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Math 8802: Functional Analysis
Homework #2

A

① Let $M(n,\mathbb{C})$ be the algebra of all $n\times n$ complex matrices. Let $T\in M(n,\mathbb{C})$ and $A$ be the subalgebra generated by $I_n$ (the identity matrix) and $T$.

(a) We determine the dimension of $A$. By definition,

$$A = \{p(T) : p\in\mathbb{C}[t]\}. \quad \checkmark$$

If $T=0$ then $A=\{0\}$ and $\dim A = 0$. Consider the case $T\neq 0$. The matrices $I_n, T, T^2, T^3, \dots$ belong to $A$. Because $A$ is a vector subspace of $M(n,\mathbb{C})$, it is finite dimensional. There exists $m\in\mathbb{N}$ such that $I_n, T, T^2, \dots, T^m$ are linearly dependent. Thus, there exists a polynomial $q\in\mathbb{C}[t]$ such that $q(T)=0$. We can take $q(t)$ to be a monic polynomial (i.e. its leading coefficient is equal to 1) and of smallest degree. It is known as the minimal polynomial of $T$. Put

$$m = \deg q \in\mathbb{N}. \quad \checkmark$$

We show that $\dim A = m$. Because $m$ is the smallest positive integer such that $I_n, T, T^2, \dots, T^m$ are linearly dependent, the matrices $I_n, T, T^2, \dots, T^{m-1}$ are linearly independent. Thus, $\dim A \geq m$. $\checkmark$

An element of $A$ is of the form $p(T)$ for some polynomial

$p \in \mathbb{C}[t]$. Write $p(t) = u(t)q(t) + r(t)$ for some $u(t), r(t) \in \mathbb{C}[t]$ and $\deg r < \deg q = m$. Then $p(T) = u(T)q(T) + r(T) = r(T)$, which is a linear combination of $I_n, T, T^2, \dots, T^{m-1}$. This implies the matrices $I_n, T, T^2, \dots, T^{m-1}$ linearly generate $A$. Therefore, $A$ has a basis $\{I_n, T, T^2, \dots, T^{m-1}\}$ and $\dim A = m$. $\surd$

We can relate $m$ to the Jordan normal form of $T$ as follows. Suppose $T$ has Jordan normal form

$$J = \begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_k \end{pmatrix}$$

where

$$J_j = \begin{pmatrix} \left.\begin{pmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix}\right\} \nu_{s_{j,n_j}} & & 0 \\ & \ddots & \\ 0 & & \left.\begin{pmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix}\right\} \nu_{s_{j,1}} \end{pmatrix},$$

$1 \leq \nu_{s_{j,1}} \leq \nu_{s_{j,2}} \leq \dots \leq \nu_{s_{j,n_j}}$ and $\lambda_1, \lambda_2, \dots, \lambda_k$ are the distinct complex eigenvalues of $T$. Then the minimal polynomial of $T$ is

$$q(t) = (t-\lambda_1)^{\nu_{s_{1,n_1}}} (t-\lambda_2)^{\nu_{s_{2,n_2}}} \cdots (t-\lambda_k)^{\nu_{s_{k,n_k}}}. \quad \surd$$

The reason is $$A \simeq \bigoplus_{j=1}^{k} \left( \mathbb{C}[t]/(t-\lambda_j)^{\nu_{s_{j,1}}} \oplus \dots \oplus \mathbb{C}[t]/(t-\lambda_j)^{\nu_{s_{j,n_j}}} \right) \quad \surd$$

(linear isomorphism)

Thus, $m = \nu_{s_{1,n_1}} + \nu_{s_{2,n_2}} + \dots + \nu_{s_{k,n_k}}$. In other words, the dimension of $A$ is the sum (over all distinct eigenvalues of $T$) of the maximum size of Jordan blocks corresponding to each eigenvalue. For example, if $T$ has Jordan normal form

$$\begin{pmatrix} \boxed{\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}} & & \\ & \boxed{1} & \\ & & \boxed{3} \end{pmatrix}$$

then $\dim A = 2 + 1 = 3$.

(b) We identify all multiplicative functionals on $A$.

Suppose $\phi : A \to \mathbb{C}$ is a multiplicative functional. Then

$$\phi(p(T)) = p(\phi(T)) \qquad \forall p \in \mathbb{C}[t].$$

This implies $\phi$ is determined by $\phi(T)$. Let $q \in \mathbb{C}[t]$ be the minimal polynomial of $T$. Then

$$q(\phi(T)) = \phi(q(T)) = \phi(0) = 0.$$

Thus, $\phi(T)$ is a root of $q$. The distinct roots of $q$ are the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ of $T$. Hence, $\phi(T) \in \{\lambda_1, \lambda_2, \dots, \lambda_k\}$.

Conversely, for each $j \in \{1, 2, \dots, k\}$, we define a map $\psi_j : A \to \mathbb{C}$,

$$\psi_j(p(T)) = p(\lambda_j) \quad \forall p \in \mathbb{C}[t].$$

First, we show that $\psi_j$ is well-defined. Suppose $p_1(T) = p_2(T)$ for some

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$p_1, p_2 \in \mathbb{C}[t]$. Then $(p_1-p_2)(T)=0$. Since $q$ is the minimal polynomial of $T$, $q$ divides $p_1-p_2$. Then $\lambda_j$ is also a root of $p_1-p_2$. Thus, $p_1(\lambda_j)=p_2(\lambda_j)$.
By definition, $\Psi_j(I_n) = 1 \neq 0$.

Next, we show that $\Psi_j$ is linear. For $p_1, p_2 \in \mathbb{C}[t]$, $c \in \mathbb{C}$,

$$\begin{aligned}\Psi_j(cp_1(T)+p_2(T)) = \Psi_j((cp_1+p_2)(T)) &= (cp_1+p_2)(\lambda_j)\\ &= cp_1(\lambda_j)+p_2(\lambda_j)\\ &= c\,\Psi_j(p_1(T)) + \Psi_j(p_2(T)).\end{aligned}$$

Next, we show that $\Psi_j$ is multiplicative. For $p_1, p_2 \in \mathbb{C}[t]$,

$$\Psi_j(p_1(T)p_2(T)) = \Psi_j((p_1p_2)(T)) = (p_1p_2)(\lambda_j) = p_1(\lambda_j)p_2(\lambda_j) = \Psi_j(p_1(T))\Psi_j(p_2(T)).$$

Therefore, $\Psi_j$ is a multiplicative functional on $A$. Each is distinct from another because

$$\Psi_j(T) = \lambda_j \neq \lambda_\ell = \Psi_\ell(T) \quad \forall\, 1 \leq j \neq \ell \leq k.$$

The set $\{\Psi_1, \Psi_2, \ldots, \Psi_k\}$ consists of all multiplicative functionals on $A$. ✓

2 Let $T$ and $A$ be as in Problem 1.

(a) We show that $A$ is a finite direct sum of local algebras.

Let $q(t) = (t-\lambda_1)^{r_1}(t-\lambda_2)^{r_2}\cdots(t-\lambda_k)^{r_k}$ be the minimal polynomial of $T$, where ✓ $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct and $r_1, r_2, \ldots, r_k \in \mathbb{N}$. Put

$$p_1(t) = (t-\lambda_2)^{r_2}(t-\lambda_3)^{r_3}\cdots(t-\lambda_k)^{r_k},$$
$$p_2(t) = (t-\lambda_1)^{r_1}(t-\lambda_3)^{r_3}\cdots(t-\lambda_k)^{r_k},$$
$$\cdots$$
$$p_k(t) = (t-\lambda_1)^{r_1}(t-\lambda_2)^{r_2}\cdots(t-\lambda_{k-1})^{r_{k-1}}.$$ ✓

These $k$ polynomials has no common roots. By Hilbert's Nullstellensatz, there exist $u_1(t), u_2(t), \dots, u_k(t) \in \mathbb{C}[t]$ such that

$$p_1(t)u_1(t) + p_2(t)u_2(t) + \dots + p_k(t)u_k(t) = 1. \qquad (1)$$

For each $j \in \{1, 2, \dots, k\}$, put

$$A_j = \{p(T) : p \in \mathbb{C}[t] \text{ and is divisible by } p_j(t)\}.$$

First, we show that $A_j$ is a subalgebra of $A$. Let $p(t)$ and $\tilde{p}(t)$ be polynomials divisible by $p_j(t)$ and let $c \in \mathbb{C}$. Then $cp(t) + \tilde{p}(t)$ is a polynomial divisible by $p_j(t)$. Thus, $cp(T) + \tilde{p}(T) \in A_j$. The product of $p(t)$ and $\tilde{p}(t)$ is a polynomial divisible by $p_j(t)$. Then $p(T)\tilde{p}(T) \in A_j$. Put $e_j(t) = p_j(t)u_j(t)$. We see that $e_j(T) \in A_j$. Because of (1), $e_j(t) - 1$ is divisible by $(t - \lambda_j)^{r_j}$. For each polynomial $p(t)$ divisible by $p_j(t)$,

$$p(t)e_j(t) - p(t) = p(t)(e_j(t) - 1),$$

which is divisible by $p_j(t)(t - \lambda_j)^{r_j} = q(t)$. Thus, $p(T)e_j(T) - p(T) = 0$. This implies $e_j(T)$ is a unit element of $A_j$. We have showed that $A_j$ is an algebra with unit element $e_j(T)$.

Next, we show that $A_j$ is a local algebra. The set

$$\mathfrak{a}_j = \{(T - \lambda_j)p(T) : p(t) \in \mathbb{C}[t] \text{ is divisible by } p_j(t)\}$$

is a linear subspace of $A_j$. Suppose by contradiction that $e_j(T) \in \mathfrak{a}_j$. Then $e_j(T) - (T - \lambda_j)p(T) = 0$ for some polynomoal $p(t)$ divisible by $p_j(t)$. Then $q(t)$ divides $e_j(t) - (t - \lambda_j)p(t)$. Then $\lambda_j$ is a root of $e_j(t)$. This is a contradiction because we know that $g(t)-1$ is divisible by $(t-\lambda_j)^{r_j}$.

For every polynomial $\tilde{p}(t)$ divisible by $p_j(t)$,

$$\tilde{p}(T)(T-\lambda_j)p(T) = (T-\lambda_j)\tilde{p}(T)p(T) \in \mathfrak{a}_j.$$

Thus, $\mathfrak{a}_j$ is an ideal of $A_j$. To show that $\mathfrak{a}_j$ is a maximal ideal of $A_j$, we show that $A_j \setminus \mathfrak{a}_j$ consists of invertible elements in $A_j$. Let $\tilde{p}(T) \in A_j \setminus \mathfrak{a}_j$ for some polynomial $\tilde{p}(t)$ divisible by $p_j(t)$. Write $\tilde{p}(t) = p_j(t)u(t)$. Since $\tilde{p}(T) \notin \mathfrak{a}_j$, $u(t)$ is not divisible by $(t-\lambda_j)$. Then $u(t)$ and $(t-\lambda_j)^{r_j}$ have no common root. By Hilbert's Nullstellensatz, there exist $v(t), w(t) \in \mathbb{C}[t]$ such that

$$u(t)v(t) + w(t)(t-\lambda_j)^{r_j} = 1.$$

Multiply both sides by $p_j(t)^2 u_j(t)^2$,

$$\underbrace{p_j(t)u(t)}_{\tilde{p}(t)}\, p_j(t)v(t)u_j(t)^2 + w(t)p_j(t)u_j(t)^2\underbrace{p_j(t)(t-\lambda_j)^{r_j}}_{q(t)} = p_j(t)^2u_j(t)^2 = g(t)^2.$$

Thus, $\tilde{p}(T)p_j(T)v(T)u_j(T)^2 = g(T)^2 = g(T)$. This implies $p_j(T)v(T)u_j(T)^2$ is an inverse of $\tilde{p}(T)$ in $A_j$. We have showed that $\mathfrak{a}_j$ is a maximal ideal of $A_j$.

