Math 8801

Fall 2014

**Homework Assignment 3**
(due December 18, 1 pm)

Do two (or more) of the following seven problems.

**1.** Let $b$ be a smooth function on $[\alpha, \beta]$ and let $L$ be the differential operator on $(\alpha, \beta)$ given by $Lu = -u_{xx} + b(x)u_x$, with the Dirichlet boundary conditions $u(\alpha) = u(\beta) = 0$. Investigate under which conditions it is possible to find a continuous function $\rho > 0$ on $[\alpha, \beta]$ such that $\int_\alpha^\beta (Lu)v\rho = \int_\alpha^\beta u(Lv)\rho$ for each smooth $u, v$ which vanish at $\alpha$ and $\beta$.
Another way to formulate the question: when can one make $L$ self-adjoint by modifying the scalar product by a suitable "weight" $\rho$? Note the possible existence of $\rho$ has important consequences for the spectrum of $L$, or - equivalently - for the spectrum of the (compact) operator $L^{-1}$. (As an optional part, you can explain why the (unbounded) operator $L$ is invertible.)

**2.** Let $X$ be a Hilbert space and let $T$ be a continuous self-adjoint operator on $X$. Show that $T$ is compact of and only if some power of $T$ is compact.

**3.** Let $a$ be a measurable function of $(\alpha, \beta)$ with $0 < c \leq a(x) \leq C < +\infty$ almost everywhere in $(\alpha, \beta)$. Show that the eigenvalue problem $-a(x)u'' = \lambda u$ with the boundary values $u(\alpha) = u(\beta) = 0$ has countably many solutions $u_j, \lambda_j$, such that $u_j$ generate a dense set in $L^2(\alpha, \beta)$ and $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, with $\lambda_j \to \infty$. (As an optional part, you can show that each eigenvalue is in fact simple.)
Hint: consider the solution operator $G$ of $-u'' = f$ with $u(\alpha) = u(\beta) = 0$ and write our eigenvalue problem as $u = \lambda G(u/a)$. The operator $u \to G(u/a)$ may not be self-adjoint in the standard $L^2$ scalar product, but one can modify the scalar product to make the operator self-adjoint.

**4.** Consider the bilinear form $\mathcal{B}(u, v) = \int_\alpha^\beta u'(x)v'(x)\,dx$ on the space $H^1(\alpha, \beta)$. Let $a, b \in \mathbf{R}$ and let $l\colon H^1(\alpha, \beta) \to \mathbf{R}$ be a linear functional on $H^1(\alpha, \beta)$ given by $l(v) = av(\alpha) + bv(\beta)$. Characterize the values of $a, b$ for which the problem $\mathcal{B}(u, v) = l(v)$ for each $v \in H^1(\alpha, \beta)$ has at least one solution. When problem is solvable, describe all its solutions.

**5.** Let $X$ be an infinite-dimensional Hilbert space and let $e_1, e_2, \ldots$ be a sequence of mutually orthogonal vectors of unit length in $X$. Consider the set $S = \{e_1, \sqrt{2}e_2, \sqrt{3}e_3, \ldots\}$. Prove that any weak neighborhood of zero in X contains infinitely many elements of $S$. Show that this implies that the weak topology on $X$ is not metrizable.

**6.** Let $X$ be a Banach space such that the dual space $X^*$ is separable. Prove that $X$ is also separable.

**7.** Let $X = L^1(0, 1)$. Give an example of a continuous linear functional on $X$ which does not attain its maximum of the closed unit ball of $X$.