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Math 8802: Functional Analysis

Homework # 1

(1) Let X be a topological linear space (TLS) over $\mathbb{R}$, and U be a neighborhood of 0 in X. We show that there exists a neighborhood V of 0 in X such that $\overline{V} \subset U$.

The map $(x,y) \mapsto x+y$ and $x \mapsto (-1)x = -x$ are continuous. Thus, the composite map $\phi: X \times X \to X$,

$$\phi(x,y) = x - y \quad \forall x,y \in X$$

is also continuous. Since $\phi(0,0) = 0$, there exists an open neighborhood $\tilde{U}$ of $(0,0)$ in $X \times X$ such that $\phi(\tilde{U}) \subset U$. We know that $\tilde{U}$ is a union of sets of the form $\tilde{U}_1 \times \tilde{U}_2$ where $\tilde{U}_1$ and $\tilde{U}_2$ are open in X. Among these sets, there must be one that contains $(0,0)$. Hence, we could assume $\tilde{U} = V_1 \times V_2$ where V_1 and V_2 are open neighborhoods of 0 in X. Then

$$V_1 - V_2 = \phi(V_1 \times V_2) = \phi(\tilde{U}) \subset U.$$

Put $V = V_1 \cap V_2$, which is also an open neighborhood of 0 in X. Then $V - V \subset V_1 - V_2 \subset U$. Let $x \in \overline{V}$. Since $x + V$ is a neighborhood of x in X,

$$(x+V) \cap V \neq \emptyset.$$

This means there exist $y, z \in V$ such that $x + y = z$. Then $x = z - y \in V - V \subset U$. We have showed that $\overline{V} \subset U$.

(2) Denote by $\mathcal{M}_0(0,1)$ the set of all Lebesgue measurable functions from $(0,1)$ to $\mathbb{R}$. It is a linear space over $\mathbb{R}$ with the operations

$$(f+g)(x) := f(x)+g(x) \quad \forall f,g \in \mathcal{M}_0(0,1),\ x\in(0,1),$$
$$(cf)(x) := c f(x) \quad \forall f\in \mathcal{M}_0(0,1),\ c\in\mathbb{R},\ x\in(0,1).$$

Define an equivalence relation on $\mathcal{M}_0(0,1)$ as follows.

$$f\sim g \Leftrightarrow f(x)=g(x) \text{ for almost every } x\in(0,1).$$

Let $\mathcal{M}(0,1)$ be the family of all equivalence classes. It is a linear space over $\mathbb{R}$ with the operations

$$[f]+[g] := [f+g] \quad \forall f,g \in \mathcal{M}_0(0,1),$$
$$c[f] := [cf] \quad \forall f\in \mathcal{M}_0(0,1),\ c\in\mathbb{R}.$$

Here $[f]$ denotes the equivalence class of f.

Let $p\in(0,1)$ and define

$$L^p(0,1) = \left\{ [f] : f\in\mathcal{M}_0(0,1),\ I_p(f) = \int_0^1 |f(x)|^p dx < \infty \right\}.$$

Then $L^p(0,1)$ is a subset of $\mathcal{M}(0,1)$. In the sequel, we view each equivalence class as any of its representatives, with a convention that two functions are considered the same if they differ by a subset of measure zero of $(0,1)$.

(a) We show that $L^p(0,1)$ is a linear space over $\mathbb{R}$.

Let $f,g\in L^p(0,1)$, $c\in\mathbb{R}$. Both $f+g$ and cf are Lebesgue measurable functions.

$$I_p(cf) = \int_0^1 |cf(x)|^p dx = |c|^p I_p(f) < \infty.$$

For $a, b \geqslant 0$, we have the inequality $(a+b)^p \leq a^p + b^p$. Then

$$I_p(f+g) = \int_0^1 |f(x)+g(x)|^p dx \leq \int_0^1 (|f(x)|+|g(x)|)^p dx \leq \int_0^1 (|f(x)|^p + |g(x)|^p) dx$$
$$= I_p(f) + I_p(g) < \infty.$$

Thus, cf and $f+g$ belong to $L^p(0,1)$.

(b) Define a map $d: L^p(0,1) \times L^p(0,1) \to \mathbb{R}$, $d(f,g) = I_p(f-g)$. First, we show that d is a metric on $L^p(0,1)$.

$$d(f,f) = I_p(f-f) = I_p(0) = 0.$$

Suppose $d(f,g) = 0$ for some $f, g \in L^p(0,1)$. Then $\int_0^1 |f(x)-g(x)|^p dx = 0$. This implies $f(x) = g(x)$ for almost every $x \in (0,1)$.

For $f, g \in L^p(0,1)$,

$$d(f,g) = \int_0^1 |f(x)-g(x)|^p dx = \int_0^1 |g(x)-f(x)|^p dx = d(g,f).$$

For $f, g, h \in L^p(0,1)$,

$$d(f,g) + d(g,h) = \int_0^1 (|f(x)-g(x)|^p + |g(x)-h(x)|^p) dx$$
$$\geqslant \int_0^1 (|f(x)-g(x)| + |g(x)-h(x)|)^p dx$$
$$\geqslant \int_0^1 |f(x)-h(x)|^p dx$$
$$= d(f,h).$$

Next, we show that $(L^p(0,1), d)$ is a complete metric space. The method to be used is the same as the method to show that $L^q(0,1)$, $q \in [1,\infty)$, is a Banach space.