Suppose by contradiction that there is another maximal ideal of $A_j$, namely $\mathfrak{a}_j'$. Let $p(T) \in \mathfrak{a}_j' \setminus \mathfrak{a}_j \subset A_j \setminus \mathfrak{a}_j$ for some polynomial $p(t)$ divisible by $p_j(t)$. As showed above, $p(T)$ is an invertible element of $A_j$. Since $\mathfrak{a}_j'$ cannot contain an invertible element, we get a contradiction. We have showed that $A_j$ is a local algebra.

Next, we show that $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$ as linear spaces over $\mathbb{C}$.

Let $p(T) \in A_1 \cap (A_2 + \cdots + A_k)$ for some polynomial $p(t)$. Then

$$p(T) = p_1(T) v_1(T),$$

$$p(T) = p_2(T) v_2(T) + \cdots + p_k(T) v_k(T)$$

for some $v_1(t), v_2(t), \ldots, v_k(t) \in \mathbb{C}[t]$. Then $p_1(T)v_1(T) - p_2(T)v_2(T) - \cdots - p_k(T)v_k(T) = 0$. Then $q(t)$ divides $p_1(t)v_1(t) - p_2(t)v_2(t) - \cdots - p_k(t)v_k(t)$. Then $(t-\lambda_1)^{r_1}$ divides the latter polynomial. Since $(t-\lambda_1)^{r_1}$ divides $p_2(t), \ldots, p_k(t)$, it also divides $p_1(t)v_1(t)$. Because $p_1(t)$ has no nontrivial common factor with $(t-\lambda_1)^{r_1}$, $(t-\lambda_1)^{r_1}$ divides $v_1(t)$. Then $q(t) = (t-\lambda_1)^{r_1} p_1(t)$ divides $v_1(t)p_1(t) = p(t)$.

Thus, $p(T) = 0$. We have showed that

$$A_1 \cap (A_2 + \cdots + A_k) = \{0\}.$$

Similarly,
$$A_j \cap \Big( \sum_{\substack{l=1 \\ l \neq j}}^{k} A_l \Big) = \{0\} \qquad \forall j \in \{1, 2, \ldots, k\}.$$

For each $p(t) \in \mathbb{C}[t]$,

$$p(t) \overset{(1)}{=\!=} p(t) \big(p_1(t)u_1(t) + \cdots + p_k(t)u_k(t)\big)$$
$$= p_1(t)u_1(t)p(t) + \cdots + p_k(t)u_k(t)p(t).$$

Then

$$p(T) = \underbrace{p_1(T)u_1(T)p(T)}_{\in A_1} + \cdots + \underbrace{p_k(T)u_k(T)p(T)}_{\in A_k}. \qquad (2)$$

Then $p(T) \in A_1 \oplus \cdots \oplus A_k$. We have showed that $A \subset A_1 \oplus \cdots \oplus A_k$. Therefore,

$$A = A_1 \oplus \cdots \oplus A_k.$$

Next, we show that this is a direct sum of algebras (not only linear spaces). ✓

That is to show

(i) the unit element of $A$ is the sum of the unit elements of $A_1, A_2, \dots, A_k$,

(ii) the multiplication in $A$ corresponds to the pointwise multiplication on each $A_1, A_2, \dots, A_k$. ✓

The identity (1) can be written as $e_1(t) + e_2(t) + \dots + e_k(t) = 1$. Thus,

$$e_1(T) + e_2(T) + \dots + e_k(T) = I_n.$$

We have showed (i). Let $p(t), \tilde{p}(t) \in \mathbb{C}[t]$. By (2),

$$p(T) = \sum_{j=1}^{k} \underbrace{p_j(T)\, u_j(T)\, p(T)}_{a_j(T) \in A_j} \qquad \text{where } a_j(t) = p_j(t)\, u_j(t)\, p(t),$$

$$\tilde{p}(T) = \sum_{j=1}^{k} \underbrace{p_j(T)\, u_j(T)\, \tilde{p}(T)}_{\tilde{a}_j(T) \in A_j} \qquad \text{where } \tilde{a}_j(t) = p_j(t)\, u_j(t)\, \tilde{p}(t).$$

For $1 \le j \ne l \le k$, $p_j(t) p_l(t)$ is divisible by $(t-\lambda_1)^{r_1}(t-\lambda_2)^{r_2}\cdots(t-\lambda_k)^{r_k} = q(t)$. Then $p_j(T) p_l(T) = 0$. Then $a_j(T)\tilde{a}_l(T) = 0$. Thus,

$$p(T)\tilde{p}(T) = \sum_{j,l=1}^{k} a_j(T)\tilde{a}_l(T) = \sum_{j=1}^{k} \underbrace{a_j(T)\tilde{a}_j(T)}_{\in A_j}.$$

In other words,

$$\left(\sum_{j=1}^{k} a_j(T)\right)\left(\sum_{l=1}^{k} \tilde{a}_l(T)\right) = \sum_{j=1}^{k} a_j(T)\tilde{a}_j(T).$$

We have showed (ii). ✓ (It can be done in a shorter way, but the proof here is natural)

(b) The minimal polynomial of $T$ is again denoted by

$$q(t) = (t-\lambda_1)^{r_1}(t-\lambda_2)^{r_2}\cdots(t-\lambda_k)^{r_k}. \quad ✓$$

It is known in linear algebra that $T$ is diagonalizable if and only if $r_1 = r_2 = \cdots = r_k = 1$. We now show that $r_1 = r_2 = \cdots = r_k = 1$ if and only if each of the algebras $A_1, A_2, \ldots, A_k$ is isomorphic to $\mathbb{C}$. ✓

($\Longrightarrow$) Suppose $r_1 = r_2 = \cdots = r_k = 1$.

For each $j \in \{1, 2, \ldots, k\}$, we define a map $\phi_j : A_j \longrightarrow \mathbb{C}$, $\phi_j(p(T)) = c$, where $c$ is the remainder in the division of $p(t)$ by $(t - \lambda_j)$. First, we show that $\phi_j$ is well-defined.

Suppose $p(T) = \tilde{p}(T)$ for some polynomials $p(t), \tilde{p}(t) \in \mathbb{C}[t]$ that are divisible by $p_j(t)$. Then ~~p(t) and p̃(t) have the same remainder~~ $(p - \tilde{p})(T) = 0$. Thus, $p(t) - \tilde{p}(t)$ is divisible by $q(t)$. Then $p(t) - \tilde{p}(t)$ is divisible by $(t - \lambda_j)$. Then $p(t)$ and $\tilde{p}(t)$ have the same remainder in the division by $(t - \lambda_j)$.

Next, we show that $\phi_j$ is a linear map. Let $p(t), \tilde{p}(t) \in \mathbb{C}[t]$ be polynomials that are divisible by $p_j(t)$ and let $\alpha \in \mathbb{C}$. Write

$$p(t) = (t - \lambda_j)\, u(t) + c, \qquad (4)$$

$$\tilde{p}(t) = (t - \lambda_j)\, \tilde{u}(t) + \tilde{c} \qquad (5)$$

for some $c, \tilde{c} \in \mathbb{C}$. Then $\alpha p(t) + \tilde{p}(t) = (t - \lambda_j)(\alpha u(t) + \tilde{u}(t)) + (\alpha c + \tilde{c})$. Thus,

$$\phi_j(\alpha p(T) + \tilde{p}(T)) = \alpha c + \tilde{c} = \alpha \phi_j(p(T)) + \phi_j(\tilde{p}(T)). \checkmark$$

Next, we show that $\phi_j$ is an algebra homomorphism. Recall that $e_j(T)$ is the unit element of $A_j$ and that $e_j(t) - 1$ is divisible by $(t - \lambda_j)$. Then $\phi_j(e_j(T)) = 1$. For $p(t)$ and $\tilde{p}(t)$ as in (4) and (5), ✓

$$p(t)\tilde{p}(t) = (t-\lambda_j)\left[(t-\lambda_j)u(t)\tilde{u}(t) + c\tilde{u}(t) + \tilde{c}u(t)\right] + c\tilde{c}.$$

Then

$$\phi_j(p(T)\tilde{p}(T)) = c\tilde{c} = \phi_j(p(T))\,\phi_j(\tilde{p}(T)).$$

Next, we show that $\phi_j$ is bijective. Because $\phi_j$ is linear and $\phi_j(\xi_j(T)) = 1$, it is surjective. Suppose $\phi_j(p(T)) = 0$ for some polynomial $p(t)$ that is divisible by $p_j(t)$. Then $p(t)$ is divisible by $(t-\lambda_j)$. Because both $p_j(t)$ and $(t-\lambda_j)$ divide $p(t)$ and have no nontrivial common factor, $p_j(t)(t-\lambda_j) = q(t)$ divides $p(t)$. Then $p(T) = 0$. Thus, $\phi_j$ is bijective.

($\Leftarrow$) Suppose each of the algebras $A_1, A_2, \ldots, A_k$ is isomorphic to $\mathbb{C}$. We show that $r_1 = r_2 = \cdots = r_k = 1$. For each $j \in \{1, 2, \ldots, k\}$, let $\Psi_j : A_j \to \mathbb{C}$ be an algebra isomorphism. Put $p(t) = (t-\lambda_j)\,p_j(t)$. Then $\tilde{p}(t) = p(t)^{r_j}$ is divisible by $(t-\lambda_j)^{r_j}p_j(t) = q(t)$. Then $\tilde{p}(T) = 0$. Then

$$\Psi_j(p(T))^{r_j} = \Psi_j(p(T)^{r_j}) = \Psi_j(\tilde{p}(T)) = \Psi_j(0) = 0.$$

Then $\Psi_j(p(T)) = 0$. Then $p(T) = 0$. Then $p(t)$ is divisible by $q(t)$. In particular, $p(t)$ is divisible by $(t-\lambda_j)^{r_j}$. Because $p_j(t)$ and $(t-\lambda_j)^{r_j}$ have no nontrivial common factor, $(t-\lambda_j)$ must divisible by $(t-\lambda_j)^{r_j}$. This happens only if $r_j = 1$. ✓

(3) Let $n \in \mathbb{N}$. A statement is said to be true for a generic matrix in $M(n, \mathbb{C})$ if it is true in an open dense subset of $M(n, \mathbb{C})$. The topology on

$M(n,\mathbb{C})$ is understood as the topology induced by norm. Since all norms on $M(n,\mathbb{C})$ are equivalent, we can use the $\infty$-norm

$$\left\| \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \right\|_\infty = \max_{1\le j,k\le n} |a_{jk}|.$$

(i) We show that for a generic matrix $T\in M(n,\mathbb{C})$, the algebra generated by $I_n$ and $T$ is isomorphic to the direct sum of $n$ copies of $\mathbb{C}$. Put

$$\mathcal{O} = \{T\in M(n,\mathbb{C}) : T \text{ has } n \text{ distinct eigenvalues}\}. \quad \checkmark$$

For each $T\in\mathcal{O}$, $T$ is diagonalizable. Moreover, the minimal polynomial of $T$ is $q(t) = (t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_n)$ where $\lambda_1,\lambda_2,\dots,\lambda_n$ are the distinct eigenvalues of $T$. By Problem 2, Part (ii), the algebra generated by $I_n$ and $T$ is isomorphic to the direct sum of $n$ copies of $\mathbb{C}$. We show that $\mathcal{O}$ is open and dense in $M(n,\mathbb{C})$. $\checkmark$

First, we show that $\mathcal{O}$ is open in $M(n,\mathbb{C})$. Each matrix $T\in M(n,\mathbb{C})$ is of the form

$$T = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = (a_{jk})_{1\le j,k\le n}.$$

The characteristic polynomial of $T$ is $p_T(t) = \det(tI_n - T) = t^n + \alpha_1 t^{n-1} + \dots + \alpha_{n-1}t + \alpha_n$.

