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Math 8802: Functional Analysis

Homework #3

(1) Let X be a separable Hilbert space over $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$) and let $\mathcal{L}(X)$ be the space of continuous linear operators on X. For $R>0$, denote $B_R(X) = \{T \in \mathcal{L}(X) : \|T\| \leq R\}$. Let σ be the strong operator topology on $\mathcal{L}(X)$, i.e. the topology defined by the system of open neighborhoods (of the zero operator),

$$\mathcal{O}_{x_1,\dots,x_n;\,\varepsilon_1,\dots,\varepsilon_n} = \{T \in \mathcal{L}(X) : \|Tx_1\| < \varepsilon_1, \dots, \|Tx_n\| < \varepsilon_n\},$$

for $n \in \mathbb{N}$, $x_1,\dots,x_n \in X$, $\varepsilon_1,\dots,\varepsilon_n > 0$. Denote by $\mathcal{B}_\sigma$ this system of neighborhoods. Let τ be the weak operator topology on $\mathcal{L}(X)$, i.e. the topology defined by the system of neighborhoods

$$\mathcal{O}_{x_1,y_1,\dots,x_n,y_n;\,\varepsilon_1,\dots,\varepsilon_n} = \{T \in \mathcal{L}(X) : |(Tx_1,y_1)| < \varepsilon_1, \dots, |(Tx_n,y_n)| < \varepsilon_n\},$$

for $n \in \mathbb{N}$, $x_1, y_1, \dots, x_n, y_n \in X$, $\varepsilon_1,\dots,\varepsilon_n > 0$. Denote by $\mathcal{B}_\tau$ this system of neighborhoods.

(i) Assume X is infinite dimensional. We show that $(\mathcal{L}(X),\sigma)$ and $(\mathcal{L}(X),\tau)$ are not metrizable. The following lemma is needed.

$\boxed{\text{Lemma 1: Let } X \text{ be an infinite dimensional Hilbert space over } \mathbb{K} \text{ (either } \mathbb{R} \text{ or } \mathbb{C}\text{). Then } X \text{ does not have a countable Hamel basis.}}$

A consequence is that X is not equal to the linear span of any countable subset.

Proof of the lemma

Suppose otherwise: X has a countable Hamel basis $\{u_1, u_2, u_3, \dots\}$. Then we can construct an orthonormal Hamel basis $\{e_1, e_2, e_3, \dots\}$ for X by the Gram-Schmidt process.

$$v_1 = u_1,$$

$$v_2 = u_2 - \frac{(u_2, v_1)}{(v_1, v_1)} v_1,$$

$$v_3 = u_3 - \frac{(u_3, v_1)}{(v_1, v_1)} v_1 - \frac{(u_3, v_2)}{(v_2, v_2)} v_2,$$

$$\dots\dots$$

$$v_n = u_n - \frac{(u_n, v_1)}{(v_1, v_1)} v_1 - \frac{(u_n, v_2)}{(v_2, v_2)} v_2 - \dots - \frac{(u_n, v_{n-1})}{(v_{n-1}, v_{n-1})} v_{n-1},$$

$$\dots\dots$$

Set $\quad e_k = \dfrac{v_k}{\|v_k\|} \quad$ for all $k \in \mathbb{N}$.

Because X is a Hilbert space and $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$, the series $\sum_{k=1}^{\infty} \frac{1}{k} e_k$ converges in X. Then it is a linear combination of $\{e_1, e_2, e_3, \dots\}$. In other words, there exist $\alpha_1, \alpha_2, \dots, \alpha_m \in K$ such that

$$\sum_{k=1}^{\infty} \frac{1}{k} e_k = \sum_{k=1}^{m} \alpha_k e_k.$$

Then $\displaystyle \sum_{k=1}^{m} \left(\frac{1}{k} - \alpha_k\right) e_k + \sum_{k=m+1}^{\infty} \frac{1}{k} e_k = 0.$

Taking the norm of both sides, we get

$$\sqrt{\sum_{k=1}^{m}\left|\frac{1}{k}-\alpha_k\right|^2+\sum_{k=m+1}^{\infty}\frac{1}{k^2}}=0.$$

This is a contradiction.

$\blacksquare$

Return to the problem. Suppose by contradiction that $(\mathcal{L}(X),\sigma)$ is metrizable. Then $(\mathcal{L}(X),\sigma)$ has a countable system of open neighborhoods (of the zero operator), namely $\{B_1, B_2, B_3, \dots\}$. Each B_j must contain an element of $\mathcal{B}_\sigma$. Then for each $j\in\mathbb{N}$, there exist $x_1^{(j)},\dots,x_{n_j}^{(j)}\in X$, $\varepsilon_1^{(j)},\dots,\varepsilon_{n_j}^{(j)}>0$ such that $O_{x_1^{(j)},\dots,x_{n_j}^{(j)};\varepsilon_1^{(j)},\dots,\varepsilon_{n_j}^{(j)}}\subset B_j$. Put

$$A=\bigcup_{j=1}^{\infty}\{x_1^{(j)},\dots,x_{n_j}^{(j)}\}\subset X.$$

Then A is a countable subset of X. By Lemma 1, there exists $x_0\in X$ that does not belong to the linear span of A. Because the set

$$O_{x_0;1}=\{T\in\mathcal{L}(X):\ \|Tx_0\|<1\}$$

is open in $(\mathcal{L}(X),\sigma)$, there exists $j\in\mathbb{N}$ such that $B_j\subset O_{x_0;1}$. This implies

$$O_{x_1^{(j)},\dots,x_{n_j}^{(j)};\varepsilon_1^{(j)},\dots,\varepsilon_{n_j}^{(j)}}\subset O_{x_0;1}. \qquad (1)$$

For convenience, we simply denote $x_1^{(j)},\dots,x_{n_j}^{(j)}$, $\varepsilon_1^{(j)},\dots,\varepsilon_{n_j}^{(j)}$ as $x_1,\dots,x_n$, $\varepsilon_1,\dots,\varepsilon_n$ respectively. Then x_0 does not belong to ~~the linear span of~~ $Y:=$ linear span of $\{x_1,x_2,\dots,x_n\}$. Denote by Y' the linear span of $\{x_0\}\cup Y$. We define a map $S: Y' \to K\{x_0\}$, $S(tx_0 + ty) = tx_0$ for all $t \in K, y \in Y$.

Because S is a linear map between two finite dimensional normed spaces, it is continuous. By Hahn-Banach theorem, S extends to a continuous linear map (still denoted by S) from $(X, \|.\|)$ to $K\{x_0\}$. Then $S \in \mathcal{L}(X)$.

We have $Sx_0 = 1$ and $Sx_1 = Sx_2 = \dots = Sx_n = 0$. Hence,

$$S \in O_{x_1, \dots, x_n; \varepsilon_1, \dots, \varepsilon_n} \setminus O_{x_0; 1}.$$

This contradicts (1).

