Math 8800 Fall 2014

Homework Assignment 2
(due November 26)

Do two (or more) of the following six problems.

1. Let $\lambda_1, \lambda_2, \ldots$ be a bounded sequence of complex numbers. Let $l^2 = \{x = (x_1, x_2, \ldots); x_k \in \mathbf{C}, \sum_k |x_k|^2 < \infty\}$. Define $T: l^2 \to l^2$ by $T(x_1, x_2, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \ldots)$. Describe the spectrum of T and also the eigenvalues of T.

2. Let A be a $n \times n$ complex matrix and let $P(\lambda) = \det(\lambda - A)$ be its characteristic polynomial. Show that $A^k = \frac{1}{2\pi i} \int_\gamma \lambda^k (\lambda - A)^{-1}\, d\lambda$, where γ is a suitable curve. Show that this immediately implies $P(A) = 0$. (As you may remember from your linear algebra class, the statement $P(A) = 0$ is known as the Caley-Hamilton Theorem.)

3. Let X be a (complex) Hilbert space and T be a continuous operator on X. We say that T is *quasi-hermitian* if there exists a continuous hermitian operator G with $(Gx, x) > 0$ for each $x \in X \setminus \{0\}$, such that $GT = T^*G$. This is equivalent to T being hermitian with respect to the scalar product $x, y \to (Gx, y)$. (Note that when the dimension of X is infinite, the space X with the new scalar product might only be a pre-Hilbert space, it may not be complete, in general.)

(i) Show that the spectrum of any compact quasi-hermitian operator is real.

(ii) Show that if the dimension of X is finite, then an operator T (which can of course be identified with a matrix) is quasi-hermitian if and only if one can find a basis of X in which the matrix of T is of the form $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where $\lambda_1, \ldots, \lambda_n$ are real numbers.

4. Let X be a Hilbert space and let $\mathcal{L}(X)$ be the space of continuous linear operators on X. We say that an operator $A \in \mathcal{L}(X)$ has a left inverse if there exists $B_1 \in \mathcal{L}(X)$ such that $B_1 A = I$. We say that A has a right inverse if there exists $B_2 \in \mathcal{L}(X)$ such that $AB_2 = I$. Finally, we recall that $A \in \mathcal{L}(X)$ is invertible if there exists $B \in \mathcal{L}(X)$ such that $AB = BA = I$.

(i) Show that if A has a left inverse B_1 and a right inverse B_2, then necessarily $B_1 = B_2$ and A is invertible.

(ii) Show that if A has a left inverse, then it must be injective and its range is closed. Moreover, $||Ax|| \geq c||x||$ for some $c > 0$. (Try to prove the last statement without using the Open Mapping Theorem.)

(iii) Show that A has a left inverse if and only if A^* has a right inverse. Show that together with (i) this implies that a self-adjoint operator is invertible if and only if it has a left inverse. (The same is true for the right inverse, as one can expect).

(iv) Show that if X is finite dimensional, then $A \in \mathcal{L}(X)$ is invertible if and only if it has a left inverse. (The same remains true for the right inverse).

5. Let X be a complex Banach space and let $T: X \to X$ be a continuous linear operator. A point $\lambda \in \sigma(T)$ (=the spectrum of T) is called a *Riesz point* if
(i) λ is an isolated point of $\sigma(T)$ and
(ii) X is the direct sum of a closed subspace Y_λ of X and a finite-dimensional subspace Z_λ of X such that $T(Y_\lambda) \subset Y_\lambda$, $T(Z_\lambda) \subset Z_\lambda$, the restriction of $\lambda - T$ to Y_λ is an isomorphism of Y_λ, and the restriction of $\lambda - T$ to Z_λ is nilpotent (i.e. a sufficiently high power of the restriction is zero).

(a) Prove that if λ and μ are two distinct Riesz points in $\sigma(T)$, then $Z_\mu \subset Y_\lambda$, and X is the direct sum of Z_λ, Z_μ, and $Y_\lambda \cap Y_\mu$.

(b) A *Riesz operator* T is a continuous linear operator on the space X such that all points $\neq 0$ of the spectrum are Riesz points. Prove that if T^m is compact for some positive integer m, then T is a Riesz operator.

(c)* Let $\mathcal{L}(X)$ be the space of all continuous linear operator on X and let $\mathcal{K}(X) \subset \mathcal{L}(X)$ be the space of all compact linear operators on X. Prove that, in order that $T \in \mathcal{L}(X)$ be a Riesz operator, it is necessary and sufficient that $\lim_{n\to\infty}\{\text{dist}(T^n, \mathcal{K}(X))\}^{\frac{1}{n}} = 0$. (The distance is measured in the operator norm.)

Statement (c) is harder than the rest. It may be proved by using the following generalization of a lemma which we did in class, which is of independent interest. Let $A \in \mathcal{L}(X)$ with $||A|| < 1/2$, let K be compact and consider $S = I - A - K$. Then S has a finite-dimensional kernel, its range is closed of finite co-dimension, and $X = Z \oplus Y$ where Z, Y are invariant under S, with Z finite-dimensional and $S|_Y$ invertible (as an operator on Y). The proof is similar to the proof of the special case when $A = 0$, which we did in class.

6. For any natural number n we define α_n in the following way: write $n = 2^l m$, where m is odd, and set $\alpha_n = e^{-l}$. Then define a linear operator A on a separable Hilbert space X as follows: choose a Hilbert basis $\{e_n\}_{n=1}^\infty$ and set $Ae_n = \alpha_n e_{n+1}$. In addition, let us define for $k = 1, 2, \ldots$ the operators $A_k \in \mathcal{L}(X)$ by $A_k e_n = 0$ when $n = 2^k m$ with m odd and $A_k e_m = \alpha_m e_{m+1}$ otherwise. Show that:
(i) $\rho(A) > 0$
(ii) A_k is nilpotent (i. e. $A_k^{m_k} = 0$ for some natural m_k), and therefore $\rho(A_k) = 0$.
(iii) $||A_k - A||_{\mathcal{L}(X)} \to 0$ as $k \to \infty$.
(Thanks to Tuan Pham for bringing this example to my attention.)