Each coefficient $\alpha_j$ is a polynomial in $a_{11}, a_{12}, \dots, a_{nn}$. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the roots, counted with multiplicity, of $p_T(t)$. The elementary symmetric polynomials of $\lambda_1, \lambda_2, \dots, \lambda_n$ are ✓

$$e_1(\lambda_1, \dots, \lambda_n) = \sum_{j=1}^{n} \lambda_j,$$

$$e_2(\lambda_1, \dots, \lambda_n) = \sum_{1 \leq j < k \leq n} \lambda_j \lambda_k,$$

$$\dots\dots\dots$$

$$e_r(\lambda_1, \dots, \lambda_n) = \sum_{1 \leq j_1 < j_2 < \dots < j_r \leq n} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_r},$$

$$\dots\dots\dots$$

$$e_n(\lambda_1, \dots, \lambda_n) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

By Vieta's formulae, $e_j(\lambda_1, \dots, \lambda_n) = (-1)^j \alpha_j$, and thus is a polynomial in $a_{11}, a_{12}, \dots, a_{nn}$. The discriminant of $p_T(t)$ is

$$\Delta(\lambda_1, \lambda_2, \dots, \lambda_n) = \prod_{1 \leq j < k \leq n} (\lambda_j - \lambda_k)^2.$$

This is a symmetric polynomial in $\lambda_1, \lambda_2, \dots, \lambda_n$. Thus, $\Delta(\lambda_1, \lambda_2, \dots, \lambda_n)$ is a polynomial in $a_{11}, a_{12}, \dots, a_{nn}$. In other words, there exists $Q \in \mathbb{C}[a_{11}, a_{12}, \dots, a_{nn}]$ such that

$$Q(a_{11}, a_{12}, \dots, a_{nn}) = \prod_{1 \leq j < k \leq n} (\lambda_j - \lambda_k)^2. \quad ✓$$

Let $T_0 = (a^0_{jk})_{1 \leq j,k \leq n} \in \mathcal{O}$ and $\lambda^0_1, \lambda^0_2, \dots, \lambda^0_n$ be the distinct eigenvalues of $T_0$. Then $Q(a^0_{11}, a^0_{12}, \dots, a^0_{nn}) = \prod_{1 \leq j < k \leq n} (\lambda^0_j - \lambda^0_k)^2 \neq 0$. Because $Q$ is a continuous function, there exists $\delta > 0$ such that

✓

$$Q(a_{11}, a_{12}, \dots, a_{nn}) \neq 0 \quad \forall a_{11}, a_{12}, \dots, a_{nn} \in \mathbb{C} \text{ satisfying } |a_{jk} - a^0_{jk}| < \delta \text{ for all } 1 \leq j, k \leq n.$$

In other words, if $T = (a_{jk}) \in M(n, \mathbb{C})$, $\|T - T_0\|_\infty < \delta$ then $Q(a_{11}, a_{12}, \dots, a_{nn}) \neq 0$. The nonvanishing of $Q(a_{11}, a_{12}, \dots, a_{nn})$ implies that all eigenvalues of $T$ are simple. If we denote by $B_\delta(T_0)$ the open ball in $M(n, \mathbb{C})$ centered at $T_0$ and with radius $\delta$ then $B_\delta(T_0) \subset \mathcal{O}$. Therefore, $\mathcal{O}$ is open in $M(n, \mathbb{C})$.

Next, we show that $\mathcal{O}$ is dense in $M(n, \mathbb{C})$. As noticed in the first part of the proof,

$$T = (a_{jk})_{1 \leq j, k \leq n} \in \mathcal{O} \Leftrightarrow Q(a_{11}, a_{12}, \dots, a_{nn}) \neq 0.$$

Because $\mathcal{O}$ is a nonempty set, $Q$ is not the trivial polynomial. Take $T_0 = (a^0_{jk})_{1 \leq j, k \leq n} \in M(n, \mathbb{C})$. We show that any neighborhood of $T_0$ contains an element in $\mathcal{O}$. Suppose by contradiction that this is not true. Then there exists a ball $B_\varepsilon(T_0)$ in $M(n, \mathbb{C})$ such that $B_\varepsilon(T_0) \subset M(n, \mathbb{C}) \setminus \mathcal{O}$. Then

$$Q(a_{11}, a_{12}, \dots, a_{nn}) = 0 \quad \forall a_{11}, a_{12}, \dots, a_{nn} \in \mathbb{C} \text{ satisfying } |a_{jk} - a^0_{jk}| < \varepsilon \text{ for all } 1 \leq j, k \leq n.$$

Thus, the set of zeros of $Q$ has nonempty interior. We get a contradiction by applying the following lemma for $m = n^2$ and $P(z_1, z_2, \dots, z_{n^2}) = Q(a_{11}, a_{12}, \dots, a_{nn})$.

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**Lemma 1**

Let $P \in \mathbb{C}[z_1,\dots,z_m]\setminus\{0\}$. Then the set of zeros of $P$

$$\Gamma = \{(z_1,\dots,z_m) : P(z_1,\dots,z_m) = 0\}$$

has empty interior in $\mathbb{C}^m$. ✓

Proof of the lemma

Note that this follows from Taylor expansion $P(z) = P(z_0) + P'(z_0)(z-z_0) + \frac{1}{2}P''(z-z_0)(z-z_0) + \dots$ etc. But the proof here is also natural

We prove by induction on $m \in \mathbb{N}$. If $m=1$ then $P = P(z)$ is a nonzero polynomial of one variable. It has finitely many zeros. Thus, $\Gamma$ has empty interior in $\mathbb{C}$. Suppose the lemma is true for $m = n-1$ for some $n \geq 2$. Let $P \in \mathbb{C}[z_1,\dots,z_n]\setminus\{0\}$. We use the $\infty$-norm on $\mathbb{C}^n$. Suppose by contradiction that $\Gamma = \{(z_1,\dots,z_n) : P(z_1,\dots,z_n) = 0\}$ contains a ball $B_\varepsilon(a_0)$ in $\mathbb{C}^n$ where $\varepsilon > 0$, $a_0 = (a_1^0, a_2^0, \dots, a_n^0)$. By rearranging the arguments $z_1, z_2, \dots, z_n$ if necessary, we can assume $z_1$ is a variable that appears in $P(z_1,\dots,z_n)$ with highest power. Then

$$P(z_1,z_2,\dots,z_n) = Q_k(z_2,\dots,z_n)z_1^k + \dots + Q_1(z_2,\dots,z_n)z_1 + Q_0(z_2,\dots,z_n), \qquad (1)$$

where $k \geq 0$ and $Q_k \not\equiv 0$. The set of zeros of $Q_k$

$$\Gamma' = \{(z_2,\dots,z_n) \in \mathbb{C}^{n-1} : Q_k(z_2,\dots,z_n) = 0\}$$

has ~~non~~ empty interior by the induction hypothesis. Thus, the set

$$B = \{(a_2,\dots,a_n) \in \mathbb{C}^{n-1} : |a_j - a_j^0| < \varepsilon\}$$

is not contained in $\Gamma'$. There exists $(a_2,\dots,a_n) \in B \setminus \Gamma'$. For every $z_1 \in \mathbb{C}$, $|z_1 - a_1^0| < \varepsilon$, we have $(z_1, a_2, \dots, a_n) \in B_\varepsilon(a_0)$.

Then $P(z_1, a_2, \dots, a_n) = 0$ for all $z_1 \in \mathbb{C}$, $|z_1 - a_1^0| < \varepsilon$. Then (1) gives us

$$Q_k(a_2, \dots, a_n) z_1^k + \dots + Q_1(a_2, \dots, a_n) z_1 + Q_0(a_2, \dots, a_n) = 0 \quad \forall z_1 \in \mathbb{C}, |z_1 - a_1^0| < \varepsilon.$$

This is a polynomial in $z_1$ that has infinitely many zeros. It must be a trivial polynomial. In particular, $Q_k(a_2, \dots, a_n) = 0$. Then $(a_2, \dots, a_n) \in \Gamma'$. This is a contradiction. ✓

(ii) We show that for every two generic matrices $S, T \in M(n, \mathbb{C})$, the only subspace of $\mathbb{C}^n$ invariant under both $S$ and $T$ are $\{0\}$ and $\mathbb{C}^n$. As in Part (i), let $\mathcal{O}$ be the set of all matrices in $M(n, \mathbb{C})$ that have $n$ distinct eigenvalues. For $\delta > 0$ and $T_0 \in M(n, \mathbb{C})$, we still denote by $B_\delta(T_0)$ the ~~unit~~ ball centered at $T_0$ with radius $\delta$ in the $\infty$-norm on $M(n, \mathbb{C})$. We need two following lemmas.

<u>Lemma 2</u>

For each $T_0 \in \mathcal{O}$, there exist $\delta = \delta_{T_0} > 0$ and a continuous map $P : B_\delta(T_0) \to GL(n, \mathbb{C})$ such that

- $B_\delta(T_0) \subset \mathcal{O}$,
- $P_T^{-1} T P_T$ is diagonal $\forall T \in B_\delta(T_0)$.

Here $P_T$ is a convenient notation for $P(T)$. ✓

For each matrix $T \in M(n, \mathbb{C})$, we denote by $\phi(T)$ the product of all entries of $T$. In other words, if $T = (a_{jk})_{1 \le k, j \le n}$ then

$$\phi(T) = \prod_{1\le j,k\le n} a_{jk}.$$

Lemma 3

For $T_0 \in \mathcal{O}$, let $\delta > 0$ and the map $P: B_\delta(T_0) \to GL(n,\mathbb{C})$ be as in Lemma 2. Suppose $S \in M(n,\mathbb{C})$ and $T \in B_\delta(T_0)$ satisfy $\phi(P_T^{-1} S P_T) \neq 0$. Then the only subspace of $\mathbb{C}^n$ that are invariant under both $S$ and $T$ are $\{0\}$ and $\mathbb{C}^n$. ✓

Assuming these lemmas, we proceed with the proof. For each $T_0 \in \mathcal{O}$, put

$$\mathcal{U}_{T_0} = \{(S,T) \in M(n,\mathbb{C}) \times B_{\delta_{T_0}}(T_0) : \phi(P_T^{-1} S P_T) \neq 0\}.$$

We show that $\mathcal{U}_{T_0}$ is open in $M(n,\mathbb{C}) \times M(n,\mathbb{C})$. Because $P: B_{\delta_{T_0}}(T_0) \to GL(n,\mathbb{C})$ and $\phi: M(n,\mathbb{C}) \to \mathbb{C}$ are continuous, the map $\Psi: M(n,\mathbb{C}) \times B_{\delta_{T_0}}(T_0) \to \mathbb{C}$,

$$\Psi(S,T) = \phi(P_T^{-1} S P_T)$$

is also continuous. Then

$$\mathcal{U}_{T_0} = \{(S,T) \in M(n,\mathbb{C}) \times B_{\delta_{T_0}}(T_0) : \Psi(S,T) \neq 0\}$$

$$= \Psi^{-1}(\mathbb{C} \setminus \{0\})$$

is open in $M(n,\mathbb{C}) \times B_{\delta_{T_0}}(T_0)$. Thus, $\mathcal{U}_{T_0}$ is open in $M(n,\mathbb{C}) \times M(n,\mathbb{C})$.

Put $\mathcal{U} = \bigcup_{T_0 \in \mathcal{O}} \mathcal{U}_{T_0}$. Then $\mathcal{U}$ is open in $M(n,\mathbb{C}) \times M(n,\mathbb{C})$. We show that $\mathcal{U}$ is a dense subset. ✓

Let $(S', T') \in M(n,\mathbb{C}) \times M(n,\mathbb{C})$ and $\varepsilon > 0$. We know from Part (c) that $\mathcal{O}$ is dense in $M(n,\mathbb{C})$. Thus, there exists $T \in \mathcal{O}$ such that $\|T' - T\|_\infty < \varepsilon$. Because $\mathcal{O} = \bigcup_{T_0 \in \mathcal{O}} B_{\delta_{T_0}}(T_0)$, there exists $T_0 \in \mathcal{O}$ such that $T \in B_{\delta_{T_0}}(T_0)$.