Let (f_n) be a Cauchy sequence in $L^p(0,1)$. To show (f_n) converges, it suffices to show that it has a convergence subsequence. There exists a subsequence (f_{n_k}) such that $d(f_{n_k}, f_{n_{k+1}}) < 2^{-k}$ for all $k \in \mathbb{N}$. By replacing (f_n) with (f_{n_k}), we can assume

$$d(f_n, f_{n+1}) < 2^{-n} \quad \forall n \in \mathbb{N}.$$

Define a map $g : (0,1) \to [0, \infty]$,

$$g(x) = |f_1(x)| + \sum_{n=1}^{\infty} |f_{n+1}(x) - f_n(x)|.$$

For each $m \in \mathbb{N}$, define a map $g_m : (0,1) \to [0, \infty)$,

$$g_m(x) = |f_1(x)| + \sum_{n=1}^{m} |f_{n+1}(x) - f_n(x)|.$$

In part (a), we showed that $I_p(u+v) \leq I_p(u) + I_p(v)$ for all $u, v \in L^p(0,1)$.

Thus,

$$I_p(g_m) \leq I_p(|f_1|) + \sum_{n=1}^{m} I_p(|f_{n+1} - f_n|) = I_p(f_1) + \sum_{n=1}^{m} 2^{-n} < I_p(f_1) + 1.$$

In other words,

$$\int_0^1 g_m(x)^p \, dx < I_p(f_1) + 1, \quad \forall m \in \mathbb{N}.$$

Because $0 \leq g_1(x) \leq g_2(x) \leq g_3(x) \leq \cdots$ and $g_m(x) \to g(x)$ as $m \to \infty$, by the Monotone Convergence Theorem,

$$\int_0^1 g(x)^p \, dx = \lim_{m \to \infty} \int_0^1 g_m(x)^p \, dx \leq I_p(f_1) + 1 < \infty.$$

This implies $g(x) < \infty$ for almost every $x \in (0,1)$. Thus, the series

$$f_1(x) + \sum_{n=1}^{\infty} (f_{n+1}(x) - f_n(x))$$

absolutely converges for almost every $x \in (0,1)$. Denote the sum of the series by $f(x)$. Then

$$f(x) = f_1(x) + \lim_{m\to\infty} \sum_{n=1}^{m} (f_{n+1}(x) - f_n(x)) = \lim_{m\to\infty} f_{m+1}(x), \quad \text{a.e. } x \in (0,1).$$

The sequence (f_n) converges almost everywhere to f. Then f is also Lebesgue measurable. We have

$$|f_m| = \left| f_1 + \sum_{n=1}^{m-1} (f_{n+1} - f_n) \right| \le |f_1| + \sum_{n=1}^{m-1} |f_{n+1} - f_n| \le g.$$

Then $|f_m|^p \le g^p$. We know $\int_0^1 g(x)^p dx < \infty$. Then by the Dominated Convergence Theorem,

$$\int_0^1 |f(x)|^p dx = \lim_{m\to\infty} \int_0^1 |f_m(x)|^p dx \le \int_0^1 g(x)^p dx < \infty.$$

Hence, $f \in L^p(0,1)$. Moreover,

$$\lim_{m\to\infty} |f_m(x) - f(x)|^p = 0 \quad \text{for almost every } x \in (0,1),$$

$$|f_m(x) - f(x)|^p \le |f_m(x)|^p + |f(x)|^p \le g(x)^p + |f(x)|^p,$$

$$\int_0^1 (g(x)^p + |f(x)|^p)\, dx < \infty.$$

By the Dominated Convergence Theorem,

$$\lim_{m\to\infty} \underbrace{\int_0^1 |f_m(x) - f(x)|^p dx}_{d(f_m, f)} = 0$$

Therefore, (f_m) converges to f in $L^p(0,1)$.

(c) First, we show that the addition map $L^p(0,1) \times L^p(0,1) \to L^p(0,1)$, $(f,g) \mapsto f+g$ is continuous. Let (f_n) and (g_n) be sequences in $L^p(0,1)$ that converge to f and g respectively. Then

$$
\begin{aligned}
0 \leq d((f_n+g_n)-(f+g)) &= d((f_n-f)+(g_n-g)) \\
&\leq d(f_n-f)+d(g_n-g) \\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

Thus, f_n+g_n tends to $f+g$ as $n\to\infty$.

Next, we show that the scalar multiplication map $\mathbb{R}\times L^p(0,1)\to L^p(0,1)$, $(c,f)\mapsto cf$ is continuous. Let (c_n) be a sequence in $\mathbb{R}$ that converges to c, and (f_n) be a sequence in $L^p(0,1)$ that converges to f. Then

$$
\begin{aligned}
d(c_nf_n-cf) = \int_0^1 |c_nf_n-cf|^p\,dx &\leq \int_0^1 \left(|c_nf_n-c_nf|^p+|c_nf-cf|^p\right)dx \\
&= |c_n|^p d(f_n-f)+|c_n-c|^p I_p(f).
\end{aligned}
$$

Because (c_n) converges, it is bounded in $\mathbb{R}$. Then both summands of the above sum converge to 0 as $n\to\infty$. Thus, c_nf_n tends to cf as $n\to\infty$.

We have showed that the metric d is compatible with the algebraic operations in $L^1(0,1)$. Therefore, $L^1(0,1)$ with the topology induced by d is a topological linear space.

(d) Let A be a nonempty open convex subset of $L^p(0,1)$. We show that $A=L^p(0,1)$. Take any $f_0\in A$. Because of Part (c), the set $A'=A-\{f_0\}$ is an open neighborhood of 0. Moreover, $A=L^p(0,1)$ if and only if $A'=L^p(0,1)$. By replacing A with A', we can assume A contains 0. Then A contains a ball of radius r centered at 0, denoted by B_r.

Put $\lambda=2^{1-p}>1$. Suppose $B_s\subset A$ for some $s>0$. We show that $B_{\lambda s}\subset A$.