Next, suppose that $(\mathcal{L}(X), \tau)$ is metrizable. We seek a contradiction by the same method as was done for $(\mathcal{L}(X), \sigma)$. $(\mathcal{L}(X), \tau)$ has a countable system of open neighborhoods (of the zero operator), namely $\{B_1, B_2, B_3, \dots\}$. Each B_j must contain an element of $\mathcal{B}_\tau$. Then for each $j \in \mathbb{N}$, there exist $x_1^{(j)}, y_1^{(j)}, \dots, x_{n_j}^{(j)}, y_{n_j}^{(j)} \in X$, $\varepsilon_1^{(j)}, \dots, \varepsilon_{n_j}^{(j)} > 0$ such that $O_{x_1^{(j)}, y_1^{(j)}, \dots, x_{n_j}^{(j)}, y_{n_j}^{(j)}; \varepsilon_1^{(j)}, \dots, \varepsilon_{n_j}^{(j)}} \subset B_j$.

Put

$$A = \bigcup_{j=1}^{\infty} \{x_1^{(j)}, \dots, x_{n_j}^{(j)}\} \subset X.$$

Then A is a countable subset of X. By Lemma 1, there exists $x_0 \in X$ that does not belong to the linear span of A. Put $y_0 = \frac{x_0}{\|x_0\|^2}$. Because the set

$$O_{x_0, y_0; 1} = \{T \in \mathcal{L}(X) : |(Tx_0, y_0)| < 1\}$$

is open in $(\mathcal{L}(X), \tau)$, there exists $j \in \mathbb{N}$ such that $B_j \subset O_{x_0, y_0; 1}$.

This implies

$$\mathcal{O}_{x_1^{(j)}, y_1^{(j)}, \dots, x_{n_j}^{(j)}, y_{n_j}^{(j)}; \varepsilon_1^{(j)}, \dots, \varepsilon_{n_j}^{(j)}} \subset \mathcal{O}_{x_0, y_0; 1}. \qquad (2)$$

For convenience, we simply denote $x_1^{(j)}, y_1^{(j)}, \dots, x_{n_j}^{(j)}, y_{n_j}^{(j)}, \varepsilon_1^{(j)}, \dots, \varepsilon_{n_j}^{(j)}$ as $x_1, y_1, \dots, x_n, y_n, \varepsilon_1, \dots, \varepsilon_n$ respectively. Then x_0 does not belong to $Y :=$ linear span of $\{x_1, x_2, \dots, x_n\}$. Denote by Y' the linear span of $\{x_0\} \cup Y$. We define a map $S : Y' \to K\{x_0\}$, $S(tx_0 + y) = tx_0$ for all $t \in K$, $y \in Y$. Because S is a linear map between two finite dimensional normed spaces, it is continuous. By Hahn-Banach theorem, S extends to a continuous linear map (still denoted by S) from $(X, \|.\|)$ to $K\{x_0\}$. Then $S \in \mathcal{L}(X)$.

We have

$$(Sx_0, y_0) = (x_0, y_0) = \left(x_0, \frac{x_0}{\|x_0\|^2}\right) = 1,$$

$$Sx_1 = Sx_2 = \dots = Sx_n = 0.$$

Hence, $S \in \mathcal{O}_{x_1, y_1, \dots, x_n, y_n; \varepsilon_1, \dots, \varepsilon_n} \setminus \mathcal{O}_{x_0, y_0; 1}$. This contradicts (2).

In case X is finite dimensional, σ and τ are the same as the topology induced by the norm on $\mathcal{L}(X)$, and thus are metrizable. We prove this claim as follows. Let γ be the topology on $\mathcal{L}(X)$ that is induced by the norm. Then $\sigma, \tau \subset \gamma$. For any $x_1, y_1, \dots, x_n, y_n \in X$, $\varepsilon_1, \dots, \varepsilon_n > 0$, put $M = \max\{\|y_1\|, \|y_2\|, \dots, \|y_n\|\} + 1$, and

$$\varepsilon_1' = \frac{\varepsilon_1}{M}, \dots, \varepsilon_n' = \frac{\varepsilon_n}{M}.$$

For $T \in \mathcal{O}_{x_1,\dots,x_n;\,\varepsilon_1',\dots,\varepsilon_n'}$,

$$|(Tx_j, y_j)| \leq \|Tx_j\| \|y_j\| \leq \varepsilon_j' \|y_j\| < \frac{\varepsilon_j}{M} M = \varepsilon_j \quad \forall 1 \leq j \leq n.$$

Thus, $T \in \mathcal{O}_{x_1,y_1,\dots,x_n,y_n;\,\varepsilon_1,\dots,\varepsilon_n}$. This implies $\mathcal{O}_{x_1,\dots,x_n;\,\varepsilon_1',\dots,\varepsilon_n'} \subseteq \mathcal{O}_{x_1,y_1,\dots,x_n,y_n;\,\varepsilon_1,\dots,\varepsilon_n}$

Then each open neighborhood of 0 in $(\mathcal{L}(X), \tau)$ contains an open neighborhood of 0 in $(\mathcal{L}(X), \sigma)$. Hence, $\tau \subset \sigma$. It now suffices to show $\gamma \subset \tau$. We know that γ has a system of open neighborhoods (of the zero operator) consisting of

$$\mathcal{U}_\varepsilon = \{ T \in \mathcal{L}(X) : \|T\| < \varepsilon \} \qquad \forall \varepsilon > 0.$$

Fix $\varepsilon > 0$. We need to show that $\mathcal{U}_\varepsilon$ contains a neighborhood of 0 in $(\mathcal{L}(X), \tau)$. Let $\{e_1, e_2, \dots, e_m\}$ be an orthonormal basis of X. Put $\delta = \frac{\varepsilon}{2m} > 0$. For each $T \in \mathcal{O}_{e_1,e_1,e_1,e_2,\dots,e_1,e_m,\dots,e_m,e_1,e_m,e_2,\dots,e_m,e_m;\,\delta,\dots,\delta}$,

$$|(Te_j, e_k)| < \delta \quad \forall 1 \leq j, k \leq m.$$

Then
$$\|Te_j\|^2 = \sum_{k=1}^{m} |(Te_j, e_k)|^2 < m\delta^2 \quad \forall 1 \leq j \leq m.$$

For each $x \in X$, $\|x\| \leq 1$, we write $x = c_1 e_1 + \dots + c_m e_m$ where $|c_1|^2 + \dots + |c_m|^2 \leq 1$.

Then
$$\begin{aligned}
\|Tx\|^2 = \|c_1 Te_1 + \dots + c_m Te_m\|^2 &\leq (|c_1| \|Te_1\| + \dots + |c_m| \|Te_m\|)^2 \\
&\leq (|c_1|^2 + \dots + |c_m|^2)(\|Te_1\|^2 + \dots + \|Te_m\|^2) \\
&\leq 1 \cdot (m\delta^2 + \dots + m\delta^2) \\
&= m^2\delta^2 = \left(\frac{\varepsilon}{2}\right)^2.
\end{aligned}$$

Thus, $$\|T\| = \sup_{\substack{x\in X\\ \|x\|\leq 1}} \|Tx\| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Then $T\in U_\varepsilon$. This shows that $O_{e_1,e_1,\dots,e_1,e_m,\dots,e_m,e_1,\dots,e_m,e_m;\delta,\dots,\delta} \subset U_\varepsilon$.