For

$$S = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in M(n,\mathbb{C}),$$

✓

each entry of $P_T^{-1} S P_T$ is a linear combination of $a_{11}, a_{12}, \dots, a_{nn}$. Then $\phi(P_T^{-1} S P_T)$, which is the product of all entries of $P_T^{-1} S P_T$, is a polynomial in $a_{11}, a_{12}, \dots, a_{nn}$. Write

$$R(a_{11}, a_{12}, \dots, a_{nn}) = \phi(P_T^{-1} S P_T).$$

At

$$S = P_T \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \\ 1 & 1 & \cdots & 1 \end{pmatrix} P_T^{-1},$$

the value of $R$ is equal to 1. Thus, $R$ is not the trivial polynomial. By Lemma 1, the set of zeros of $R$, denoted by $\Gamma(R)$, has empty interior. Then $B_\varepsilon(S') \not\subset \Gamma(R)$. There exists $S \in B_\varepsilon(S') \setminus \Gamma(R)$. That is $\|S - S'\|_\infty < \varepsilon$ and $\phi(P_T^{-1} S P_T) \neq 0$. Then $(S,T) \in \mathcal{U}_{T_0} \subset \mathcal{U}$ and

$$\|(S', T') - (S, T)\|_\infty = \max\{\|S' - S\|_\infty, \|T' - T\|_\infty\} < \varepsilon.$$

✓

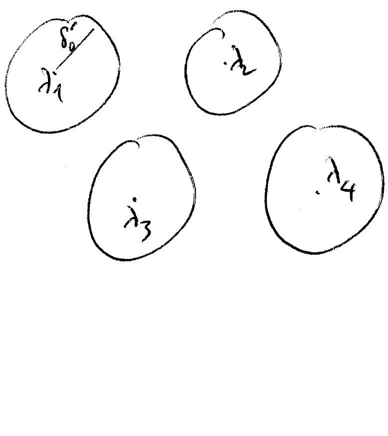
We have showed that $\mathcal{U}$ is dense in $M(n,\mathbb{C})\times M(n,\mathbb{C})$.

Now let $(S,T)\in\mathcal{U}$. There exists $T_0\in\mathcal{O}$ such that $(S,T)\in\mathcal{U}_{T_0}$. Then $S\in M(n,\mathbb{C})$, $T\in B_{\delta_{T_0}}(T_0)$ and $\phi(P_T^{-1}ST)\neq 0$. By Lemma 3, the only subspaces of $\mathbb{C}^n$ that are invariant under both $S$ and $T$ are $\{0\}$ and $\mathbb{C}^n$. We conclude that $\mathcal{U}$ is an open dense subset of $M(n,\mathbb{C})\times M(n,\mathbb{C})$ such that for every $(S,T)\in\mathcal{U}$, the only subspaces of $\mathbb{C}^n$ that are invariant under both $S$ and $T$ are $\{0\}$ and $\mathbb{C}^n$. ✓

Proof of Lemma 2

Let $T_0=(a_{jk}^0)_{1\le j,k\le n}\in\mathcal{O}$. Denote by $\lambda_1^0,\lambda_2^0,\dots,\lambda_n^0$ the distinct eigenvalues of $T_0$. For $n=1$, a matrix is simply a complex number; the lemma is proved by choosing $\delta=1$ and $P(T)\equiv 1$. Consider the case $n\geqslant 2$. We show that the eigenvalues depend continuously on the matrix. ✓

Put $\delta_0'=\frac{1}{3}\min\{|\lambda_j^0-\lambda_k^0| : 1\le j<k\le n\}$. The disks $\mathbb{D}(\lambda_1,\delta_0')$, $\mathbb{D}(\lambda_2,\delta_0'),\dots,\mathbb{D}(\lambda_n,\delta_0')$ in $\mathbb{C}$ are pairwise disjoint.




Because $\mathcal{O}$ is open in $M(n,\mathbb{C})$, there exists $\delta_0>0$ such that $B_{\delta_0}(T_0)\subset\mathcal{O}$. For each $T=(a_{jk})_{1\le j,k\le n}\in M(n,\mathbb{C})$, we define a function $p:\mathbb{C}^n\times\mathbb{C}\to\mathbb{C}$, ✓

$$p(a_{11}, a_{12}, \dots, a_{nn}, t) = \det(tI_n - T).$$

If we fix $a_{11}, a_{12}, \dots, a_{nn}$ and view $p$ as a polynomial in $t$ then its roots are the eigenvalues of $T$. For $T \in B_{\delta_0}(T_0)$, $p(a_{11}, a_{12}, \dots, a_{nn}, \cdot)$ only has simple roots. Then for each $j \in \{1, 2, \dots, n\}$,

$$\begin{cases} p(a_{11}^0, a_{12}^0, \dots, a_{nn}^0, \lambda_j^0) = 0, \\ \dfrac{\partial p}{\partial t}(a_{11}^0, a_{12}^0, \dots, a_{nn}^0, \lambda_j^0) \neq 0. \end{cases}$$

By the Implicit Function Theorem, there exist $\delta_j > 0$ and $\delta_j' \in (0, \delta_0']$ such that for $T = (a_{jk})_{1 \le j,k \le n} \in B_{\delta_j}(T_0)$ the equation

$$p(a_{11}, a_{12}, \dots, a_{nn}, t) = 0$$

has only one solution $t$ in $D(\lambda_j^0, \delta_j')$. Moreover, this solution depends $C^1$ in $(a_{11}, a_{12}, \dots, a_{nn})$.

$$t = \lambda_j(a_{11}, a_{12}, \dots, a_{nn}).$$

Put $\delta = \min\{\delta_0, \delta_1, \dots, \delta_n\} > 0$. Then each $T \in B_\delta(T_0)$ has $n$ distinct eigenvalues $\lambda_j \in D(\lambda_j^0, \delta_0')$, $1 \le j \le n$. Each can be viewed as a $C^1$ function $\lambda_j : B_\delta(T_0) \to \mathbb{C}$.

Next, we show that the ~~eigenvalue~~ eigenvectors of $T \in B_\delta(T_0)$ can be chosen to depend continuously on $T$. Let $v_j^0 \in \mathbb{C}^n$ be an eigenvector corresponding to the eigenvalue $\lambda_j^0$ of $T_0$.

$$\mathbb{C}^n = \underbrace{\mathbb{C}v_1^0}_{X_1} \oplus \underbrace{\mathbb{C}v_2^0 \oplus \cdots \oplus \mathbb{C}v_n^0}_{Y_1}$$

Then $(T_0 - \lambda_1^0 I_n)|_{X_1} = 0$ and $(T_0 - \lambda_1^0 I_n)|_{Y_1} : Y_1 \to Y_1$ is a linear isomorphism.

Put $S_0 = T_0 - \lambda_1^0 I_n$ and $S = T - \lambda_1 I_n$. Then $S$ is a (continuous) perturbation of $S_0$ as $T$ deviates from $T_0$. Lecture 02/20/2015 gives us a method to solve the equation $Sx = 0$. ✓

Each $x \in \mathbb{C}^n$ is represented by a pair $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ where $x_1 \in X_1$, $y_1 \in Y_1$ and $x_1 + y_1 = x$. Similarly, $S$ is represented by a matrix $\begin{pmatrix} S_{XX} & S_{XY} \\ S_{YX} & S_{YY} \end{pmatrix}$

where $S_{XX} : X_1 \to X_1$, $S_{XY} : Y_1 \to X_1$, $S_{YX} : X_1 \to Y_1$, $S_{YY} : Y_1 \to Y_1$,

$$Sx = \begin{pmatrix} S_{XX} & S_{XY} \\ S_{YX} & S_{YY} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} S_{XX} x_1 + S_{XY} y_1 \\ S_{YX} x_1 + S_{YY} y_1 \end{pmatrix}.$$

We have

$$S_0 = \begin{pmatrix} 0 & 0 \\ 0 & S_{YY}^0 \end{pmatrix}$$

where $S_{YY}^0 : Y_1 \to Y_1$ is a linear isomorphism. Since $S$ is a perturbation of $S_0$,

$$S = S_0 + \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ C & S_{YY}^0 + D \end{pmatrix}$$

where the operator norms of $A, B, C, D$ are small. If $\|D\|$ is small enough, $S_{YY}^0 + D$ is still an isomorphism, i.e. its determinant is still nonzero. ✓

This amounts to a further shrinking of $\delta > 0$.

$$S x = \begin{pmatrix} A & B \\ C & S_{YY}^0 + D \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} A x_1 + B y_1 \\ C x_1 + (S_{YY}^0 + D) y_1 \end{pmatrix}.$$

The equation $Sx = 0$ becomes

$$\begin{cases} A x_1 + B y_1 = 0, \\ C x_1 + (S_{YY}^0 + D) y_1 = 0. \end{cases}$$

The second equation gives $y_1 = -(S_{YY}^0 + D)^{-1} C x_1$. Then the first equation becomes

$$\left(A - B (S_{YY}^0 + D)^{-1} C\right) x_1 = 0. \qquad (2)$$

In our circumstance, $X = \mathbb{C} v_1^0$. Thus, $x_1 = \alpha v_1^0$. To get $x = x_1 + y_1 \neq 0$, $\alpha$ must be nonzero. Then (2) is satisfied for any choice of $\alpha \in \mathbb{C} \setminus \{0\}$. We choose $\alpha = 1$. Then

$$x = x_1 + y_1 = \left(I_n - (S_{YY}^0 + D)^{-1} C\right) v_1^0.$$

Because $D$ is a perturbation of $0$, $(S_{YY}^0 + D)^{-1}$ is a perturbation of $(S_{YY}^0)^{-1}$. Thus, $x$ is a perturbation of $\left(I_n - (S_{YY}^0)^{-1} 0\right) v_1^0 = v_1^0$.

We have chosen a solution of $S = (T - \lambda_1 I_n)$ as a perturbation of $v_1^0$. It is an eigenvector of $T$ corresponding to eigenvalue $\lambda_1$, which we denote by $v_1 = v_1(a_{11}, a_{12}, \ldots, a_{nn})$. Similarly to other eigenvalues $\lambda_2, \lambda_3, \ldots, \lambda_n$ : a corresponding eigenvector can be chosen to depend continuously on $v_2^0, v_3^0, \dots, v_n^0$ respectively. Write

$$v_j = v_j(a_{11}, a_{12}, \dots, a_{nn}).$$

Then $(v_1, v_2, \dots, v_n)$ is a basis of $\mathbb{C}^n$ that diagonalizes $T$. Put

$$P_T = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad \forall T = (a_{jk})_{1 \le j,k \le n} \in B_\delta(T_0).$$

Then the map $P: B_\delta(T_0) \to GL(n, \mathbb{C})$, $P(T) = P_T$ is continuous. Moreover,

$$P_T^{-1} T P_T = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

✓

Proof of Lemma 3

Let $T_0 \in \mathcal{O}$. Let $\delta > 0$ and $P: B_\delta(T_0) \to GL(n, \mathbb{C})$ be as in the statement of Lemma 2. Let $S \in M(n, \mathbb{C})$ and $T \in B_\delta(T_0)$ such that $\phi(P_T^{-1} S P_T) \neq 0$. Recall that $\phi(A)$ is simply the product of all entries of matrix $A$.

Suppose by contradiction that there is a subspace $Y$ of $\mathbb{C}^n$, $1 \le \dim Y \le n-1$, that is invariant under both $S$ and $T$. As constructed in Lemma 2,

$$P_T = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad P_T^{-1} T P_T = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of $T$ and $v_1, v_2, \dots, v_n$ are the corresponding eigenvectors. $P_T^{-1} S P_T$ is the matrix representing $S$ in the

✓

basis $\mathcal{B} = (v_1, v_2, \ldots, v_n)$. In other words, if we write

$$P_T^{-1} S P_T = (u_{jk})_{1 \le j,k \le n}$$

then $Sv_j = \sum_{k=1}^{n} u_{jk} v_k$. We have

$$\phi(P_T^{-1} S P_T) = \prod_{j,k=1}^{n} u_{jk}.$$

Thus, $u_{jk} \neq 0$ for all $1 \le j,k \le n$.