Let $h \in B_{\lambda s}$. Then
$$\int_0^1 |h(x)|^p dx = d(0,h) < \lambda s.$$
Because $|h|^p$ is integrable, the map $\phi : [0,1] \to \mathbb{R}$, $\displaystyle \phi(t) = \int_0^t |h(x)|^p dx$ is continuous. Since $\varphi(0) = 0$ and $\varphi(1) = \int_0^1 |h(x)|^p dx$, there exists $a \in (0,1)$ such that
$$\phi(a) = \frac{1}{2}\int_0^1 |h(x)|^p dx.$$
Then
$$\int_0^a |h(x)|^p dx = \frac{1}{2}\int_0^1 |h(x)|^p dx = \frac{1}{2} d(0,h) < \frac{\lambda s}{2},$$
$$\int_a^1 |h(x)|^p dx = \frac{1}{2}\int_0^1 |h(x)|^p dx = \frac{1}{2} d(0,h) < \frac{\lambda s}{2}.$$
Put $f = 2h\chi_{(0,a)}$ and $g = 2h\chi_{(a,1)}$, where χ_M denotes the characteristic function of $M \subset (0,1)$. Then f and g are Lebesgue measurable. We have
$$d(0,f) = \int_0^1 |f(x)|^p dx = \int_0^1 2^p |h(x)|^p \chi_{(0,a)} dx = 2^p \int_0^a |h(x)|^p dx < 2^p \frac{\lambda s}{2} = s,$$
$$d(0,g) = \int_0^1 |g(x)|^p dx = \int_0^1 2^p |h(x)|^p \chi_{(a,1)} dx = 2^p \int_a^1 |h(x)|^p dx < 2^p \frac{\lambda s}{2} = s.$$
Thus, $f, g \in B_s \subset A$. Since A is convex,
$$h = h\chi_{(0,a)} + h\chi_{(a,1)} = \frac{1}{2}f + \frac{1}{2}g \in A.$$
We have showed that $B_{\lambda s} \subset A$.

Applying this result for $s = r$, $s = \lambda r$, $s = \lambda^2 r$, $s = \lambda^3 r, \ldots$ consecutively, we get $B_{\lambda^n s} \subset A$ for every $n \in \mathbb{N}$. Then
$$A \supset \bigcup_{n=1}^{\infty} B_{\lambda^n r} = L^p(0,1).$$
Therefore, $A = L^p(0,1)$.

(e) Let $T: L^p(0,1) \longrightarrow \mathbb{R}$ be a continuous linear functional. We show that $T=0$. Suppose by contradiction that $T \neq 0$, i.e. there exists $f_0 \in L^p(0,1)$ such that $Tf_0 \neq 0$. By replacing f_0 with $-f_0$ if necessary, we can assume $Tf_0 = \alpha > 0$. Put

$$A = \{ f \in L^p(0,1) : Tf < 2\alpha \} = T^{-1}((-\infty, 2\alpha)).$$

Then A is open in $L^p(0,1)$ because T is continuous. Also, A is not empty because $f_0 \in A$. If $f, g \in A$, $t \in [0,1]$ then

$$T(tf + (1-t)g) = tTf + (1-t)Tg \leq \max\{Tf, Tg\} < 2\alpha.$$

This implies $tf + (1-t)g \in A$. Thus, A is convex. By Part (d), $A = L^p(0,1)$. Since $T(3f_0) = 3Tf_0 = 3\alpha > 2\alpha$, $3f_0 \in L^p(0,1) \setminus A$. This is a contradiction.

(3) Let X, Y be Banach spaces over $\mathbb{R}$ and $T: X \to Y$ be a linear map. Let w_X, w_Y respectively be the weak topologies on X, Y. We show that the two following statements are equivalent.

(i) $T: (X, \|.\|) \longrightarrow (Y, \|.\|)$ is continuous.

(ii) $T: (X, w_X) \longrightarrow (Y, w_Y)$ is continuous.

<u>Show (i) $\Rightarrow$ (ii)</u>

Suppose $T: (X, \|.\|) \to (Y, \|.\|)$ is continuous. We know that $T: (X, w_X) \to (Y, w_Y)$ is a linear map between two topological linear spaces. To show it is continuous, we only need to show it is continuous at 0. Let V be an open neighborhood of 0 in (Y, w_Y). One basis of neighborhoods of 0 in (Y, w_Y)

is the family

$$\{V_{g_1,\dots,g_n,\varepsilon} = g_1^{-1}((-\varepsilon,\varepsilon)) \cap g_2^{-1}((-\varepsilon,\varepsilon)) \cap \dots \cap g_n^{-1}((-\varepsilon,\varepsilon)) : n \in \mathbb{N}, g_1,\dots,g_n \in Y^*, \varepsilon > 0\}.$$

Hence, there exist $n \in \mathbb{N}$, $g_1,\dots,g_n \in Y^*$ and $\varepsilon > 0$ such that

$$V_{g_1, g_2, \dots, g_n, \varepsilon} \subset V.$$

Put $f_j = g_j \circ T$ for all $1 \le j \le n$. Since $g_j \in Y^*$, $f_j \in X^*$. Then

$$U = f_1^{-1}((-\varepsilon,\varepsilon)) \cap f_2^{-1}((-\varepsilon,\varepsilon)) \cap \dots \cap f_n^{-1}((-\varepsilon,\varepsilon))$$

is an open neighborhood of 0 in (X, w_X). For each $1 \le j \le n$,

$$T(U) \subset T(f_j^{-1}((-\varepsilon,\varepsilon))) = T(T^{-1}(g_j^{-1}((-\varepsilon,\varepsilon)))) \subset g_j^{-1}((-\varepsilon,\varepsilon)).$$

Thus,

$$T(U) \subset g_1^{-1}((-\varepsilon,\varepsilon)) \cap g_2^{-1}((-\varepsilon,\varepsilon)) \cap \dots \cap g_n^{-1}((-\varepsilon,\varepsilon)) = V_{g_1,\dots,g_n,\varepsilon} \subset V.$$

<u>Show (ii) $\Rightarrow$ (i)</u>

Suppose $T: (X, w_X) \to (Y, w_Y)$ is continuous. To show that $T: (X, \|\cdot\|) \to (Y, \|\cdot\|)$ is continuous, by the Closed Graph theorem, we only need to show that the graph of T

$$\Gamma(T) = \{(x, T(x)) : x \in X\}$$

is closed in $(X \times Y, \|\cdot\|)$. Let (u_n) be a sequence in $\Gamma(T)$ that converges to $(x,y) \in X \times Y$ in the product norm. Write $u_n = (x_n, Tx_n)$. Then $x_n \to x$ in $(X, \|\cdot\|)$ and $Tx_n \to y$ in $(Y, \|\cdot\|)$.