(ii) Let $R>0$. If $X=\{0\}$ then the ~~topo~~ only topology on $B_R(X)=\{0\}$ is the trivial topology, which is metrizable and compact. Consider the case $X\neq\{0\}$. Because X is separable, it has a dense countable subset $\{a_k : k\in\mathbb{N}\}$. We can assume that this set does not contain 0.

We show that $(B_R(X),\sigma)$ is metrizable. Define a map $d_\sigma : B_R(X)\times B_R(X) \to \mathbb{R}$,

$$d_\sigma(T,S) = \sum_{k=1}^{\infty} \frac{\|(T-S)a_k\|}{\|a_k\|} 2^{-k} \quad \forall T,S\in B_R(X). \tag{3}$$

First, we verify that d_σ is well-defined.

$$\frac{\|(T-S)a_k\|}{\|a_k\|} \leq \frac{\|T-S\|\,\|a_k\|}{\|a_k\|} \leq \|T\|+\|S\| \leq 2R.$$

Thus, $$d_\sigma(T,S) \leq \sum_{k=1}^{\infty} (2R)2^{-k} = 2R < \infty.$$

Next, we verify that d_σ is a metric on $B_R(X)$. It is clear that

$$d_\sigma(T,S)\geq 0,\quad d_\sigma(T,S) = d_\sigma(S,T) \quad \forall T,S\in B_R(X).$$

For $T_1,T_2,T_3\in B_R(X)$,

$$d_\sigma(T_1,T_2)+d_\sigma(T_2,T_3) = \sum_{k=1}^{\infty} \frac{\|T_1a_k - T_2a_k\| + \|T_2a_k - T_3a_k\|}{\|a_k\|} 2^{-k} \geq$$

$$\geq \sum_{k=1}^{\infty} \frac{\|(T_1 a_k - T_2 a_k) + (T_2 a_k - T_3 a_k)\|}{\|a_k\|} 2^{-k}$$

$$= \sum_{k=1}^{\infty} \frac{\|(T_1 - T_3) a_k\|}{\|a_k\|} 2^{-k} = d_\sigma(T_1, T_3).$$

Suppose $d_\sigma(T,S) = 0$. Then $Ta_k = Sa_k$ for every $k \in \mathbb{N}$. Take $x \in X$. There exists a sequence (x_n) in $\{a_k : k \in \mathbb{N}\}$ that converges to x. We have $Tx_n = Sx_n$ for all $n \in \mathbb{N}$. Then $Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = Sx$. Hence, $T = S$. We have showed that d_σ is a metric on $B_R(X)$.

Next, we show that the topology on $B_R(X)$ induced by d_σ is the same as $(B_R(X), \sigma)$. Denote

$$B_\sigma(S, \varepsilon) = \{T \in B_R(X) : d_\sigma(T, S) < \varepsilon\} \quad \forall S \in B_R(X), \forall \varepsilon > 0.$$

To show that the topology $(B_R(X), d_\sigma)$ is coarser than $(B_R(X), \sigma)$, we show that $B_\sigma(S, \varepsilon)$ contains a neighborhood of S in $(B_R(X), \sigma)$.

$$B_\sigma(S,\varepsilon) = \left\{ T \in \mathcal{L}(X) : \|T\| \leq R, \ \sum_{k=1}^{\infty} \frac{\|(T-S)a_k\|}{\|a_k\|} 2^{-k} < \varepsilon \right\}.$$

There exists $m \in \mathbb{N}$ such that

$$\sum_{k=m+1}^{\infty} 2^{-k} < \frac{\varepsilon}{4R}.$$

Put $\varepsilon' = \frac{\varepsilon}{2} \left(\sum_{k=1}^{m} \frac{1}{\|a_k\|} \right)^{-1} > 0$. For every $T \in (S + O_{a_1, a_2, \dots, a_m; \varepsilon', \dots, \varepsilon'}) \cap$ $\cap B_R(X)$, we have $\|T\| \leq R$ and $\|(T-S)a_1\|, \dots, \|(T-S)a_m\| < \varepsilon'$. Then

$$\sum_{k=1}^{m} \frac{\|(T-S)a_k\|}{\|a_k\|} 2^{-k} < \varepsilon' \sum_{k=1}^{m} \frac{1}{\|a_k\|} 2^{-k} \leq \varepsilon' \sum_{k=1}^{m} \frac{1}{\|a_k\|} = \frac{\varepsilon}{2},$$

$$\sum_{k=m+1}^{\infty} \frac{\|(T-S)a_k\|}{\|a_k\|} 2^{-k} \leq \sum_{k=m+1}^{\infty} \|T-S\| 2^{-k} \leq \sum_{k=m+1}^{\infty} (2R) 2^{-k} < 2R \frac{\varepsilon}{4R} = \frac{\varepsilon}{2}.$$

Then $\displaystyle\sum_{k=1}^{\infty} \frac{\|(T-S)a_k\|}{\|a_k\|} 2^{-k} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

This shows that $\left(S + O_{a_1,\dots,a_m;\varepsilon',\dots,\varepsilon'}\right) \cap B_R(X) \subset B_\sigma(S,\varepsilon)$. Thus, $B_\sigma(S,\varepsilon)$ contains a neighborhood of S in $(B_R(X), \sigma)$.

To show that the topology $(B_R(X),\sigma)$ is coarser than $(B_R(X), d_\sigma)$, we show that each neighborhood $\mathcal{U}$ of S in $(B_R(X),\sigma)$ contains a ball $B_\sigma(S,\varepsilon)$. We can assume $\mathcal{U} = (S + O_{x_1,\dots,x_n;\varepsilon_1,\dots,\varepsilon_n}) \cap B_R(X)$ for some $x_1, x_2, \dots, x_n \in X$, $\varepsilon_1, \dots, \varepsilon_n > 0$. Because the set $\{a_m : m \in \mathbb{N}\}$ is dense in X, there exist $a_{m_1}, a_{m_2}, \dots, a_{m_n}$ such that

$$\|a_{m_k} - x_k\| < \frac{\varepsilon_k}{4R} \quad \forall\, 1 \leq k \leq n.$$

Put $\displaystyle\varepsilon = \min\left\{ \frac{\varepsilon_k}{2\|a_{m_k}\|} 2^{-m_k} : 1 \leq k \leq n \right\} > 0$. For every $T \in B_\sigma(S,\varepsilon)$ we have $\|T\| \leq R$ and $\displaystyle\sum_{k=1}^{\infty} \frac{\|(T-S)a_k\|}{\|a_k\|} 2^{-k} < \varepsilon.$

Thus, $$\frac{\|(T-S)a_{m_k}\|}{\|a_{m_k}\|} 2^{-m_k} < \varepsilon \leq \frac{\varepsilon_k}{2\|a_{m_k}\|} 2^{-m_k} \quad \forall\, 1 \leq k \leq n.$$

Then $\quad \|(T-S)a_{m_k}\| < \frac{\varepsilon_k}{2} \quad \forall 1 \le k \le n.$

Then
$$\|(T-S)x_k\| \le \|(T-S)a_{m_k}\| + \|(T-S)(a_{m_k}-x_k)\|$$
$$< \frac{\varepsilon_k}{2} + \underbrace{\|T-S\|}_{\le 2R}\ \underbrace{\|a_{m_k}-x_k\|}_{<\frac{\varepsilon_k}{4R}}$$
$$< \frac{\varepsilon_k}{2} + \frac{\varepsilon_k}{2} = \varepsilon_k \quad \forall 1 \le k \le n.$$

Thus, $T-S \in O_{x_1,\dots,x_n;\,\varepsilon_1,\dots,\varepsilon_n}$. This shows $B_\sigma(S,\varepsilon) \subset \mathcal{U}$. We have showed that the topologies on $(B_R(X),\sigma)$ and on $(B_R(X), d_\sigma)$ are the same.