Since $T$ is diagonalizable and $T(Y) \subset Y$, the linear map $T_1 = T|_Y : Y \to Y$ is also diagonalizable. Then $Y$ has a basis consisting of eigenvectors of $T_1$. An eigenvalue of $T_1$ is also an eigenvalue of $T$. Thus, an eigenvector of $T_1$ is an eigenvector of $T$. We know that all eigenvectors of $T$ are the nonzero scalar multiples of $v_1, v_2, \ldots, v_n$. Hence, $Y$ has a basis of the form $(v_{j_1}, v_{j_2}, \ldots, v_{j_r})$, $1 \le r \le n-1$. Take $1 \le \ell \le n$, $\ell \notin \{j_1, j_2, \ldots, j_r\}$. We have

$$\sum_{k=1}^{n} u_{j_1 k} v_k = S v_{j_1} \in Y = \operatorname{span}\{v_{j_1}, v_{j_2}, \ldots, v_{j_r}\}.$$

Thus, $u_{j_1 \ell} = 0$. This is a contradiction. ✓

(iii) For every two generic matrices $S, T \in M(n, \mathbb{C})$, we show that the algebra generated by $I_n$, $S$ and $T$ coincides with $M(n, \mathbb{C})$. For $n = 1$, this is true for all $S, T \in M(n, \mathbb{C})$. Consider the case $n \geq 2$.

Recall that $\mathcal{O}$ denotes the set of all matrices that have $n$ distinct eigenvalues. Let $\mathcal{U} \subset M(n,\mathbb{C}) \times \mathcal{O}$ be the set as constructed in Part (ii). We know that $\mathcal{U}$ is an open dense subset of $M(n,\mathbb{C}) \times M(n,\mathbb{C})$ and that for any $(S,T) \in \mathcal{U}$, the only subspaces of $\mathbb{C}^n$ that are invariant under both $S$ and $T$ are $\{0\}$ and $\mathbb{C}^n$. ✓

Now take $(S,T) \in \mathcal{U}$. We show that the algebra $\mathcal{A}$ generated by $I_n, S$ and $T$ is equal to $M(n,\mathbb{C})$. Because $T \in \mathcal{O}$, it has $n$ distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Let $v_1, v_2, \dots, v_n \in \mathbb{C}^n$ be (arbitrarily chosen) corresponding eigenvectors. They form a basis for $\mathbb{C}^n$. For each $j, k \in \{1,2,\dots,n\}$, let $E_{jk} \in M(n,\mathbb{C})$ be the matrix such that

$$E_{jk} v_\ell = \begin{cases} v_k & \text{if } \ell = j, \\ 0 & \text{if } \ell \neq j. \end{cases}$$

Then $(E_{jk})_{1 \le j,k \le n}$ is a linear basis of $M(n,\mathbb{C})$. Thus, it suffices to show $E_{jk} \in \mathcal{A}$ for all $1 \le j,k \le n$. We will show $E_{1k} \in \mathcal{A}$ for each $k \in \{1,2,\dots,n\}$. ✓ Each of the cases $j \in \{2,3,\dots,n\}$ is dealt similarly.

Take $k \in \{1,2,\dots,n\}$. Denote

$$\mathcal{F} = \{ B \in \mathcal{A} : Bv_2 = Bv_3 = \dots = Bv_n = 0 \}.$$

Then $\mathcal{F}$ is a linear space and $I_n \notin \mathcal{F}$. For any $B' \in \mathcal{A}, B \in \mathcal{F}$, we have $B'B \in \mathcal{F}$. Thus, $\mathcal{F}$ is a left ideal of $\mathcal{A}$. Consider the matrix

$$B_0 = (\lambda_2 I_n - T)(\lambda_3 I_n - T) \cdots (\lambda_n I_n - T) \in \mathcal{A}.$$ ✓

For $2 \leq \ell \leq n$,

$$B_0 v_\ell = \prod_{j=2}^{n} (\lambda_j I_n - T) v_\ell = \prod_{\substack{j=2 \\ j\neq \ell}}^{n} (\lambda_j I_n - T) \underbrace{(\lambda_\ell I_n - T) v_\ell}_{=0} = 0.$$

Thus, $B_0 \in \mathcal{F}$.

$$\begin{aligned} B_0 v_1 &= (\lambda_2 I_n - T) \cdots (\lambda_{n-1} I_n - T)(\lambda_n I_n - T) v_1 \\ &= (\lambda_2 I_n - T) \cdots (\lambda_{n-1} I_n - T)(\lambda_n - \lambda_1) v_1 \\ &= \cdots \\ &= (\lambda_2 - \lambda_1) \cdots (\lambda_{n-1} - \lambda_1)(\lambda_n - \lambda_1) \neq 0. \end{aligned}$$

Put $Y = \{ B v_1 : B \in \mathcal{F} \}$. Then $Y$ is a linear subspace of $\mathbb{C}^n$. It is not the trivial subspace because $B_0 v_1 \in Y$ and $B_0 v_1 \neq 0$. Since $\mathcal{F}$ is a left ideal of $\mathcal{A}$,

$$SB, TB \in \mathcal{F} \quad \forall B \in \mathcal{F}.$$

Then

$$\left.\begin{aligned} S(Bv_1) &= (SB) v_1 \in Y, \\ T(Bv_1) &= (TB) v_1 \in Y \end{aligned}\right\} \quad \forall B \in \mathcal{F}.$$

Thus, $Y$ is invariant under both $S$ and $T$. Then $Y = \mathbb{C}^n$. In particular, $v_k \in Y$. Then there exists $B_k \in \mathcal{F}$ such that $B_k v_1 = v_k$. Because $B_k v_1 = v_k$, $B_k v_2 = B_k v_3 = \cdots = B_k v_n = 0$, we get $B_k = E_{1k}$. Therefore, $E_{1k} \in \mathcal{A}$.

The idea of introducing $\mathcal{F}$ and $Y$ was learned from T. Y. Lam in his article "A Theorem of Burnside on Matrix Rings", 1998.

4) Let $(\Omega, \Sigma, \mu)$ be a measure space in which every subset with positive measure contains a subset with positive and strictly smaller measure.

Let $X = L^2(\Omega, \mu)$ and $m \in L^\infty(\Omega, \mu)$. Consider the map $T : X \to X$, $Tf = mf$. It is well-defined because for each $f \in X$, $mf$ is a measurable function and

$$\|mf\|_X = \left(\int_\Omega m^2 f^2 \, d\mu\right)^{1/2} \le \|m\|_\infty \left(\int_\Omega f^2 \, d\mu\right)^{1/2} = \|m\|_\infty \|f\|_X < \infty.$$

$T$ is also a linear operator. We show that $T$ is compact if and only if $m = 0$ almost everywhere.

($\Leftarrow$) Suppose $m = 0$ almost everywhere. Then $Tf = mf = 0$ almost everywhere. This means $T$ is the trivial linear operator. It is a compact operator.

($\Rightarrow$) Suppose $T$ is a compact operator. Then $-T$ is also a compact operator. Suppose by contradiction that the set

$$A = \{x \in \Omega : m(x) \neq 0\}$$

has positive measure. Put

$$A_+ = \{x \in \Omega : m(x) > 0\},$$
$$A_- = \{x \in \Omega : m(x) < 0\}.$$

Because $A_+, A_- \in \Sigma$, $A_+ \cap A_- = \emptyset$, $A_+ \cup A_-$, we have $\mu(A_+) + \mu(A_-) = \mu(A) > 0$. Then either $\mu(A_+) > 0$ or $\mu(A_-) > 0$. By replacing $m$ with $-m$ (thereby replacing $T$ with $-T$) if necessary, we can assume $\mu(A_+) > 0$.

Put

$$A_n = \{x \in \Omega : m(x) > \tfrac{1}{n}\} \quad \forall n \in \mathbb{N}.$$

Then $A_n \in \Sigma$ and $\bigcup_{n=1}^{\infty} A_n = A_+$.

$$0 < \mu(A_+) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

There exists $n_0 \in \mathbb{N}$ such that $\mu(A_{n_0}) > 0$. Denote $E = A_{n_0}$ and $Y = L^2(E,\mu)$.
We identify each function $f \in L^2(E,\mu)$ with its extension

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in E, \\ 0 & \text{if } x \in \Omega \setminus E. \end{cases}$$

Then $Y$ can be viewed as a subset of $X$. We show two following properties.

(i) $B_Y\left(\frac{1}{n_0}\right) \subset T(B_X(1))$,

(ii) $Y$ is infinitely dimensional.

Here $B_X(r)$ and $B_Y(r)$ denote the balls centered at $0$ with radius $r$ in $X$ and $Y$ respectively. Suppose (i) and (ii) were proved. Because $T: X \to X$ is a compact operator, $\overline{T(B_X(1))}$ is compact. By (i), $\overline{B_Y\left(\frac{1}{n_0}\right)} \subset \overline{T(B_X(1))}$. Here the closures of $B_Y\left(\frac{1}{n_0}\right)$ in $X$ and in $Y$ are the same because $Y$ is closed in $X$. Then $\overline{B_Y\left(\frac{1}{n_0}\right)}$ is compact. This happens only if $Y$ is finite dimensional, which contradicts with (ii).

Proof of (i)

Let $f \in B_Y\left(\frac{1}{n_0}\right)$. For $x \in E$, $m(x) > \frac{1}{n_0}$. The function $g : \Omega \to \mathbb{C}$,

$$g(x) = \begin{cases} \dfrac{f(x)}{m(x)} & \text{if } x \in E, \\ 0 & \text{if } x \in \Omega \setminus E \end{cases}$$

is well-defined and measurable.

$$\|g\|_X = \left(\int_E \frac{f^2}{m^2}\,d\mu\right)^{1/2} \le \left(\int_E n_0^2 f^2\,d\mu\right)^{1/2} = n_0\|f\|_Y < 1.$$

Thus, $g \in B_X(1)$. We have

$$(Tg)(x) = m(x)g(x) = \begin{cases} f(x) & \text{if } x \in E, \\ 0 & \text{if } x \in \Omega\setminus E \end{cases} \quad \text{a.e.}$$

$$= f(x) \quad \text{a.e.}$$

Hence, $f = Tg \in T(B_X(1))$.

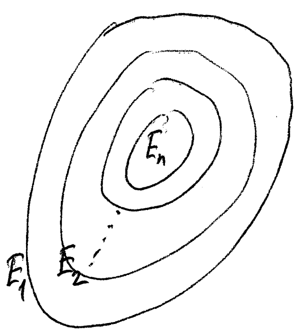
Proof of (ii)

Because $\mu(E) > 0$, there exists $E_1 \in \Sigma$, $E_1 \subset E$ such that $\mu(E) > \mu(E_1) > 0$. Then there exists $E_2 \in \Sigma$, $E_2 \subset E_1$ such that $\mu(E_1) > \mu(E_2) > 0$. Then there exists $E_3 \in \Sigma$, $E_3 \subset E_2$ such that $\mu(E_2) > \mu(E_3) > 0$. Repeating this process gives us a decreasing sequence $(E_n)$ in $\Sigma$ such that

$$\infty \geq \mu(E) > \mu(E_1) > \mu(E_2) > \mu(E_3) > \dots > 0.$$

Then the characteristic functions $\chi_{E_1}, \chi_{E_2}, \chi_{E_3}, \dots$ belong to $L^2(E,\mu)$. To show that $Y$ is infinite dimensional, it suffices to show that $\chi_{E_1}, \chi_{E_2}, \chi_{E_3}, \dots$ are linearly independent. Let $n \in \mathbb{N}$ and $c_1, \dots, c_n \in \mathbb{C}$ be such that

$$c_1\chi_{E_1} + c_2\chi_{E_2} + \dots + c_n\chi_{E_n} = 0 \quad \text{a.e.}$$




We have $$0 = (c_1\chi_{E_1} + c_2\chi_{E_2} + \dots + c_n\chi_{E_n})\chi_{E_n}$$
$$= (c_1 + c_2 + \dots + c_n)\chi_{E_n} \quad \text{a.e.}$$

Integrating both sides over $\Omega$, we get $0=(c_1+c_2+\dots+c_n)\mu(E_1)$. Then $c_1+c_2+\dots+c_n=0$. For each $1\le k\le n-1$,

$$\chi_{E_j}\,\chi_{E_k\setminus E_{k+1}} = \begin{cases} \chi_{E_k\setminus E_{k+1}} & \text{if } 1\le j\le k, \\ 0 & \text{if } k<j\le n. \end{cases}$$

Then

$$0=\Big(\sum_{j=1}^{n} c_j\chi_{E_j}\Big)\chi_{E_k\setminus E_{k+1}} = \sum_{j=1}^{n} c_j\chi_{E_j}\chi_{E_k\setminus E_{k+1}} = \sum_{j=1}^{k} c_j\,\chi_{E_k\setminus E_{k+1}}.$$

Integrating both sides over $\Omega$, we get

$$0=(c_1+c_2+\dots+c_k)\mu(E_k\setminus E_{k+1}) = (c_1+c_2+\dots+c_k)\underbrace{(\mu(E_k)-\mu(E_{k+1}))}_{>0}.$$

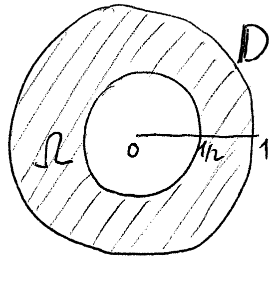
Then $c_1+c_2+\dots+c_k=0$ for all $1\le k\le n$. Then

$$c_k=\sum_{j=1}^{k} c_j-\sum_{j=1}^{k-1} c_j = 0-0=0 \quad \forall\, 1\le k\le n.$$

Therefore, $\chi_{E_1},\chi_{E_2},\chi_{E_3},\dots$ are linearly independent in $Y$.