Because (x_n) converges in norm to x, (x_n) converges weakly to x.

Since $T:(X,w_X)\to(Y,w_Y)$ is continuous, the sequence (Tx_n) converges weakly to Tx. Because (Tx_n) converges in norm to y, (Tx_n) also converges weakly to y. Thus, $y=Tx$. This implies $(x,y)=(x,Tx)\in\Gamma(T)$.

④ Let $X_1, X_2, \dots, X_n, Y$ be Banach spaces over $\mathbb{R}$ and $M: X_1\times X_2\times\dots\times X_n\to Y$ be a multilinear map. We show that the following statements are equivalent.

(i) There exists a number $c>0$ such that

$$\|M(x_1,\dots,x_n)\|\leq c\|x_1\|\|x_2\|\dots\|x_n\| \quad \forall (x_1,\dots,x_n)\in X_1\times\dots\times X_n.$$

(ii) M is continuous.

(iii) M is separately continuous (i.e. continuous in each variable).

<u>Show (i)$\Rightarrow$(ii)</u>

Suppose there exists a number $c>0$ such that

$$\|M(x_1,\dots,x_n)\|\leq c\|x_1\|\|x_2\|\dots\|x_n\| \quad \forall (x_1,\dots,x_n)\in X_1\times\dots\times X_n.$$

We show that M is continuous. Let $(a^{(m)})$ be a sequence in $X_1\times X_2\times\dots\times X_n$ that converges to a. Write

$$a^{(m)}=(a_1^{(m)}, a_2^{(m)}, \dots, a_n^{(m)}),$$
$$a=(a_1, a_2, \dots, a_n).$$

Then for each $1\leq j\leq n$, $a_j^{(m)}\to a_j$ in X_j as $m\to\infty$. These n sequences are bounded. Thus, there exists a number $N>0$ such that

$$\|a_j^{(m)}\|\leq N \quad \forall 1\leq j\leq n, \forall m\in\mathbb{N}.$$

We can pick N such that $N\geq\|a_1\|, \|a_2\|, \dots, \|a_n\|$.

$$
\begin{aligned}
\| M(a^{(m)}) - M(a) \| &= \| M(a_1^{(m)}, a_2^{(m)}, \dots, a_n^{(m)}) - M(a_1, a_2, \dots, a_n) \| \\
&= \| M(a_1^{(m)}, a_2^{(m)}, \dots, a_n^{(m)}) - M(a_1, a_2^{(m)}, \dots, a_n^{(m)}) + \\
&\quad + M(a_1, a_2^{(m)}, \dots, a_n^{(m)}) - M(a_1, a_2, a_3^{(m)}, \dots, a_n^{(m)}) + \\
&\quad + M(a_1, a_2, a_3^{(m)}, \dots, a_n^{(m)}) - M(a_1, a_2, a_3, a_4^{(m)}, \dots, a_n^{(m)}) + \\
&\quad + \dots \\
&\quad + M(a_1, \dots, a_{n-1}, a_n^{(m)}) - M(a_1, \dots, a_{n-1}, a_n) \| \\
&= \| M(a_1^{(m)} - a_1, a_2^{(m)}, \dots, a_n^{(m)}) + \\
&\quad + M(a_1, a_2^{(m)} - a_2, a_3^{(m)}, \dots, a_n^{(m)}) + \\
&\quad + M(a_1, a_2, a_3^{(m)} - a_3, a_4^{(m)}, \dots, a_n^{(m)}) + \\
&\quad \dots \\
&\quad + M(a_1, \dots, a_{n-1}, a_n^{(m)} - a_n) \| \\
&\le \| M(a_1^{(m)} - a_1, a_2^{(m)}, \dots, a_n^{(m)}) \| + \| M(a_1, a_2^{(m)} - a_2, a_3^{(m)}, \dots, a_n^{(m)}) \| \\
&\quad + \dots + \| M(a_1, \dots, a_{n-1}, a_n^{(m)} - a_n) \| \\
&\le c \| a_1^{(m)} - a_1 \| \, \| a_2^{(m)} \| \dots \| a_n^{(m)} \| + c \| a_1 \| \, \| a_2^{(m)} - a_2 \| \, \| a_3^{(m)} \| \dots \| a_n^{(m)} \| \\
&\quad + \dots + c \| a_1 \| \dots \| a_{n-1} \| \, \| a_n^{(m)} - a_n \| \\
&\le c N^{n-1} \| a_1^{(m)} - a_1 \| + c N^{n-1} \| a_2^{(m)} - a_2 \| + \dots + c N^{n-1} \| a_n^{(m)} - a_n \| \\
&\le c n N^{n-1} \max \{ \| a_j^{(m)} - a_j \| : 1 \le j \le n \} .
\end{aligned}
$$

Thus, $\| M(a^{(m)}) - M(a) \| \le C n N^{-1} \underbrace{\max \{ \| a_j^{(m)} - a_j \| : 1 \le j \le n \}}_{A_m}$.

