Math 8800 Fall 2014

Homework Assignment 1
(due October 20)

Do two (or more) of the following five problems.

1. Let X be a normed space over $\mathbf{R}$ (resp. $\mathbf{C}$) such that every linear functional $f: X \to \mathbf{R}$ (resp. $\mathbf{C}$) is continuous. Prove that X is finite-dimensional.

2. Let $X = \mathbf{R}[\mathbf{x}]$ (= all real polynomials of one variable). Consider X as a vector space over $\mathbf{R}$. Is is possible to define a norm on X such that X is complete in the metric given by the norm? (In other words, can X be the underlying linear space of a Banach space?)

3. Let X be a Banach space and let $T: X \to X$ be a compact linear operator. Assume that $S = I - T$ is invertible. Show that $S^{-1} = I - Q$, where Q is compact.

4. Let $X = \mathcal{C}[0,1]$ (considered with the standard norm $||f|| = \sup_{x\in[0,1]} |f(x)|$). Let $K: X \to X$ be an integral operator of the form $Kf(x) = \int_0^1 k(x,y)f(y)\,dy$, where k is a continuous function on $[0,1] \times [0,1]$, and let $A: X \to X$ be an arbitrary continuous linear operator. Prove that the operator $B = AK$ (defined, as usual, by $Bf = A(Kf)$ is also an integral operator with a continuous kernel, i. e. there exists a continuous function b on $[0,1] \times [0,1]$ such that $Bf(x) = \int_0^1 b(x,y)f(y)\,dy$ for each $f \in X$.

Remark: note that this implies that if $I - K$ is invertible, then we can write $(I - K)^{-1} = I - H$, where H is an integral operator with continuous kernel.

5. True or false ? (Please give a reason for your answer.)
Let X be a Hilbert space and T a continuous operator on X.
Then
(a) *If $T^2 = TT$ is compact, then T is compact.*
(b) *If T^*T is compact, then T is compact.*