⑤ Denote

$$\mathbb{D}=\{z\in\mathbb{C}: |z|<1\},$$
$$A=\{f:\mathbb{D}\to\mathbb{C} \text{ holomorphic, can be continuously extended to } \overline{\mathbb{D}}\},$$
$$\Omega=\{z\in\mathbb{C}: \tfrac{1}{2}<|z|<1\},$$
$$B=\{g:\Omega\to\mathbb{C} \text{ holomorphic, can be continuously extended to } \overline{\Omega}\}.$$




Then $A$ and $B$ are complex algebras in which the multiplication of two functions is defined as pointwise multiplication. The unit element of each algebra is the constant function 1. Moreover, $A$ and $B$ are Banach algebras with norms

$$\|f\|_A = \sup_{z\in\mathbb{D}} |f(z)|, \qquad \|g\|_B = \sup_{z\in\Omega} |g(z)|.$$

(i) Let $f\in A$. We show that $\|f|_\Omega\|_B = \|f\|_A$. Because $f|_\Omega : \Omega\to\mathbb{C}$ is holomorphic and extends to a continuous function $f|_{\bar\Omega} : \bar\Omega\to\mathbb{C}$, $f|_\Omega \in B$. ✓

$$\|f|_\Omega\|_B = \sup_{z\in\Omega} |f|_\Omega(z)| = \sup_{z\in\Omega}|f(z)| = \sup_{z\in\bar\Omega}|f(z)|,$$

$$\|f\|_A = \sup_{z\in\mathbb{D}}|f(z)| = \sup_{z\in\bar{\mathbb{D}}}|f(z)|.$$ ✓

Since $\bar\Omega\subset\bar{\mathbb{D}}$, $\|f|_\Omega\|_B \le \|f\|_A$. The function $f$ is continuous in $\bar{\mathbb{D}}$ and holomorphic in $\mathbb{D}$. By the Maximum principle, the maximum of $|f(z)|$ in $\bar{\mathbb{D}}$ is assumed on the boundary of $\bar{\mathbb{D}}$. Thus, there exists $z_0\in\mathbb{C}$, $|z_0|=1$ such that $\|f\|_A = |f(z_0)|$. Because $z_0\in\bar\Omega$, $|f(z_0)| \le \sup_{z\in\bar\Omega}|f(z)| = \|f|_\Omega\|_B$.

Then $\|f\|_A \le \|f|_\Omega\|_B$. We conclude $\|f\|_A = \|f|_\Omega\|_B$.

(ii) Thanks to Part (i), the map $f\in A\mapsto f|_\Omega\in B$ is isometric. It is also an algebra morphism. Through this map, $A$ can be considered as a subalgebra of $B$. We show that $B^\times\cap A\not\subset A^\times$. ✓

The function $h(z)=z$ and $k(z)=\frac{1}{z}$ are holomorphic in $\mathbb{C}\setminus\{0\}$. Thus, $h|_\Omega, k|_\Omega\in B$ and $h|_{\mathbb{D}}\in A$. Since $h(z)k(z)=1$ for all $z\in\Omega$, $k|_\Omega$ is the inverse of $h|_\Omega$ in $B$. Then $h|_\Omega\in B^\times$. Then $h|_\Omega\in B^\times\cap A$. ✓

Suppose by contradiction that $h|_\Omega \in A^\times$. Then there exists $k_1 \in A$ such that $h(z)k_1(z) = 1$ for all $z \in \mathbb{D}$. Then

$$1 = h(0)k_1(0) = 0 \cdot k_1(0) = 0.$$

This is a contradiction. ✓

⑥ (a) Let $X_m = C^m([0,1], \mathbb{C})$ for each $m = 0, 1, 2, \ldots$ It is a complex Banach space with norm
$$\|f\|_m = \sum_{k=0}^{m} \sup_{x \in [0,1]} |f^{(k)}(x)|.$$

Let $m \in \mathbb{N}$ and $a_1, a_2, \ldots, a_m \in \mathbb{C}$. Define a map $L : X_m \to X_0$,
$$Lf = f^{(m)} + a_1 f^{(m-1)} + \cdots + a_{m-1} f^{(1)} + a_m f.$$

We show that $L$ is a Fredholm operator. By the definition, $L$ is linear.
$$\|Lf\|_0 = \Big\| f^{(m)} + \sum_{k=1}^{m} a_k f^{(m-k)} \Big\|_0 \leq \|f^{(m)}\|_0 + \sum_{k=1}^{m} |a_k| \, \|f^{(m-k)}\|_0$$
$$\leq \|f\|_m + \sum_{k=1}^{m} |a_k| \, \|f\|_m$$
$$= \underbrace{\Big(1 + \sum_{k=1}^{m} |a_k|\Big)}_{C} \|f\|_m \quad \forall f \in X_m.$$

Thus, $L$ is continuous. Next, we show that $\dim(\ker L) = m$. ✓ For $f \in \ker L$, we denote $f_1 = f$, $f_2 = f'$, $f_3 = f''$, $\ldots$, $f_m = f^{(m-1)}$, and $u = (f_1, f_2, \ldots, f_m)$. Then

$$\begin{cases} f_1' = f_2, \\ f_2' = f_3, \\ \cdots \\ f_{m-1}' = f_m, \\ f_m' = -a_1 f_m - \cdots - a_{m-1} f_2 - a_m f_1. \end{cases} \qquad (I)$$

In matrix form,

$$\underbrace{\begin{pmatrix} f_1' \\ f_2' \\ \vdots \\ f_m' \end{pmatrix}}_{u'(t)} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots \; 0 & 1 \\ -a_m & -a_{m-1} & \cdots & \cdots & -a_1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{m-1} \\ f_m \end{pmatrix}}_{u(t)}$$

By the system (I), a solution $u \in [C^1([0,1], \mathbb{C})]^m$ to the equation $u'(t) = Au(t)$ is determined by its first coordinate. Therefore, the correspondence between $f$ and $u$ is linear and bijective. Put $u(0) = u_0$. We know from the theory of ordinary differential equations that the problem

$$\begin{cases} u'(t) = Au(t), \\ u(0) = u_0 \end{cases}$$

has a unique solution, which is $u(t) = e^{At} u_0$. Here the exponential of a matrix $B \in M(m, \mathbb{C})$ is defined as

$$e^B = I_m + \frac{B}{1!} + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{B^k}{k!}.$$

Each choice of $u_0 \in \mathbb{C}^m$ determines $u$. The dependence is linear and bijective. Thus, the space of solutions $u \in [C^1([0,1], \mathbb{C})]^m$ to the equation $u'(t) = Au(t)$, $t \in [0,1]$, is of dimension $m$. Thus, $\ker L$ is of dimension $m$.

Next, we show that $L$ is surjective. Let $g \in X_0$. With the same definition for $f_1, f_2, \ldots, f_m$ as above, the equation $Lf = g$ is equivalent to

$$\underbrace{\begin{pmatrix} f_1' \\ f_2' \\ \vdots \\ f_m' \end{pmatrix}}_{u'(t)} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_m & -a_{m-1} & \cdots & \cdots & \cdots & -a_1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}}_{u(t)} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g \end{pmatrix}}_{v(t)}.$$

We seek for a solution $u \in [C^1([0,1], \mathbb{C})]^m$ to the equation $u'(t) = Au(t) + v(t)$ in the form $u(t) = e^{At} u_0(t)$. This method is known as the variation-of-constants method.

$$u'(t) = Au(t) + v(t) \iff Ae^{At}u_0(t) + e^{At}u_0'(t) = Ae^{At}u_0(t) + v(t)$$
$$\iff e^{At}u_0'(t) = v(t)$$
$$\iff u_0'(t) = e^{-At}v(t).$$

Because $v$ is a continuous function, we can take $u_0(t) = \int_0^t e^{-As} v(s)\,ds$ to satisfy the above equation. Since $u_0$ belongs to $[C^1([0,1], \mathbb{C})]^m$, so is $u$. We have showed that $L$ is surjective.

Then range $L$ is closed in $X_0$ and $\operatorname{codim}(\operatorname{range} L) = 0$. We conclude that $L$ is a Fredholm operator and

$$\operatorname{ind}(L) = \dim(\ker L) - \operatorname{codim}(\operatorname{range} L) = m - 0 = m. \checkmark$$

(b) Let $\Omega \subset \mathbb{C}$ be a smooth bounded domain. A domain is understood as a nonempty open connected subset of $\mathbb{C}$. For $m \geq 0$, we denote

$$X_m = \{ f : \Omega \to \mathbb{C} \mid f \text{ is holomorphic and } f, f^{(1)}, \ldots, f^{(m)} \text{ can be continuously extended to } \overline{\Omega} \}.$$

Then $X_m$ equipped with the norm $\|f\|_m = \sum_{k=0}^{m} \sup_{z \in \Omega} |f^{(k)}(z)|$ is a Complex Banach space. ✓

Let $L : X_1 \to X_0$ be an operator given by $Lf = f'$. We show that $L$ is a Fredholm operator. It is clear that $L$ is well-defined and linear.

$$\|Lf\|_0 = \sup_{z \in \Omega} |f'(z)| \leq \sup_{z \in \Omega} |f(z)| + \sup_{z \in \Omega} |f'(z)| = \|f\|_1 \quad \forall f \in X_1.$$

Hence, $L$ is continuous. For each $f \in \ker L$, $f' = Lf = 0$. Because $\Omega$ is connected, $f$ is a constant function in $\Omega$. Then $\ker L$ consists of constant functions. Thus, $\dim(\ker L) = 1$. ✓

Next, we show that $\operatorname{range}(L)$ is closed in $X_0$. Let $(g_k)$ be a sequence in $\operatorname{range}(L)$ such that $g_k \to g$ in $X_0$. For each closed rectifiable curve $\gamma$ in $\Omega$,

$$\left| \int_\gamma g_k(z)\,dz - \int_\gamma g(z)\,dz \right| = \left| \int_\gamma (g_k(z) - g(z))\,dz \right| \leq \text{length}(\gamma)\, \|g_k - g\|_0 \to 0 \quad \text{as } k \to \infty.$$

Thus, $\int_\gamma g(z)\,dz = \lim_{k\to\infty} \int_\gamma g_k(z)\,dz$.

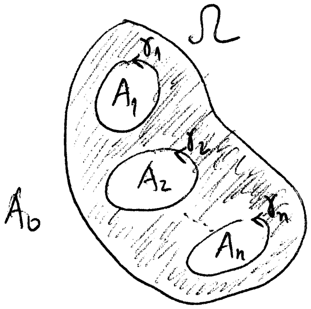
Because $g_k$ has an antiderivative in $\Omega$, the integral of $g_k$ along any closed rectifiable curve is zero. Then $\int_\gamma g(z)\,dz = 0$. ✓

This means the integral of $g$ along any closed rectifiable curve in $\Omega$ depends only on the end points of that path. Then $g$ has an antiderivative

$$f(z) = \int_{z_0}^{z} g(w)\,dw \quad \forall z \in \Omega,$$

where $z_0$ is some fixed point in $\Omega$. This function is continuous in $\bar{\Omega}$. Then $f \in X_1$ and $g = Lf \in \text{range}(L)$. We have showed that range $(L_0)$ is closed in $X_0$.