Because each sequence $(a_j^{(m)})_m$ converges to a_j, $\lim_{m \to \infty} A_m = 0$. Then

$$\lim_{m\to\infty} \|M(a^{(m)}) - M(a)\| = 0.$$

Show (ii) $\Rightarrow$ (iii)

Suppose M is continuous. First, fix the last $(n-1)$ arguments. Let $a_2 \in X_2, \ldots, a_n \in X_n$. Consider the map $M_1 : X_1 \to Y$; $M_1(z) = M(z, a_2, \ldots, a_n)$ for all $z \in X_1$. Let (z_m) be a sequence in X_1 that converges to $z_0 \in X_1$. Then $(z_m, a_2, \ldots, a_n) \to (z_0, a_2, \ldots, a_n)$ in $X_1 \times \cdots \times X_n$ as $m \to \infty$. By the continuity of M,

$$M(z_m, a_2, \ldots, a_n) \longrightarrow M(z_0, a_2, \ldots, a_n) \quad \text{as } m \to \infty.$$

In other words, $M_1(z_m) \to M_1(z_0)$ as $m \to \infty$. We have showed that M_1 is continuous.

In case that another tuple of $(n-1)$ arguments is chosen to be fixed, we still deal the same way.

Show (iii) $\Rightarrow$ (i)

We show by induction in $n \in \mathbb{N}$ that for each separately continuous multilinear map $M : X_1 \times \cdots \times X_n \to Y$, there exists a number $c > 0$ such that

$$\|M(x_1, x_2, \ldots, x_n)\| \leq c\|x_1\|\|x_2\| \cdots \|x_n\| \quad \forall (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n. \quad (1)$$

For $n = 1$, (1) is equivalent to that M is continuous. Since M has only one argument, the separate continuity and continuity are the same. Thus, (1) is true.

Suppose (1) is true for some $n \in \mathbb{N}$. Let $M : X_1 \times \cdots \times X_n \times X_{n+1} \to Y$ be a multilinear map that is separately continuous. For each $x_1 \in X_1$, the map $\tilde{M} : X_2 \times X_3 \times \cdots \times X_{n+1} \to Y$, $\tilde{M}(x_2, \ldots, x_{n+1}) = M(x_1, x_2, \ldots, x_{n+1})$ is multilinear and separately continuous. By the induction hypothesis, there exists a number $c = c(x_1) > 0$ such that

$$\|\tilde{M}(x_2, \dots, x_{n+1})\| \leq c(x_1)\|x_2\| \cdots \|x_{n+1}\| \quad \forall (x_2, \dots, x_{n+1}) \in X_2 \times \cdots \times X_{n+1}.$$

This means

$$\|M(x_1, x_2, \dots, x_{n+1})\| \leq c(x_1)\|x_2\| \cdots \|x_{n+1}\| \quad \forall (x_2, \dots, x_{n+1}) \in X_2 \times \cdots \times X_{n+1}. \quad (2)$$

Put $I = X_2 \times \cdots \times X_{n+1} \setminus \{(0, \dots, 0)\}$. For each $(x_2, \dots, x_{n+1}) \in I$, the map

$$M_{x_2, \dots, x_{n+1}} : X_1 \longrightarrow Y,$$

$$M_{x_2, \dots, x_{n+1}}(x_1) = \frac{M(x_1, x_2, \dots, x_{n+1})}{\|x_2\| \cdots \|x_{n+1}\|} \quad \forall x_1 \in X_1$$

is linear and continuous. Because of (2), the set

$$\{M_{x_2, \dots, x_{n+1}}(x_1) : (x_2, \dots, x_{n+1}) \in I\}$$

is bounded in Y for each $x_1 \in X_1$. By the Banach-Steinhaus theorem, the family ~~of~~ of continuous linear maps $\{M_{x_2, \dots, x_{n+1}}\}_{(x_2, \dots, x_{n+1}) \in I}$ is uniformly bounded. That is, there exists a number $c_0 > 0$ such that

$$\cancel{\|M(} \; \|M_{x_2, \dots, x_{n+1}}(x_1)\| \leq c_0 \|x_1\| \quad \forall (x_2, \dots, x_{n+1}) \in I, \forall x_1 \in X_1.$$

Thus,
$$\frac{\|M(x_1, x_2, \dots, x_{n+1})\|}{\|x_2\| \cdots \|x_{n+1}\|} \leq c_0 \|x_1\| \quad \forall (x_2, \dots, x_{n+1}) \in I, \forall x_1 \in X_1.$$

Therefore, $\|M(x_1, x_2, \dots, x_{n+1})\| \leq c_0 \|x_1\| \|x_2\| \cdots \|x_{n+1}\|$ for all $x_1 \in X_1$, $x_2 \in X_2$, $\dots$, $x_{n+1} \in X_{n+1}$.

5 Let $m, n \in \mathbb{N}$, $p, q \in [1, \infty]$. Suppose $(\mathbb{R}^m, d_p)$ and $(\mathbb{R}^n, d_q)$ are isometric, i.e. there exists a bijective map $\phi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ such that

$$d_q(\phi(x), \phi(y)) = d_p(x,y) \quad \forall x,y \in \mathbb{R}^m. \qquad (1)$$

First, we show $m=n$. Suppose by contradiction that $m \neq n$. By replacing ϕ with ϕ^{-1} if necessary, we can assume $m<n$. We know that all norms in a finite-dimensional ~~normed~~ vector space are equivalent. Thus, there exists a number $C = C(m,n,p,q) > 0$ such that

$$C^{-1}\|u\|_2 \leq \|u\|_p \leq C\|u\|_2 \qquad \forall u \in \mathbb{R}^m,$$

$$C^{-1}\|v\|_2 \leq \|v\|_q \leq C\|v\|_2 \qquad \forall v \in \mathbb{R}^n.$$

The topologies induced on $\mathbb{R}^n$ by the metrics d_q and d_2 are the same. Similarly, d_p and d_2 induce the same topology on $\mathbb{R}^m$. Then (1) implies

$$d_2(\phi(x), \phi(y)) \leq C\, d_q(\phi(x), \phi(y)) = C d_p(x,y) \leq C^2 d_2(x,y) \quad \forall x,y \in \mathbb{R}^m. \quad (2)$$

Because of (1), ϕ is continuous. (1) also implies

$$d_q(x', y') = d_p(\phi^{-1}(x'), \phi^{-1}(y')) \quad \forall x', y' \in \mathbb{R}^m,$$

which implies ϕ^{-1} is continuous. Thus, ϕ is a homeomorphism. In particular, ϕ is a Borel-measurable map.