Next, we show that $(B_R(X),\tau)$ is also metrizable. Define a map $d_\tau : B_R(X) \times B_R(X) \to \mathbb{R}$,
$$d_\tau(T,S) = \sum_{k,j=1}^{\infty} \frac{|((T-S)a_k, a_j)|}{\|a_k\|\,\|a_j\|} 2^{-k-j} \quad \forall T,S \in B_R(X). \quad (4)$$

First, we verify that d_τ is well-defined.
$$\frac{|((T-S)a_k, a_j)|}{\|a_k\|\,\|a_j\|} \le \frac{\|(T-S)a_k\|\,\|a_j\|}{\|a_k\|\,\|a_j\|} \le \|T-S\| \le 2R \quad \forall T,S \in B_R(X).$$

Thus,
$$d_\tau(T,S) \le 2R \sum_{k,j=1}^{\infty} 2^{-k-j} = 2R\Big(\sum_{k=1}^{\infty} 2^{-k}\Big)\Big(\sum_{k=1}^{\infty} 2^{-j}\Big) = 2R < \infty.$$

Next, we verify that d_τ is a metric on $B_R(X)$. It is clear that
$$d_\tau(T,S) \ge 0, \quad d_\tau(T,S) = d_\tau(S,T) \quad \forall T,S \in B_R(X).$$

For $T_1, T_2, T_3 \in B_R(X)$,

$$d_\tau(T_1,T_2)+d_\tau(T_2,T_3)=\sum_{k,j=1}^{\infty}\frac{|(T_1a_k-T_2a_k,a_j)|+|(T_2a_k-T_3a_k,a_j)|}{\|a_k\|\,\|a_j\|}2^{-k-j}$$

$$\geq\sum_{k,j=1}^{\infty}\frac{|(T_1a_k-T_2a_k,a_j)+(T_2a_k-T_3a_k,a_j)|}{\|a_k\|\,\|a_j\|}2^{-k-j}$$

$$=\sum_{k,j=1}^{\infty}\frac{|(T_1a_k-T_3a_k,a_j)|}{\|a_k\|\,\|a_j\|}2^{-k-j}=d_\tau(T_1,T_3).$$

Suppose $d_\tau(T,S)=0$. Then $(Ta_k,a_j)=(Sa_k,a_j)$. for every $k,j\in\mathbb{N}$. Let $x,y\in X$. There exist sequences (x_n) and (y_n) in $\{a_k : k\in\mathbb{N}\}$ such that $x_n\to x$ and $y_n\to y$. We have $(Tx_n,y_n)=(Sx_n,y_n)$ for all $n\in\mathbb{N}$.

$$|(Tx,y)-(Sx,y)|\leq\underbrace{|(Tx,y)-(Tx_n,y)|}_{\{1\}}+\underbrace{|(Tx_n,y)-(Tx_n,y_n)|}_{\{2\}}+\underbrace{|(Tx_n,y_n)-(Sx,y)|}_{\{3\}}\quad\forall n\in\mathbb{N}.$$

$\{1\}\to 0$ as $n\to\infty$,

$\{2\}\leq\|Tx_n\|\,\|y-y_n\|\longrightarrow\|Tx\|.0=0$ as $n\to\infty$,

$$\{3\}=|(Sx_n,y_n)-(Sx,y)|\leq|(Sx_n,y_n-y)|+|(Sx_n-Sx,y)|$$

$$\leq\|Sx_n\|\,\|y_n-y\|+\|Sx_n-Sx\|\,\|y\|$$

$$\to\|Sx\|.0+0.\|y\|=0\quad\text{as } n\to\infty.$$

Thus, $(Tx,y)=(Sx,y)$ for every $x,y\in X$. Then

$$\|Tx-Sx\|^2=(Tx-Sx,\underbrace{Tx-Sx}_{y})=(Tx,y)-(Sx,y)=0\quad\forall x\in X.$$

Therefore, $T=S$. We have showed that d_τ is a metric on $B_R(X)$.

Next, we show that the topology on $B_R(X)$ induced by d_τ is the same as $(B_R(X),\tau)$. Denote

$$B_\tau(S,\varepsilon)=\{T\in B_R(X):\ d_\tau(T,S)<\varepsilon\}\quad \forall S\in B_R(X),\ \forall\varepsilon>0.$$

To show that the topology $(B_R(X),d_\tau)$ is coarser than $(B_R(X),\tau)$, we show that $B_\tau(S,\varepsilon)$ contains a neighborhood of S in $(B_R(X),\tau)$.

$$B_\tau(S,\varepsilon)=\Big\{T\in\mathcal{L}(X):\ \|T\|\le R,\ \sum_{k,j=1}^{\infty}\frac{|((T-S)a_k,a_j)|}{\|a_k\|\,\|a_j\|}2^{-k-j}<\varepsilon\Big\}.$$

There exists $m\in\mathbb{N}$ such that
$$\sum_{k,j=1}^{\infty}2^{-k-j}-\sum_{k,j=1}^{m}2^{-k-j}<\frac{\varepsilon}{4R}.$$

Put
$$\varepsilon'=\frac{\varepsilon}{2}\Big(\sum_{k,j=1}^{m}\frac{1}{\|a_k\|\,\|a_j\|}\Big)^{-1}>0,$$

$$U=O_{a_1,a_1,\,a_1,a_2,\dots,\,a_1,a_m,\,a_2,a_1,\,a_2,a_2,\dots,\,a_2,a_m,\dots,\,a_m,a_1,\,a_m,a_2,\dots,\,a_m,a_m;\ \varepsilon',\varepsilon',\dots,\varepsilon'}$$

For every $T\in(S+U)\cap B_R(X)$,

$$\sum_{k,j=1}^{m}\frac{|((T-S)a_k,a_j)|}{\|a_k\|\,\|a_j\|}2^{-k-j}<\sum_{k,j=1}^{m}\frac{\varepsilon'}{\|a_k\|\,\|a_j\|}2^{-k-j}\le\varepsilon'\sum_{k,j=1}^{m}\frac{1}{\|a_k\|\|a_j\|}=\frac{\varepsilon}{2},$$

$$\sum_{\substack{(k,j)\\ k>m\text{ or }j>m}}\frac{|((T-S)a_k,a_j)|}{\|a_k\|\,\|a_j\|}2^{-k-j}\le\sum_{\substack{(k,j)\\ k>m\text{ or }j>m}}\frac{\|T-S\|\,\|a_k\|\,\|a_j\|}{\|a_k\|\,\|a_j\|}2^{-k-j}$$
$$=\underbrace{\|T-S\|}_{\le 2R}\sum_{\substack{(k,j)\\ k>m\text{ or }j>m}}2^{-k-j}$$

$$\le 2R\Big(\underbrace{\sum_{k,j=1}^{\infty} 2^{-k-j} - \sum_{k,j=1}^{m} 2^{-k-j}}_{<\frac{\varepsilon}{4R}}\Big)$$

$$< \frac{\varepsilon}{2}.$$

Then $$\sum_{k,j=1}^{\infty} \frac{|((T-S)a_k, a_j)|}{\|a_k\|\,\|a_j\|}\, 2^{-k} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows $(S+\mathcal{U}) \cap B_R(X) \subset B_\tau(S,\varepsilon)$. Thus, $B_\tau(S,\varepsilon)$ contains a neighborhood of S in $(B_R(X),\tau)$.