Next, we compute the codimension of range$(L)$ in $X_0$. Suppose that $\mathbb{C}\setminus\bar{\Omega}$ has finitely many connected components $A_0, A_1, A_2, \dots, A_n$. Only one of them is unbounded. We can assume $A_0$ is unbounded. Take




$z_1 \in A_1, z_2 \in A_2, \dots, z_n \in A_n$ and let $Y$ be the linear span of the functions $\frac{1}{z-z_1}, \frac{1}{z-z_2}, \dots, \frac{1}{z-z_n}$.

We show that $X_0 = \text{range}(L) + Y$.

The function $\frac{1}{z-z_j}$ is holomorphic in $\mathbb{C}\setminus\{z_j\}$. In particular, it is holomorphic in $\Omega$ and can be continuously extended to $\bar{\Omega}$. Thus, $\frac{1}{z-z_j} \in X_0$. We get range$(L) + Y \subset X_0$.

Because $\Omega$ is a smooth domain, the boundary of each $A_j$, $1 \leq j \leq n$, is a smooth curve. We parametrize the boundary of $A_j$ as a closed curve $\gamma_j : [0,1] \to \mathbb{C}$ with winding number $1$, i.e.

$$\frac{1}{2\pi i}\int_{\gamma_j} \frac{1}{z-z_j}\,dz = 1.$$

Let $g \in X_0$. Put

$$c_j = \frac{1}{2\pi i}\int_{\gamma_j} g(z)\,dz \qquad \forall 1 \le j \le n$$

and
$$h(z) = g(z) - \frac{c_1}{z-z_1} - \frac{c_2}{z-z_2} - \cdots - \frac{c_n}{z-z_n} \in X_0 .$$

Let $\gamma$ be any closed curve in $\Omega$ that encloses $A_{j_1}, A_{j_2}, \ldots, A_{j_k}$. According to ~~g~~ a general form of Cauchy's theorem (Ahlfors "Complex Analysis", 1979, page 147),

$$\int_\gamma h(z)\,dz = \alpha_{j_1}\int_{\gamma_{j_1}} h(z)\,dz + \alpha_{j_2}\int_{\gamma_{j_2}} h(z)\,dz + \cdots + \alpha_{j_k}\int_{\gamma_{j_k}} h(z)\,dz . \qquad (1)$$

where $\alpha_k$ is the winding number of $\gamma$ with respect to $z_k$.

We have
$$\int_{\gamma_{j_1}} h(z)\,dz = \int_{\gamma_{j_1}} g(z)\,dz - \int_{\gamma_{j_1}} \frac{c_1}{z-z_{j_1}}\,dz = c_{j_1} - c_{j_1} = 0.$$

Similarly, other terms on RHS(1) are also zero. Thus, $\int_\gamma h(z)dz = 0$. This implies the integral of $h$ along any path in $\Omega$ depends only on the end points of that path. Then $h$ has an antiderivative in $\Omega$:

$$f(z) = \int_{z_0}^{z} h(w)\,dw \qquad \forall z \in \Omega$$

where $z_0$ is some fixed point in $\Omega$. Then $f \in X_1$ and $Lf = h$.

$$g(z) = \underbrace{h(z)}_{\in \text{Range}(L)} + \underbrace{\frac{c_1}{z-z_1} + \cdots + \frac{c_n}{z-z_n}}_{\in Y} .$$

We have showed that $X_0 = \text{range}(L) + Y$.

Now let $g \in \text{range}(L) \cap Y$. Then there are $c_1, c_2, \ldots, c_n \in \mathbb{C}$ such

that
$$g(z) = \frac{c_1}{z-z_1} + \dots + \frac{c_n}{z-z_n} \quad \forall z \in \Omega.$$

Because $g \in \text{range}(L)$, it has an antiderivative in $\Omega$. Then the integral of $g$ over every closed path is zero. Then

$$0 = \int_{\gamma_1} g(z)\,dz = c_1 \int_{\gamma_1} \frac{1}{z-z_1}\,dz + \sum_{j=2}^{n} c_j \int_{\gamma_1} \frac{1}{z-z_j}\,dz. \qquad (2)$$

The integral $\int_{\gamma_1} \frac{1}{z-z_j}\,dz$, $2 \le j \le n$, is zero because the function $\frac{1}{z-z_j}$ is ~~an~~ holomorphic and $z_j$ is not enclosed in $\gamma_1$. Then (2) implies $c_1 = 0$. Similarly, $c_2 = c_3 = \dots = c_n = 0$. Then $g(z) \equiv 0$ in $\Omega$. Then $\text{range}(L) \cap Y = \{0\}$. We get $\text{range}(L) \oplus Y = X_0$.

Suppose there are numbers $c_1, c_2, \dots, c_n \in \mathbb{C}$ such that

$$\frac{c_1}{z-z_1} + \frac{c_2}{z-z_2} + \dots + \frac{c_n}{z-z_n} = 0 \quad \forall z \in \Omega.$$

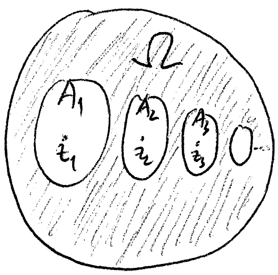
The right hand side is a function in range$(L) \cap Y$. It was showed above that $c_1 = c_2 = \dots = c_n = 0$. Thus, the functions $\frac{1}{z-z_1}, \frac{1}{z-z_2}, \dots, \frac{1}{z-z_n}$ are linearly independent. Then $\text{codim}(\text{range}\, L) = \dim Y = n$.

We conclude that $L$ is a Fredholm operator and

$$\text{ind}(L) = \dim(\ker L) - \text{codim}(\text{range}\, L) = 1-n, \qquad \checkmark$$

where $n$ is the number of bounded connected components of $\mathbb{C} \setminus \bar{\Omega}$.

In case $\mathbb{C} \setminus \bar{\Omega}$ has infinitely many bounded connected component, range$(L)$ has infinite codimension (and thus is not a Fredholm operator). $\checkmark$




$$\operatorname{range}(L) \oplus \underbrace{\text{linear span}\left\{ \frac{1}{z-z_j} : j = 1,2,3,\dots \right\}}_{\text{infinite dimensional space}} \subset X_0 .$$

(c) Let $\Omega \subset \mathbb{C}$ be a smooth bounded domain such that $\mathbb{C} \setminus \overline{\Omega}$ has finitely many connected components. For $m \geq 0$, we define the complex Banach space $X_m$ as in Part (b). Let $m \geq 1$ and $a_1, a_2, \dots, a_m \in \mathbb{C}$. Consider the map $L : X_m \to X_0$, $Lf = f^{(m)} + a_1 f^{(m-1)} + \dots + a_{m-1} f^{(1)} + a_m$. We show that $L$ is a Fredholm operator by following the same method as in Part (a).

It is clear that $L$ is well-defined and linear.

$$\|f\|_0 = \Big\| f^{(m)} + \sum_{k=1}^{m} a_k f^{(m-k)} \Big\|_0 \leq \|f^{(m)}\|_0 + \sum_{k=1}^{m} |a_k| \, \|f^{(m-k)}\|_0$$

$$\leq \|f^{(m)}\|_m + \sum_{k=1}^{m} |a_k| \, \|f\|_m$$

$$= \underbrace{\Big(1 + \sum_{k=1}^{m} |a_k|\Big)}_{C} \|f\|_m \quad \forall f \in X_m .$$

Thus, $L$ is continuous.

To characterize the kernel and range of $L$, we introduce $f_1 = f$, $f_2 = f'$, $f_3 = f''$, $\dots$, $f_m = f^{(m-1)}$ and $u = (f_1, f_2, \dots, f_m)$. As in Part (a), the equation $Lf = 0$ is equivalent to

$$u'(z) = A\,u(z) \qquad \forall z \in \Omega,$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & \cdots & 0 & 1 \\ -a_m & -a_{m-1} & \cdots & \cdots & -a_1 \end{pmatrix}.$$

Fix a point $z_0 \in \Omega$ and put $u_0 = u(z_0)$. We show that given $u_0 \in \mathbb{C}^m$, the problem

$$\begin{cases} u'(z) = A\,u(z) & \forall z \in \Omega \\ u(z_0) = u_0 \end{cases} \qquad (\mathrm{II})$$

has a unique solution $u \in X_1^m$. The function $z \in \mathbb{C} \mapsto e^{Az} \in M(m, \mathbb{C})$ is defined by

$$e^{Az} = I_m + zA + \frac{z^2}{2!}A^2 + \frac{z^3}{3!}A^3 + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!}A^k.$$

Denote by $\|.\|$ the Euclidean norm on $M(m, \mathbb{C})$. The above series converges because

$$\sum_{k=0}^{\infty} \left\| \frac{z^k}{k!} A^k \right\| = \sum_{k=0}^{\infty} \frac{|z|^k}{k!} \|A^k\| \leqslant \sum_{k=0}^{\infty} \frac{|z|^k}{k!} \|A\|^k$$

$$= \sum_{k=0}^{\infty} \frac{(|z|\,\|A\|)^k}{k!} = e^{|z|\,\|A\|} < \infty.$$

For each $j \in \mathbb{N}$, put $\quad g_j(z) = \sum_{k=0}^{j} \frac{z^k}{k!} A^k.$

Then $g_j : \mathbb{C} \to M(m, \mathbb{C})$ is a holomorphic function. For $R > 0$,

$$\|e^{Az} - g_j(z)\| = \left\| \sum_{k=j+1}^{\infty} \frac{z^k}{k!} A^k \right\| \leq \sum_{k=j+1}^{\infty} \frac{(|z|\,\|A\|)^k}{k!}$$

$$\leq \sum_{k=j+1}^{\infty} \frac{(R\|A\|)^k}{k!} \quad \forall\, |z| \leq R.$$

The last series converges to $0$ as $j \to \infty$ because $\displaystyle\sum_{k=0}^{\infty} \frac{(R\|A\|)^k}{k!} = e^{R\|A\|} < \infty$.

Thus, the sequence of holomorphic functions $(g_j)_{j\in\mathbb{N}}$ converges uniformly on every compact subset of $\mathbb{C}$ to $e^{Az}$. By Theorem 1, Ahlfors "Complex Analysis" 1979, page 177, $e^{Az}$ is holomorphic in $\mathbb{C}$ and $(g_j')$ converges to $(e^{Az})'$ uniformly on every compact subset of $\mathbb{C}$. Put $u_1(z) = e^{A(z-z_0)} u_0$.

Then $u_1 \in X_1^m$ and

$$u_1'(z) = \left(e^{A(z-z_0)}\right)' u_0 = \lim_{j\to\infty} g_j'(z-z_0)\, u_0$$

$$= \lim_{j\to\infty} \sum_{k=1}^{j} \frac{(z-z_0)^{k-1}}{(k-1)!} A^k u_0 = \lim_{j\to\infty} A \sum_{k=0}^{j-1} \frac{(z-z_0)^k}{k!} A^k u_0$$

$$= A\, e^{A(z-z_0)} u_0 = A u_1(z),$$

$$u_1(z_0) = e^{A.0} u_0 = u_0.$$

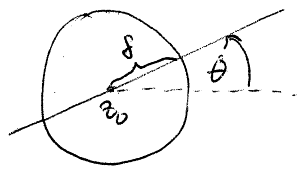
Thus, $u_1$ solves Problem (II). Suppose $u_2 \in X_1^m$ is also a solution to (II).

Put $v = u_1 - u_2 \in X_1^m$. Then

$$\begin{cases} v'(z) = Av(z) & \forall z \in \Omega \\ v(z_0) = 0. \end{cases}$$

Since $\Omega$ is open, it contains a disk centered at $z_0$ with radius $\delta > 0$.