Since $m<n$, we can view $\mathbb{R}^m$ as a subset of $\mathbb{R}^n$ via the injection

$$(x_1, x_2, \dots, x_m) \longmapsto (x_1, x_2, \dots, x_m, 0, \dots, 0).$$

Let B be the open unit ball centered at 0 in $(\mathbb{R}^m, d_2)$. Then $\phi(B)$ is open in $\mathbb{R}^n$. Let μ be the usual Lebesgue measure in $\mathbb{R}^n$. Then $\mu(B) = 0$. ~~By the definition of~~ For each $\varepsilon > 0$, we can cover B by balls

$B_1, B_2, B_3, \ldots$ in $(\mathbb{R}^n, d_2)$, each centered at a point in $\mathbb{R}^m$, such that $\sum_{k=1}^{\infty} \mu(B_k) < \varepsilon$. Because $B \subset \left(\bigcup_{k=1}^{\infty} B_k\right) \cap \mathbb{R}^m = \bigcup_{k=1}^{\infty} (B_k \cap \mathbb{R}^m)$

we have

$$\phi(B) \subset \phi\left(\bigcup_{k=1}^{\infty} (B_k \cap \mathbb{R}^m)\right) \subset \bigcup_{k=1}^{\infty} \phi(B_k \cap \mathbb{R}^m). \qquad (3)$$

Let $a_k \in \mathbb{R}^m$ be the center of B_k and $r_k > 0$ be its radius. For each $y \in \phi(B_k \cap \mathbb{R}^m)$, write $y = \phi(x)$ for some $x \in B_k \cap \mathbb{R}^m$. Since $x, a_k \in \mathbb{R}^m$,

$$\|x - a_k\|_{(\mathbb{R}^m, \|\cdot\|_2)} = \|x - a_k\|_{(\mathbb{R}^n, \|\cdot\|_2)} < r_k.$$

Then

$$d_2(\underbrace{\phi(x)}_{y}, \phi(a_k)) \overset{(2)}{\leq} C^2 d_2(x, a_k) = C^2 \|x - a_k\|_{(\mathbb{R}^m, \|\cdot\|_2)} < C^2 r_k.$$

Thus, y belongs to the ball centered at $\phi(a_k)$ with radius $C^2 r_k$ in $\mathbb{R}^n$. Denote this ball by B_k'. We have showed that $\phi(B_k \cap \mathbb{R}^m) \subset B_k'$. Moreover,

$$\mu(B_k') = C^{2n} \mu(B_k).$$

By (3), $\phi(B) \subset \bigcup_{k=1}^{\infty} \phi(B_k \cap \mathbb{R}^m) \subset \bigcup_{k=1}^{\infty} B_k'$. Thus,

$$\mu(\phi(B)) \leq \sum_{k=1}^{\infty} \mu(B_k') = C^{2n} \sum_{k=1}^{\infty} \mu(B_k) < \varepsilon C^{2n}.$$

Because $\varepsilon > 0$ was chosen arbitrarily, $\mu(\phi(B)) = 0$. This is a contradiction because $\phi(B)$ is a nonempty open subset of $\mathbb{R}^n$. We have showed $m = n$.

Now assume $n \geq 2$. We show that $p = q$. The map $\Psi: (\mathbb{R}^n, d_p) \to (\mathbb{R}^n, d_q)$, $\Psi(x) = \phi(x) - \phi(0)$ is bijective and isometric. Replacing ϕ with

Ψ, we can assume $\phi(0)=0$. Suppose by contradiction that $p \neq q$. By replacing ϕ with ϕ^{-1} if necessary, we can assume $1 \leq q < p \leq \infty$. Consider two cases: $p=\infty$ and $p<\infty$.

- $p=\infty$:

We seek contradiction from the fact that the unit ball in $(\mathbb{R}^n, d_q)$ is strictly convex while the unit ball in $(\mathbb{R}^n, d_\infty)$ is not. Take

$$x = (1, 0, \ldots, 0) \in \mathbb{R}^n,$$
$$y_\alpha = (-1, \alpha, 0, \ldots, 0) \in \mathbb{R}^n, \quad 0<\alpha<1.$$

Then $\|x\|_\infty = \|y_\alpha\|_\infty = \left\|\frac{x-y_\alpha}{2}\right\|_\infty = 1$.

$$\left\|\frac{\phi(x)-\phi(y_\alpha)}{2}\right\|_q = \left\|\frac{x-y_\alpha}{2}\right\|_\infty = 1.$$

Put $u=\phi(x)$ and $v_\alpha = \phi(y_\alpha)$. Then $u \neq v_\alpha$ and

$$\|u\|_q = \|x\|_\infty = 1,$$
$$\|v_\alpha\|_q = \|y_\alpha\|_\infty = 1,$$
$$\left\|\frac{u-v_\alpha}{2}\right\|_q = 1.$$

Then $\|u\|_q + \|v_\alpha\|_q = \|u-v_\alpha\|_q$. Because $q<\infty$, u and $-v_\alpha$ are parallel. Write $u = c_\alpha v_\alpha$ for some $c_\alpha \in \mathbb{R}$. Because $\|u\|_q = \|v_\alpha\|_q = 1$, $|c_\alpha| = 1$. Thus, $u = \pm v_\alpha$. Because $u \neq v_\alpha$, u must be equal to $-v_\alpha$. Then $v_\alpha = v_\beta$ for all $\alpha, \beta \in (0,1)$. This is a contradiction because $y_\alpha \neq y_\beta$ for $\alpha \neq \beta$.