To show that the topology $(B_R(X),\tau)$ is coarser than $(B_R(X), d_\tau)$, we show that each neighborhood V of S in $(B_R(X),\tau)$ contains a ball $B_\tau(S,\varepsilon)$. We can assume $V = (S + O_{x_1,y_1,\dots,x_n,y_n;\,\varepsilon_1,\dots,\varepsilon_n}) \cap B_R(V)$ for some $x_1,y_1,\dots,x_n,y_n \in X$, $\varepsilon_1,\dots,\varepsilon_n > 0$. Because the set $\{a_m : m \in \mathbb{N}\}$ is dense in X, there exist $a_{m_1}, a_{m_2}, \dots, a_{m_n}$ such that

$$\|x_\ell - a_{m_\ell}\| < \frac{\varepsilon_\ell}{6R(\|y_\ell\|+1)} \qquad \forall\, 1 \le \ell \le n. \qquad (5)$$

There exist $a_{r_1}, a_{r_2}, \dots, a_{r_n}$ such that

$$\|y_\ell - a_{r_\ell}\| < \frac{\varepsilon_\ell}{6R\,\|a_{m_\ell}\|} \qquad \forall\, 1 \le \ell \le n. \qquad (6)$$

Put $$\varepsilon = \min\left\{ \frac{\varepsilon_\ell}{3}\, \frac{2^{-m_\ell - r_\ell}}{\|a_{m_\ell}\|\,\|a_{r_\ell}\|} : 1 \le \ell \le n \right\} > 0.$$

For every $T \in B_\tau(S,\varepsilon)$, we have $\|T\| \le R$ and

$$\sum_{k,j=1}^{\infty} \frac{|((T-S)a_k, a_j)|}{\|a_k\| \, \|a_j\|} 2^{-k-j} < \varepsilon.$$

Thus,
$$\frac{|((T-S)a_{m_\ell}, a_{r_\ell})|}{\|a_{m_\ell}\| \, \|a_{r_\ell}\|} 2^{-m_\ell - r_\ell} < \varepsilon \le \frac{\varepsilon_\ell}{3} \frac{2^{-m_\ell - r_\ell}}{\|a_{m_\ell}\| \, \|a_{r_\ell}\|}.$$

Then
$$|((T-S)a_{m_\ell}, a_{r_\ell})| < \frac{\varepsilon_\ell}{3} \quad \forall 1 \le \ell \le n.$$

Then
$$\begin{aligned}
|((T-S)x_\ell, y_\ell)| &\le |((T-S)(x_\ell - a_{m_\ell}), y_\ell)| + |((T-S)a_{m_\ell}, y_\ell - a_{r_\ell})| \\
&\quad + |((T-S)a_{m_\ell}, a_{r_\ell})| \\
&< \|T-S\| \, \|x_\ell - a_{m_\ell}\| \, \|y_\ell\| + \|T-S\| \, \|a_{m_\ell}\| \, \|y_\ell - a_{r_\ell}\| + \frac{\varepsilon_\ell}{3} \\
&\le 2R \|x_\ell - a_{m_\ell}\| \, \|y_\ell\| + 2R \|a_{m_\ell}\| \, \|y_\ell - a_{r_\ell}\| + \frac{\varepsilon_\ell}{3} \\
&\overset{(5),(6)}{\le} \frac{\varepsilon_\ell}{3} + \frac{\varepsilon_\ell}{3} + \frac{\varepsilon_\ell}{3} = \varepsilon_\ell \quad \forall 1 \le \ell \le n.
\end{aligned}$$

Thus, $T - S \in \mathcal{O}_{x_1, y_1, \dots, x_n, y_n; \varepsilon_1, \dots, \varepsilon_n}$. This shows $B_\tau(S,\varepsilon) \subset V$. We have showed that the topology on $(B_R(X), \tau)$ and the topology $(B_R(X), d_\tau)$ are the same.

By the definition of d_σ at (3) and d_τ at (4), we deduce that for any $T, T_1, T_2, T_3, \dots \in B_R(X)$,

(a) $T_n \to T$ in $(B_R(X), d_\sigma)$ if and only if $\lim_{n\to\infty} T_n a_k = T a_k$ for all $k \in \mathbb{N}$;

(b) $T_n \to T$ in $(B_R(X), d_\tau)$ if and only if $\lim_{n\to\infty} (T_n a_k, a_j) = (T a_k, a_j)$ for all $k, j \in \mathbb{N}$.

Next, we show that $(B_R(X), \tau)$ is compact. That is to show $(B_R(X), d_\tau)$ is a compact metric space. Let (T_n) be a sequence in $(B_R(X), d_\tau)$. We show that it has a convergent subsequence. Let $\{a_k : k \in \mathbb{N}\}$ be a dense subset of X. For each $k, j \in \mathbb{N}$,

$$|(T_n a_k, a_j)| \leq \|T_n a_k\| \, \|a_j\| \leq \|T_n\| \, \|a_k\| \, \|a_j\| \leq R \|a_k\| \, \|a_j\| \quad \forall n \in \mathbb{N}.$$

Thus, the sequence $\{(T_n a_k, a_j)\}_{n\in\mathbb{N}}$ is bounded in K, which is either $\mathbb{R}$ or $\mathbb{C}$. Then it has a convergent subsequence. Choose a way to enumerate the elements of $\mathbb{N} \times \mathbb{N}$, for example as in the following diagram.

$$\begin{tikzcd}
(1,1) \arrow[r] & (1,2) \arrow[d] & (1,3) \arrow[r] & (1,4) \arrow[d] & \vdots \\
(2,1) \arrow[d] & (2,2) \arrow[l] & (2,3) \arrow[u] & (2,4) \arrow[d] & \\
(3,1) \arrow[r] & (3,2) \arrow[r] & (3,3) \arrow[u] & (3,4) \arrow[d] & \\
(4,1) \arrow[d] & (4,2) \arrow[l] & (4,3) \arrow[l] & (4,4) \arrow[l] & \\
\cdots & & & &
\end{tikzcd}$$

Let $(T_n^{(1,1)})$ be a subsequence of (T_n) such that $\{(T_n^{(1,1)} a_1, a_1)\}$ converges.

