is also convex. Because $\overline{Y_1}$ is convex and closed in X and $x \in \overline{Y_1}$, we can separate x ~~and~~ from $\overline{Y_1}$ by a hyperplane.

In other words, there exists $f \in X^*$ such that $f(x) < \inf_{y \in Y_1} f(y)$. Consequently, $f(x) < \inf_{y \in Y_1} f(y) = \inf f(Y_1)$. Since $f(Y_1)$ is a vector subspace of $\mathbb{R}$, it is either $\{0\}$ or $\mathbb{R}$. Because $f(Y_1)$ is bounded from below, $f(Y_1) = \{0\}$. Thus, f vanishes on Y_1 and $f(x) < 0$.

Because the set $S = \{f_n : n \in \mathbb{N}\}$ is dense in X^*, there exists a sequence (g_k) in S such that $f = \lim_{k\to\infty} g_k$. Then

$$\sup_{\|y\|=1} |g_k(y) - f(y)| \to 0 \quad \text{as} \quad k \to \infty.$$

Write $g_k = f_{n_k}$. Then $g_k(x_{n_k}) = f_{n_k}(x_{n_k}) \geq \frac{\|f_{n_k}\|}{2} = \frac{\|g_k\|}{2}$.

We have

$$0 = f(x_{n_k}) = g_k(x_{n_k}) - (g_k(x_{n_k}) - f(x_{n_k})) \geq \frac{\|g_k\|}{2} - \sup_{\|y\|=1} |g_k(y) - f(y)|.$$

Then
$$\frac{\|g_k\|}{2} \leq \sup_{\|y\|=1} |g_k(y) - f(y)| \longrightarrow 0 \quad \text{as} \quad k \to \infty.$$

Thus, $\lim_{k\to\infty} g_k = 0$. We get $f = 0$. This is a contradiction because $f(x) < 0$.

7 Let $X = L^1(0,1)$. We give an example of a continuous linear functional on X which does not attain maximum on the closed unit ball of X.

Consider the map $f : X \to \mathbb{R}$, $f(x) = \int_0^1 t x(t)\, dt$. It is well-defined because $|tx(t)| \leq |x(t)|$, which is integrable on $(0,1)$. Moreover, f is linear.

$$f(x) \leq \int_0^1 |t|\, |x(t)|\, dt \leq \int_0^1 |x(t)|\, dt = \|x\| \quad \forall x \in X.$$

Thus, f is continuous on X. Let B_1 be the closed unit ball of X. Then

$$f(x) \leq 1 \quad \forall x \in B_1 \qquad (1)$$

We show that f does not attain maximum in B_1. For each $n \in \mathbb{N}$, we define $x_n : (0,1) \to \mathbb{R}$,

$$x_n(t) = \begin{cases} n & \text{if } t \in (1-\frac{1}{n}, 1), \\ 0 & \text{if } t \in (0, 1-\frac{1}{n}]. \end{cases}$$

Then $\|x_n\| = \int_0^1 |x_n(t)|\,dt = \int_{1-1/n}^1 n\,dt = 1$. Thus, $x_n \in B_1$.

$$f(x_n) = \int_0^1 t x_n(t)\,dt = \int_{1-1/n}^1 nt\,dt = \frac{n}{2} t^2 \Big|_{1-1/n}^1 = \frac{n}{2}\left[1 - \left(1-\frac{1}{n}\right)^2\right]$$
$$= 1 - \frac{1}{2n}.$$

Hence, $\lim_{n\to\infty} f(x_n) = 1$. Combining this result with (1), we conclude that $\sup_{x \in B_1} f(x) = 1$.

Suppose by contradiction that f attains maximum in B_1. Then there exists $x_0 \in B_1$ such that $f(x_0) = 1$. For each $\varepsilon \in (0,1)$, we have

$$1 = f(x_0) \leq \int_0^1 t|x_0(t)|\,dt = \int_0^{1-\varepsilon} t|x_0(t)|\,dt + \int_{1-\varepsilon}^1 t|x_0(t)|\,dt$$
$$\leq (1-\varepsilon)\int_0^{1-\varepsilon} |x_0(t)|\,dt + \int_{1-\varepsilon}^1 |x_0(t)|\,dt$$
$$= \int_0^1 |x_0(t)|\,dt - \varepsilon \int_0^{1-\varepsilon} |x_0(t)|\,dt$$
$$\leq 1 - \varepsilon \int_0^{1-\varepsilon} |x_0(t)|\,dt.$$

Thus, $\int_0^{1-\varepsilon} |x_0(t)|\,dt = 0$. Replacing ε with $\frac{1}{n}$ for $n \in \mathbb{N}, n \geq 2$, we get

$$\int_0^1 |x_0(t)|\,\chi_{(0,1-\frac{1}{n})}(t)\,dt = \int_0^{1-1/n} |x_0(t)|\,dt = 0.$$

By the Monotone Convergence Theorem,

$$\|x_0\| = \int_0^1 |x_0(t)|\,dt = \lim_{n\to\infty} \int_0^1 |x_0(t)|\,\chi_{(0,1-\frac{1}{n})}(t)\,dt = 0.$$

Thus, $x_0 = 0$. This contradicts the fact that $f(x_0) = 1$.