$p < \infty$:

We show that ϕ is a linear map. First, we show $\phi(-x) = -\phi(x)$ for all $x \in \mathbb{R}^n$. Fix $x \in \mathbb{R}^n \setminus \{0\}$ and put $v = \dfrac{\phi(x)+\phi(-x)}{2}$. Then

$$d_q(v, \phi(x)) = \|v - \phi(x)\|_q = \frac{\|\phi(-x)-\phi(x)\|_q}{2} = \frac{\|(-x)-x\|_p}{2} = \|x\|_p, \quad (4),$$

$$d_q(v, \phi(-x)) = \|v - \phi(-x)\|_q = \frac{\|\phi(x)-\phi(-x)\|_q}{2} = \frac{\|x-(-x)\|_p}{2} = \|x\|_p. \quad (5)$$

Because ϕ is bijective, $v = \phi(u)$ for some $u \in \mathbb{R}^n$. Then

$$d_q(v, \phi(x)) = d_q(\phi(u), \phi(x)) = d_p(u, x),$$

$$d_q(v, \phi(-x)) = d_q(\phi(u), \phi(-x)) = d_p(u, -x).$$

By (4) and (5), $d_p(u,x) = d_p(u,-x) = \|x\|_p$.

$$d_p(u,x) + d_p(u,-x) = \|x-u\|_p + \|u+x\|_p \geq \|(x-u)+(u+x)\|_p = 2\|x\|_p.$$

The equality must hold. Because $p < \infty$, $x-u$ and $u+x$ are parallel vectors. Then u and x are also parallel. Because $x \neq 0$, $u = cx$ for some $c \in \mathbb{R}$.

$$d_p(u,x) = \|cx - x\|_p = |c-1|\,\|x\|_p,$$

$$d_p(u,-x) = \|cx + x\|_p = |c+1|\,\|x\|_p.$$

Thus, $|c-1| = |c+1| = 1$. This happens only if $c=0$. Then $u=0$. We get

$$\phi(x) + \phi(-x) = 2v = 2\phi(u) = 0.$$

Next, for $x, y \in \mathbb{R}^n$, we show that $\phi\left(\frac{x+y}{2}\right) = \dfrac{\phi(x)+\phi(y)}{2}$.

The map $\tilde{\phi} : (\mathbb{R}^n, d_p) \longrightarrow (\mathbb{R}^n, d_q)$, $\tilde{\phi}(z) = \phi\left(z + \frac{x+y}{2}\right) - \phi\left(\frac{x+y}{2}\right)$ is bijective, isometric, and $\tilde{\phi}(0) = 0$. By the previous result,

$$\tilde{\phi}\left(\frac{x-y}{2}\right) = -\tilde{\phi}\left(\frac{y-x}{2}\right),$$

which means

$$\phi(x) - \phi\left(\frac{x+y}{2}\right) = -\left(\phi(y) - \phi\left(\frac{x+y}{2}\right)\right).$$

Therefore,

$$\phi\left(\frac{x+y}{2}\right) = \frac{\phi(x)+\phi(y)}{2}. \qquad (6)$$

As a consequence, $\phi\left(\frac{z+w}{2}\right) = \frac{\phi(z+w)+\phi(0)}{2} = \frac{\phi(z+w)}{2} \quad \forall z, w \in \mathbb{R}^n$.

Replacing z with x, and w with y, we get

$$\phi\left(\frac{x+y}{2}\right) = \frac{\phi(x+y)}{2}. \qquad (7)$$

By (6) and (7), $\phi(x) + \phi(y) = \phi(x+y)$ for all $x, y \in \mathbb{R}^n$. In other words, ϕ is additive. Let $x \in \mathbb{R}^n$. We show that $\phi(cx) = c\phi(x)$ for all $c \in \mathbb{R}$. The proof is the same as the well-known proof for the fact that a continuous additive function from $\mathbb{R}$ to $\mathbb{R}$ is linear. For $l \in \mathbb{N}$,

$$\phi(lx) = \phi(\underbrace{x + \cdots + x}_{l}) = \underbrace{\phi(x) + \cdots + \phi(x)}_{l} = l\phi(x),$$

For $l \in \mathbb{Z}$, $l < 0$,

$$\phi(lx) = \phi(-|l|x) = -\phi(|l|x) = -|l|\phi(x) = l\phi(x).$$

For $l \in \mathbb{Z} \setminus \{0\}$, $\quad \phi(x) = \phi\left(l \frac{1}{l} x\right) = l\phi\left(\frac{1}{l}x\right),$

which implies $\phi\left(\frac{1}{\ell}x\right) = \frac{1}{\ell}\phi(x)$.

For $s \in \mathbb{Z}$, $\ell \in \mathbb{Z}\setminus\{0\}$,

$$\phi\left(\frac{s}{\ell}x\right) = s\,\phi\left(\frac{1}{\ell}x\right) = s\,\frac{1}{\ell}\phi(x) = \frac{s}{\ell}\phi(x).$$

Hence $\phi(rx) = r\phi(x)$ for every $r \in \mathbb{Q}$. Let $c \in \mathbb{R}$. There exists a sequence (r_n) in $\mathbb{Q}$ such that $r_n \to c$. Then

$$c\,\phi(x) = \lim_{n\to\infty} r_n \phi(x) = \lim_{n\to\infty} \phi(r_n x)$$

$$= \phi(cx) \quad \text{(because } \phi \text{ is continuous).}$$

We have showed ϕ is a linear map.