Let $(T_n^{(1,2)})$ be a subsequence of $(T_n^{(1,1)})$ such that $\{(T_n^{(1,2)} a_1, a_2)\}$ converges.

Let $(T_n^{(2,2)})$ be a subsequence of $(T_n^{(1,2)})$ such that $\{(T_n^{(2,2)} a_2, a_2)\}$ converges.

$\cdots$

By Cantor's diagonal method, there is a subsequence of (T_n), called (T_{n_ℓ}), such that $\{(T_{n_\ell} a_k, a_j)\}_{\ell\in\mathbb{N}}$ converges for all $k,j\in\mathbb{N}$. For an explicit expression for (T_{n_ℓ}), we can take $(T_{n_\ell})_{\ell\in\mathbb{N}} = (T^{(k,j)}_{k+j})_{(k,j)\in\mathbb{N}\times\mathbb{N}}$ where the order in $\mathbb{N}\times\mathbb{N}$ is indicated as in the above diagram. By replacing $(T_n)_{n\in\mathbb{N}}$ with $(T_{n_\ell})_{\ell\in\mathbb{N}}$, we can assume $\{(T_n a_k, a_j)\}_{n\in\mathbb{N}}$ converges for all $k,j\in\mathbb{N}$. Let V be the linear span of $\{a_k : k\in\mathbb{N}\}$. Then $\{(T_n x, y)\}_{n\in\mathbb{N}}$ converges for all $x,y\in V$. Denote

$$(Tx,y) = \lim_{n\to\infty}(T_n x, y) \qquad \forall x,y\in V. \qquad (7)$$

Then for every $x\in V$, Tx can be viewed as a linear map from V to K.

$$|(T_n x, y)| \leq \|T_n\|\,\|x\|\,\|y\| \leq R\|x\|\,\|y\| \quad \forall n\in\mathbb{N}.$$

Thus, $|(Tx,y)| \leq R\|x\|\,\|y\|$ for all $y\in V$. This implies $Tx: V\to K$ is a continuous linear map. Since $\overline{V}=X$, Tx can be extended to a continuous linear map from X to K. Because X is a Hilbert space, Tx can be identified with an element of X. Then $T: V\to X$ is a linear map.

$$\|Tx\| = \sup_{\substack{y\in X\\ \|y\|=1}} |(Tx,y)| \leq \sup_{\substack{y\in X\\ \|y\|=1}} R\|x\|\,\|y\| = R\|x\| \quad \forall x\in V. \quad (8)$$

This implies T is continuous. Because $\overline{V}=X$, T can extend to a continuous linear map from X to X. Then $T\in\mathcal{L}(X)$. The estimate (8) implies $\|T\|\leq R$.

Then $T \in B_R(X)$. By (7) and statement (b), we conclude that (T_n) converges to T in $(B_R(X), d_\tau)$. Thus, $(B_R(X), d_\tau)$ is a compact metric space.

In case X is finite dimensional, $B_R(X)$ is a compact subset of $(\mathcal{L}(X), \|\cdot\|)$. We showed on pages 5-6 that $(B_R(X), \sigma)$ is the same as the topology induced by the norm. Thus, $(B_R(X), \sigma)$ is compact. Now assume X is infinite dimensional. We show that $(B_R(X), \sigma)$ is not compact. Specifically, we construct a sequence (T_n) in $(B_R(X), d_\sigma)$ that has no convergent subsequence.

Let $\{e_n : n \in \mathbb{N}\}$ be a Hilbert basis of X. By replacing the set $\{a_k : k \in \mathbb{N}\}$ with $\{a_k : k \in \mathbb{N}\} \cup \{e_1\}$, we can assume that $e_1 \in \{a_k : k \in \mathbb{N}\}$. Let S be the 1-translation operator of X, i.e. $S \in \mathcal{L}(X)$ and $Se_k = e_{k+1}$ for every $k \in \mathbb{N}$. Then $\|S\| = 1$. Denote $S^n = S \circ S \circ \cdots \circ S$ (n times). Then

$$\|S^n\| \le \underbrace{\|S\| \, \|S\| \cdots \|S\|}_{n \text{ times}} \le 1.$$

Put $T_n = R \cdot S^n$ for every $n \in \mathbb{N}$. Then $T_n \in \mathcal{L}(X)$ and $\|T_n\| \le R$.

$$T_n e_1 = R \cdot S^n e_1 = R\, e_{n+1} \quad \forall n \in \mathbb{N}.$$

Then

$$\|T_m e_1 - T_n e_1\| = R\, \|e_{m+1} - e_{n+1}\| = R\sqrt{2} \quad \forall m, n \in \mathbb{N},\ m \ne n.$$

This implies the sequence $(T_n e_1)$ has no convergent subsequence. By statement (a), the sequence (T_n) has no convergent subsequence.

(iii) Suppose X is infinite dimensional. We show that the composition $(S,T) \mapsto S \circ T$ is not continuous as a map from $(\mathcal{L}(X),\sigma) \times (\mathcal{L}(X),\sigma)$ to $(\mathcal{L}(X),\sigma)$.

Suppose otherwise. Take any $x_0 \in X \setminus \{0\}$. Then the preimage of

$$O_{x_0;1} = \{L \in \mathcal{L}(X) : \|Lx_0\| < 1\}$$

under the composition map is a neighborhood of $(0,0)$ in $(\mathcal{L}(X),\sigma) \times (\mathcal{L}(X),\sigma)$. There exist $x_1, x_2, \dots, x_n \in X$, $x_1', \dots, x_m' \in X$, $\varepsilon_1, \dots, \varepsilon_n > 0$, $\varepsilon_1', \dots, \varepsilon_m' > 0$ such that $S \circ T \in O_{x_0;1}$ for all $S \in O_{x_1,\dots,x_n;\varepsilon_1,\dots,\varepsilon_n}$ and $T \in O_{x_1',\dots,x_m';\varepsilon_1',\dots,\varepsilon_m'}$

In other words, for any $S, T \in \mathcal{L}(X)$ such that

$$\|Sx_j\| < \varepsilon_j \quad \forall 1 \le j \le n,$$

$$\|Tx_k'\| < \varepsilon_k' \quad \forall 1 \le k \le m$$

we have $\|S(Tx_0)\| < 1$. Put

$$\varepsilon = \frac{\min\{\varepsilon_1', \varepsilon_2', \dots, \varepsilon_m'\}}{\frac{1}{\|x_0\|^2}\max\{|(x_1', x_0)|, |(x_2', x_0)|, \dots, |(x_m', x_0)|\} + 1},$$

$$Y = \text{linear span}\{x_1, x_2, \dots, x_n\},$$

$$Z = \text{linear span}\{x_1', x_2', \dots, x_m'\}.$$

Because X is infinite dimensional, there exists $y_0 \in X \setminus Y$. By rescaling y_0 if necessary, we can assume $\|y_0\| = \varepsilon$. We show that there exists $B \in O_{x_1',\dots,x_m';\varepsilon_1',\dots,\varepsilon_m'}$ such that $Bx_0 = y_0$. Consider two following cases.