For $\theta \in [0, 2\pi)$, put $v_\theta : (-\delta, \delta) \to \mathbb{C}$, $v_\theta(t) = v(z_0 + te^{i\theta})$. Then




$$\begin{cases} v_\theta'(t) = A v_\theta(t) \quad \forall t \in (-\delta, \delta) \\ v_\theta(0) = 0. \end{cases}$$

We know from the theory of ordinary differential equations that this problem has only trivial solution. Thus, $v_\theta \equiv 0$ in $(-\delta, \delta)$. Then $v = 0$ on the intersection between the line of angle $\theta$ to the horizontal direction and the disk $D(z_0, \delta)$. Since this is true for all $\theta \in [0, 2\pi)$, $v = 0$ in $\mathbb{D}(z_0, \delta)$. Because $v$ is holomorphic in $\Omega$, it must vanish in $\Omega$. Then $u_1 = u_2$ in $\Omega$. We have showed that Problem (II) has a unique solution. The solution then depends bijectively and linearly on the data $u_0 \in \mathbb{C}^m$. Thus, the solution space of $u'(z) = Au(z)$ is of dimension $m$. We get $\dim(\ker L) = m$. ✓

Next, we show that range$(L)$ is closed in $X_0$. The equation $Lf = g \in X_0$ is equivalent to

$$u'(z) = A u(z) + v(z) \quad \forall z \in \Omega,$$

where $v(z) = (0, 0, \ldots, 0, g(z))$. Suppose $(g_k)$ is a sequence in range$(L)$ that converges to $g_0 \in X_0$. Put

$$v_k = (0, \ldots, g_k), \qquad v_0 = (0, \ldots, 0, g_0).$$

Then $\|v_k - v_0\|_{X_0^m} = \|g_k - g_0\|_{X_0} \to 0$ as $k \to \infty$. There exists $u_k \in X_1^m$ such that

$$u_k'(z) = A u_k(z) + v_k(z) \quad \forall z \in \Omega. \quad ✓ \qquad (3)$$

We showed earlier that there exists $\tilde{u}_k \in X_1^m$ such that

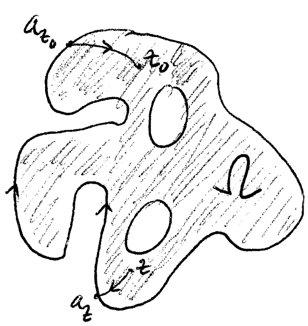
$$\begin{cases} \tilde{u}_k'(z) = A\tilde{u}_k(z) & \forall z \in \Omega, \\ \tilde{u}_k(z_0) = u_0(z_0). \end{cases}$$

By replacing $u_k$ with $u_k - \tilde{u}_k$, we can assume $u_k(z_0) = 0$. Multiplying both sides of (3) by $e^{-Az}$, we get

$$\frac{d}{dz}\left(e^{-Az}u_k(z)\right) = e^{-Az}v_k(z) \qquad \forall z \in \Omega.$$

Then
$$u_k(z) = e^{Az}\int_{z_0}^{z} e^{-A\zeta}v_k(\zeta)\,d\zeta \qquad \forall z \in \Omega. \tag{4}$$

The integral is taken along any path in $\Omega$ that connects $z_0$ to $z$.




Each $z \in \Omega$ can be connected to a point $a_z \in \partial\Omega$ by a path with length $\leq \operatorname{diam}(\Omega)$. Point $a_z$ can be connected to point $a_{z_0}$ by a path along the boundary of $\Omega$. This path has length $\leq$ length$(\partial\Omega)$, which is finite because $\Omega$ is a smooth domain. Thus, $z$ can be connected to $z_0$ by a path in $\bar{\Omega}$ of length $\leq l := \operatorname{diam}(\Omega) + \text{length}(\partial\Omega) + \operatorname{diam}(\Omega) < \infty$.

Suppose $\bar{\Omega}$ is contained in a disk $D(0,R)$. We deduce from (4)

$$\left| u_k(z) - e^{Az}\int_{z_0}^{z} e^{-A\zeta}v_0(\zeta)\,d\zeta \right| = \left| e^{Az}\int_{z_0}^{z} e^{-A\zeta}(v_k(\zeta) - v_0(\zeta))\,d\zeta \right| \leq$$

$$\leq \|e^{Az}\| \left| \int_{z_0}^{z} e^{-A\zeta}(v_k(\zeta) - v_0(\zeta))\, d\zeta \right|$$

$$\leq e^{R\|A\|} \, l \sup_{\substack{|\zeta| \leq R \\ \zeta \in \Omega}} \left| e^{-A\zeta}(v_k(\zeta) - v_0(\zeta)) \right|$$

$$\leq e^{R\|A\|} \, l \, e^{R\|A\|} \sup_{\zeta \in \Omega} |v_k(\zeta) - v_0(\zeta)|$$

$$= l e^{2R\|A\|} \|v_k - v_0\|_{X_0}$$

$$\longrightarrow 0 \quad \text{as } k \to \infty.$$

Thus, $(u_k)$ converges uniformly in $\overline{\Omega}$. Because each $u_k$ is a holomorphic function, the limit function $u : \Omega \to \mathbb{C}^m$,

$$u(z) = e^{Az} \int_{z_0}^{z} e^{-A\zeta} v_0(\zeta)\, d\zeta$$

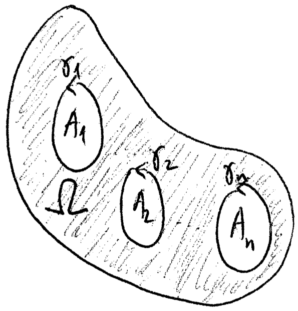
is also holomorphic. Moreover, $(u_k')$ converges to $u'$ uniformly on $\overline{\Omega}$. Then $u \in X_1^m$ and

$$u'(z) = \lim_{k\to\infty} u_k'(z) = \lim_{k\to\infty} (Au_k(z) + v_k(z)) = Au(z) + v_0(z).$$

This means the equation $Lf = g_0$ has a solution $f \in X_m$. Thus, $g_0 \in$ range$(L)$. We have showed that range$(L)$ is closed in $X_0$.

We now compute the codimension of range$(L)$. As in Part (b), let $A_1, A_2, \ldots, A_n$ be the bounded connected components of $\mathbb{C} \setminus \overline{\Omega}$. Take $z_1 \in A_1$, $z_2 \in A_2, \ldots, z_n \in A_n$. Parametrize the boundary of $A_j$ as a closed path $\gamma_j : [0,1] \to \mathbb{C}$ with winding number 1 with respect to $z_j$, i.e.

$$\frac{1}{2\pi i}\int_{\gamma_j}\frac{1}{z-z_j}\,dz = 1 \qquad \forall\, 1\le j\le n.$$




For $f\in X_m$ and $g\in X_0$, the equation $Lf=g$ is equivalent to

$$u'(z) = Au(z) + v(z) \quad \forall z\in\Omega \qquad (5)$$

where $u=(f_1,f_2,\dots,f_m) = (f,f',\dots,f^{(m-1)})\in X_1^m$,

$$v=(0,\dots,0,g)\in X_0^m.$$

We know that $e^{-Az}e^{Az} = e^{Az}e^{-Az} = I_m$. Thus, $e^{-Az}\in GL(m,\mathbb{C})$ for every $z\in\mathbb{C}$. Then

$$(5) \Leftrightarrow e^{-Az}u'(z) = e^{-Az}Au(z) + e^{-Az}v(z) \quad \forall z\in\Omega$$

$$\Leftrightarrow \frac{d}{dz}\left[e^{-Az}u(z)\right] = e^{-Az}v(z) \qquad \forall z\in\Omega.$$

Given $g\in X_0$, the equation has a solution $u\in X_1^m$ only if $e^{-Az}v(z)$ has an antiderivative in $\Omega$. Conversely, if $e^{-Az}v(z)$ has an antiderivative in $\Omega$, say $w(z)$, then we can choose $u(z)=e^{Az}w(z)$. The function $e^{-Az}v(z)$ has an antiderivative in $\Omega$ if and only if

$$\int_\gamma e^{-Az}v(s)\,dz = 0 \qquad \forall \text{ closed curve } \gamma \text{ in } \Omega.$$

By the general Cauchy's theorem,

$$\int_\gamma e^{-Az}v(s)\,dz = \alpha_1\int_{\gamma_1} e^{-Az}v(z)\,dz + \alpha_2\int_{\gamma_2} e^{-Az}v(z)\,dz + \dots + \alpha_n\int_{\gamma_n} e^{-Az}v(z)\,dz,$$

where $\alpha_j$ is the winding number of $\gamma$ with respecte to $z_j$, i.e.

$$\alpha_j = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_j}\,dz.$$

Thus, the equation (5)' is solvable (for u') if and only if

$$\int_{\gamma_j} e^{-Az} v(z)\, dz = 0 \qquad \forall 1 \leq j \leq n. \qquad (6)$$

Write

$$e^{-Az} = \left( \begin{array}{c|c} \ast & \begin{matrix} p_1(z) \\ p_2(z) \\ \vdots \\ p_m(z) \end{matrix} \end{array} \right) \in GL(m, \mathbb{C}). \qquad (7)$$

The functions $p_1(z), p_2(z), \ldots, p_m(z)$ are holomorphic in $\mathbb{C}$.

$$-Ae^{-Az} = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & & -1 \\ a_m & a_{m-1} & \cdots & & a_1 \end{pmatrix} \left( \begin{array}{c|c} \ast & \begin{matrix} p_1(z) \\ p_2(z) \\ \vdots \\ p_m(z) \end{matrix} \end{array} \right)$$

$$= \left( \begin{array}{c|c} \ast & \begin{matrix} -p_2(z) \\ -p_3(z) \\ \vdots \\ -p_{m-1}(z) \\ a_m p_1(z) + a_{m-1} p_2(z) + \cdots + a_1 p_m(z) \end{matrix} \end{array} \right). \qquad (8)$$

On the other hand, $-Ae^{-Az}$ can be obtained by differentiating both sides of (7) with respect to $z$.

$$-Ae^{-Az} = \left( \begin{array}{c|c} \ast & \begin{matrix} p_1'(z) \\ p_2'(z) \\ \vdots \\ p_m'(z) \end{matrix} \end{array} \right) \qquad (9)$$

Equating the last columns in (8) and (9), we get

$$\begin{cases} p_1'(z) = -p_2(z) \\ p_2'(z) = -p_3(z) \\ \cdots\cdots \\ p_{m-1}'(z) = -p_m(z) \\ p_m'(z) = a_m p_1(z) + a_{m-1} p_2(z) + \cdots + a_1 p_m(z). \end{cases}$$

Thus, $p_k(z) = (-1)^{k-1} p_1^{(k-1)}(z)$ for $2 \le k \le m$. (10)

We have

$$e^{-Az} v(z) = \begin{pmatrix} \boxed{\quad * \quad} & \begin{matrix} p_1(z) \\ p_2(z) \\ \vdots \\ p_m(z) \end{matrix} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g(z) \end{pmatrix} = \begin{pmatrix} p_1(z) g(z) \\ p_2(z) g(z) \\ \vdots \\ p_m(z) g(z) \end{pmatrix}.$$

Then the condition (6) is equivalent to

$$\int_{\gamma_j} p_k(z) g(z)\, dz = 0 \qquad \forall\, 1 \le j \le n,\ 1 \le k \le m. \qquad (11)$$ ✓

For $2 \le k \le m$,

$$\int_{\gamma_j} p_k(z) g(z)\, dz \overset{(10)}{=\!=} \int_{\gamma_j} (-1)^{k-1} p_1^{(k-1)}(z) g(z)\, dz = \int_{\gamma_j} p_1(z) g^{(k-1)}(z)\, dz.$$

Then the condition (11) is equivalent to

$$\int_{\gamma_j} p_1(z)\, g^{(l)}(z)\, dz = 0 \qquad \forall\, 1 \le j \le n,\ 0 \le l \le m-1.$$ ✓

[illegible]

what is the conclusion?