Recall that a consequence to the fact $\phi(0) = 0$ is that $\|\phi(x)\|_q = \|x\|_p$ for all $x \in \mathbb{R}^n$. Consider elements $x \in \mathbb{R}^n$ of the form

$$x = (t, s, 0, \dots, 0), \quad t, s \in \mathbb{R}.$$

Because of the linearity of ϕ, the equation $\|\phi(x)\|_q = \|x\|_p$ becomes

$$\left(\sum_{k=1}^{n} |a_k t + b_k s|^q\right)^{\frac{1}{q}} = \left(|t|^p + |s|^p\right)^{\frac{1}{p}} \quad \forall t, s \in \mathbb{R}, \qquad (8)$$

where $a_1, \dots, a_n, b_1, \dots, b_n$ are constants in $\mathbb{R}$.

In (8), we can replace any pair (a_k, b_k) with $(-a_k, -b_k)$, thereby can assume $a_k \geq 0$ for all $1 \leq k \leq n$. The numbers $a_1, a_2, \dots, a_n$ cannot be equal to 0 at once. Otherwise, LHS (8) does not depend on t while RHS(8) is increasing in $t \in (0, \infty)$. We can relabel the pairs $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)$ so that $a_1, a_2, \dots, a_r > 0$, $a_{r+1} = \dots = a_n = 0$ for some $1 \le r \le n$. Consider $0<s<2$ and $t > R = \max\left\{ \frac{2|b_k|}{a_k} : 1 \le k \le r \right\} + 1$.

Then (8) becomes

$$\left(cs^q + \sum_{k=1}^{r} (a_k t + b_k s)^q \right)^{\frac{1}{q}} = (t^p + s^p)^{\frac{1}{p}} \quad \forall t > R, 0<s<2, \quad (9)$$

where $c = \sum_{k=r+1}^{n} |b_k|^q \ge 0$. Taking both sides of (9) to the power p, we get

$$t^p + s^p = \left(cs^q + \sum_{k=1}^{r} (a_k t + b_k s)^q \right)^{p/q} \quad \forall t > R, 0<s<2.$$

Take derivative both sides with respect to s,

$$p s^{p-1} = \frac{p}{q} \left[cq s^{q-1} + \sum_{k=1}^{r} q b_k (a_k t + b_k s)^{q-1} \right] \left(cs^q + \sum_{k=1}^{r} (a_k t + b_k s)^q \right)^{\frac{p}{q}-1}$$

$$= \frac{p}{q} q \left[cs^{q-1} + \sum_{k=1}^{r} b_k (a_k t + b_k s)^{q-1} \right] (t^p + s^p)^{\frac{p/q - 1}{p/q}}.$$

Thus,

$$s^{p-1} = \left[cs^{q-1} + \sum_{k=1}^{r} b_k (a_k t + b_k s)^{q-1} \right] (t^p + s^p)^{1-\frac{q}{p}} \quad \forall t > R, 0<s<2.$$

Take $s = 1$,

$$1 = \left[c + \sum_{k=1}^{r} b_k (a_k t + b_k)^{q-1} \right] (t^p + 1)^{1-\frac{q}{p}} \quad \forall t > R$$

Change the variable t to $\frac{1}{t}$,

$$1 = \left[c + \sum_{k=1}^{r} b_k \left(\frac{a_k}{t} + b_k \right)^{q-1} \right] \left(\frac{1}{t^p} + 1 \right)^{1-\frac{q}{p}} \quad \forall 0 < t < R^{-1},$$

which implies

$$1 = \frac{ct^{q-1} + \sum_{k=1}^{r} b_k (a_k + t b_k)^{q-1}}{t^{p-1}} (1 + t^p)^{1-\frac{q}{p}} \quad \forall 0 < t < R^{-1}. \quad (10)$$

Put
$$f(t)=ct^{q-1}+\sum_{k=1}^{r} b_k(a_k+tb_k)^{q-1}.$$

Taking the limit of both sides of (10) as $t\to 0^+$, we get
$$\lim_{t\to 0^+}\frac{f(t)}{t^{p-1}}=1.$$

The function $t\mapsto \sum_{k=1}^{r} b_k(a_k+tb_k)^{q-1}$ is analytic in $(-R^{-1};R^{-1})$. Thus, we can write $f(t)=ct^{q-1}+\alpha_0+\alpha_1 t+\alpha_2 t^2+\alpha_3 t^3+\cdots$ for $t\in(0,R^{-1})$.

If $p-1\notin\{q-1,0,1,2,3,\dots\}$ then
$$\frac{f(t)}{t^{p-1}}=ct^{q-p}+\sum_{0\leqslant l<p-1}\alpha_l t^{l-(p-1)}+\sum_{l>p-1}\alpha_l t^{l-(p-1)}.$$

The limit as $t\to 0^+$ exists only if the sum of the first two summands on the right hand side is zero. Then
$$\lim_{t\to 0^+}\frac{f(t)}{t^{p-1}}=\lim_{t\to 0^+}\sum_{l>p-1}\alpha_l t^{l-(p-1)}=0,$$

which is a contradiction. Thus, $p-1\in\{q-1,0,1,2,3,\dots\}$. Because $p>q\geqslant 1$, $p\in\{2,3,4,\dots\}$. Then
$$\frac{f(t)}{t^{p-1}}=\underbrace{ct^{q-p}+\sum_{l<p-1}\alpha_l t^{l-(p-1)}}_{\{1\}}+\alpha_{p-1}+\underbrace{\sum_{l>p-1}\alpha_l t^{l-(p-1)}}_{\{2\}}.$$

The limit as $t\to 0^+$ is equal to 1 if and only if $\{1\}\equiv 0$, $\{2\}\equiv 0$ and $\alpha_{p-1}=1$. Then $f(t)=t^{p-1}$ for all $t\in(0,R^{-1})$. Then (10) becomes
$$1=(1+t^p)^{1-\frac{q}{p}}\quad \forall\, 0<t<R^{-1}!$$

This is a contradiction because the right hand side is increasing in t.