▪ $\underline{x_0 \in Z}$:

Let $\{x_0, z_1, z_2, \dots, z_k\}$ be an othogonal basis of Z. Define a linear map $B: Z \to K\{y_0\}$,

$$Bx_0 = y_0, \quad Bz_1 = Bz_2 = \dots = Bz_k = 0.$$

By Hahn-Banach theorem, B can extend to a continuous linear map from X to $K\{y_0\}$, which is still denoted by B. We can regard $B \in \mathcal{L}(X)$. Then

$$Bx_j' = \frac{(x_0, x_j')}{\|x_0\|^2} \underbrace{Bx_0}_{=y_0} + \frac{(z_1, x_j')}{\|z_1\|^2} \underbrace{Bz_1}_{=0} + \dots + \frac{(z_k, x_j')}{\|z_k\|^2} \underbrace{Bz_k}_{=0}$$
$$= \frac{(x_0, x_j')}{\|x_0\|^2} y_0 .$$

Then $\quad \|Bx_j'\| = |(x_0, x_j')| \frac{\|y_0\|}{\|x_0\|^2} = \frac{|(x_0, x_j')|}{\|x_0\|^2} \varepsilon < \varepsilon_j' \quad \forall 1 \le j \le m.$

Thus, $B \in O_{x_1', \dots, x_m'; \varepsilon_1', \dots, \varepsilon_m'}$.

▪ $\underline{x_0 \notin Z}$:

Define a linear map $B: Z \oplus K\{x_0\} \to K\{y_0\}$,

$$Bx_0 = y_0, \quad B|_Z = 0.$$

By Hahn-Banach theorem, B can extend to a continuous linear map from X to $K\{y_0\}$. We can regard $B \in \mathcal{L}(X)$. Since $B|_Z = 0$, $Bx_1' = Bx_2' = \dots = Bx_m' = 0$.

Then $B \in O_{x_1', \dots, x_m'; \varepsilon_1', \dots, \varepsilon_m'}$. The existence of B is proved.

Next, define a linear map $A: Y \oplus K\{y_0\} \to K\{y_0\}$,

$$Ay_0 = \frac{y_0}{\varepsilon}, \quad A|_Y = 0.$$

By Hahn-Banach theorem, A can extend to a continuous linear map from X to $K\{y_0\}$. We can regard $A \in \mathcal{L}(X)$. Because $Ax_1 = Ax_2 = \cdots = Ax_n = 0$, $A \in \mathcal{O}_{x_1,\dots,x_n;\,\varepsilon_1,\dots,\varepsilon_n}$.

$$A(Bx_0) = Ay_0 = \frac{y_0}{\varepsilon}.$$

Then $\|A(Bx_0)\| = \frac{\|y_0\|}{\varepsilon} = 1$, which implies $A \circ B \notin \mathcal{O}_{x_0;1}$. This is a contradiction.

(iv) We show that the composition $(S,T) \mapsto S \circ T$ is continuous as a map from $(B_{R_1}(X), \sigma) \times (B_{R_2}(X), \sigma)$ to $(B_{R_1R_2}(X), \sigma)$.

First, this map is well-defined because $\|S \circ T\| \le \|S\|\,\|T\| \le R_1 R_1$. Because $(B_R(X), \sigma)$ is metrizable for every $R > 0$, it suffices to show that the composition map is sequentially continuous. Let (S_n) be a sequence in $(B_{R_1}(X), \sigma)$ that converges to $S \in B_{R_1}(X)$. Let (T_n) be a sequence in $(B_{R_2}(X), \sigma)$ that converges to $T \in B_{R_2}(X)$. We show that $(S_n \circ T_n)$ converges to $S \circ T$ in $(B_{R_1R_2}(X), \sigma)$. By statement (a) on page 15, it suffices to show that $S_n \circ T_n(y) \to S \circ T(y)$ for every $y \in X$.

Fix $y \in X$. ~~By~~ Because $T_n \to T$ in $(B_{R_2}(X), \sigma)$, $T_n y \to Ty$. Put $Ty = z$. Because $S_n \to S$ in $(B_{R_1}(X), \sigma)$, $S_n z \to Sz$. Then

$$\begin{aligned}
\|S_n \circ T_n(y) - S \circ T(y)\| &= \|S_n(T_n y - Ty) + (S_n - S) \circ T(y)\| \\
&\le \|S_n\|\,\|T_n y - Ty\| + \|S_n z - Sz\| \\
&\le R_1 \underbrace{\|T_n y - Ty\|}_{\to 0} + \underbrace{\|S_n z - Sz\|}_{\to 0} \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} S_n \circ T_n(y) = S \circ T(y)$.

(v). We show that the composition $(S,T) \mapsto S\circ T$ is continuous as a map from $(B_{R_1}(X),\tau)\times(B_{R_2}(X),\sigma)$ to $(B_{R_1R_2}(X),\tau)$.

Because $(B_R(X),\tau)$ and $(B_R(X),\sigma)$ are metrizable for every $R>0$, it suffices to show that the composition map is sequentially continuous. Let

$$S_n \longrightarrow S \quad \text{in } (B_{R_1}(X),\tau),$$
$$T_n \longrightarrow T \quad \text{in } (B_{R_2}(X),\sigma).$$

We show that $S_n\circ T_n \longrightarrow S\circ T$ in $(B_{R_1R_2}(X),\tau)$. By statement (b) on page 15, it suffices to show that $(S_n\circ T_n(x),y) \longrightarrow (S\circ T(x),y)$ for all $x,y\in X$.

Fix $x,y\in X$. Because $T_n\to T$ in $(B_{R_2}(X),\sigma)$, $T_nx\longrightarrow Tx$. Put $z=Tx$. Because $S_n\to S$ in $(B_{R_1}(X),\tau)$, $(S_nz,y)\longrightarrow(Sz,y)$. We have

$$\begin{aligned}
|(S_n\circ T_n(x),y)-(S\circ T(x),y)| &= |(S_n(T_nx)-S(Tx),y)| \\
&= |(S_n(T_nx-Tx),y)+(S_n(Tx),y)-(S(Tx),y)| \\
&\le |(S_n(T_nx-Tx),y)| + |(S_n(Tx),y)-(S(Tx),y)| \\
&\le \|S_n\|\,\|T_nx-Tx\|\,\|y\| + |(S_nz,y)-(Sz,y)| \\
&\le R_1 \underbrace{\|T_nx-Tx\|}_{\to 0}\,\|y\| + \underbrace{|(S_nz,y)-(Sz,y)|}_{\longrightarrow 0} \quad \text{as } n\to\infty.
\end{aligned}$$

Therefore, $\lim_{n\to\infty}(S_n\circ T_n(x),y) = (S\circ T(x),y)$.

(vi) Suppose X is infinite dimensional. We show that the composition $(S,T) \mapsto S\circ T$ is not continuous as a map from $(B_{R_1}(X),\sigma)\times(B_{R_2}(X),\tau)$ to $(B_{R_1R_2}(X),\tau)$.

Let (e_n) be a Hilbert basis of X. For each $x\in X$, we have the Parseval's identity

$$\|x\|^2 = \sum_{n=1}^{\infty} |(x,e_n)|.$$

Thus, $\lim_{n\to\infty}(x,e_n)=0$. Consider the linear maps $S_n, T_n : X \to X$,

$$S_n x = R_1(x,e_n)\,e_1 \quad \forall x\in X,$$

$$T_n x = R_2(x,e_1)\,e_n \quad \forall x\in X.$$

Then

$$\|S_n x\| = R_1\,|(x,e_n)| \le R_1\|x\|\,\|e_n\| = R_1\|x\|,$$

$$\|T_n x\| = R_2\,|(x,e_1)| \le R_2\|x\|\,\|e_1\| = R_2\|x\|.$$

Thus, $S_n, T_n \in \mathcal{L}(X)$ and $\|S_n\|\le R_1$, $\|T_n\|\le R_2$. For each $x\in X$, $S_n x\to 0$ because $(x,e_n)\to 0$ as $n\to\infty$. Thus $S_n\to 0$ in $(B_{R_1}(X),\sigma)$ as $n\to\infty$. For $x,y\in X$,

$$(T_n x, y) = (R_2(x,e_1)e_n, y) = R_2(x,e_1)(e_n,y) \longrightarrow 0 \quad \text{as } n\to\infty.$$

Thus, $T_n\to 0$ in $(B_{R_2}(X),\tau)$ as $n\to\infty$.

$$S_n\circ T_n(x) = S_n(R_2(x,e_1)e_n) = R_2(x,e_1)\,S_n e_n$$
$$= R_2(x,e_1)R_1(e_n,e_n)\,e_1 = R_1R_2(x,e_1)\,e_1 \quad \forall n\in\mathbb{N}.$$

Then

$$(S_n \circ T_n(e_1), e_1) = (R_1 R_2 (e_1, e_1) e_1, e_1) = R_1 R_2 > 0 \quad \forall n \in \mathbb{N}.$$

This shows that $S_n \circ T_n$ does not tend to 0 in $(B_{R_1 R_2}(X), \tau)$.

(vii) Suppose X is infinite dimensional. We show that the involution $S \mapsto S^*$ is not continuous as a map from $(B_R(X), \sigma)$ to $(B_R(X), \sigma)$.

Let (e_n) be a Hilbert basis of X and $S_n \in B_R(X)$ be the map as in Part (vi):

$$S_n x = R(x, e_n) e_1 \qquad \forall x \in X.$$

We showed that $S_n \to 0$ in $(B_R(X), \sigma)$. For $x, y \in X$,

$$(S_n x, y) = (R(x, e_n) e_1, y) = R(x, e_n)(e_1, y) = (x, R\overline{(e_1, y)}\, e_n)$$
$$= (x, R(y, e_1) e_n).$$

Thus, $S_n^* y = R(y, e_1) e_n$. Then $S_n^* e_1 = R(e_1, e_1) e_n = R e_n$. Since $\|S_n^* e_1\| = R$, $S_n^* e_1 \not\to 0$ in X as $n \to \infty$. This shows that $S_n^* \not\to 0$ in $(B_R(X), \sigma)$.

(viii) We show that the involution $T \mapsto T^*$ is continuous as a map from $(B_R(X), \tau)$ to $(B_R(X), \tau)$.

Let (T_n) be a sequence in $(B_R(X), \tau)$ that converges to T. For every $x, y \in X$,

$$(T_n^* x, y) = (x, T_n y) = \overline{(T_n y, x)} \qquad \forall n \in \mathbb{N}.$$

Taking the limit both sides, we get

$$\lim_{n \to \infty} (T_n^* x, y) = \overline{(Ty, x)} = (x, Ty) = (T^* x, y).$$

This shows that (T_n^*) converges to T^* in $(B_R(X), \tau)$.

(ix) We show that the involution $T \mapsto T^*$ is continuous as a map from $(\mathcal{L}(X), \tau)$ to $(\mathcal{L}(X), \tau)$. Because the map is linear, it suffices to show that it is continuous at 0. Let $\mathcal{U}$ be a neighborhood of 0 in $(\mathcal{L}(X), \tau)$. There exist $x_1, y_1, \ldots, x_n, y_n \in X$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that

$$\mathcal{O}_{x_1, y_1, \ldots, x_n, y_n; \varepsilon_1, \ldots, \varepsilon_n} \subset \mathcal{U}.$$

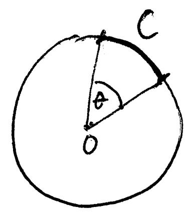
Put $V = \mathcal{O}_{y_1, x_1, \ldots, y_n, x_n; \varepsilon_1, \ldots, \varepsilon_n}$. For each $T \in V$, $|(Ty_j, x_j)| < \varepsilon_j$ for all $1 \leq j \leq n$.

Then

$$|(T^* x_j, y_j)| = |(x_j, Ty_j)| = |(Ty_j, x_j)| < \varepsilon_j \quad \forall 1 \leq j \leq n.$$

Thus, $T^* \in \mathcal{O}_{x_1, y_1, \ldots, x_n, y_n; \varepsilon_1, \ldots, \varepsilon_n} \subset \mathcal{U}$. We conclude that the involution map is continuous at 0.

(2) Let $S^1 \subset \mathbb{C}$ be the unit circle centered at 0, and λ be the Lebesgue measure on S^1 normalized such that $\lambda(S^1) = 2\pi$. (It is the measure of angle). Put $X = L^2(S^1, \lambda)$, which is a Hilbert space over $\mathbb{C}$. Define a linear map $T: X \to X$, $Tf(z) = x f(z)$ for $f \in X$, $z \in S^1$, where $z = x + iy$, $x, y \in \mathbb{R}$.



$\lambda(C) = \theta$

$$\|Tf\|_X = \left(\int_{S^1} |x f(z)|^2 d\lambda \right)^{1/2} \leq \left(\int_{S^1} |f(z)|^2 d\lambda \right)^{1/2} = \|f\|_X \quad \forall f \in X.$$

Thus, T is continuous and $\|T\| \leq 1$. We compute T^*. For $f, g \in X$,

$$(Tf, g) = \int_{S^1} Tf(z) \overline{g(z)} \, d\lambda = \int_{S^1} x f(z) \overline{g(z)} d\lambda = \int_{S^1} f(z) \overline{x g(z)} d\lambda = (f, Tg).$$

Thus, $T^*g(z) = x\overline{g(z)}$. Then

$$T(T^*g)(z) = T(z \mapsto x\overline{g(z)})(z) = x^2\overline{g(z)},$$

$$T^*(Tg)(z) = T^*(z \mapsto xg(z))(z) = x\,\overline{xg(z)} = x^2\overline{g(z)}.$$

Then $TT^* = T^*T$, i.e. T is a normal operator of X.

We compute the spectrum of T. For $\alpha \in \mathbb{C}$,

$$(\alpha Id_X - T)f(z) = \alpha f(z) - Tf(z) = (\alpha - x)f(z) \quad \forall f \in X, \forall z \in S^1.$$

If $\alpha \notin [-1,1]$ then $\alpha Id_X - T$ has a continuous inverse, which is

$$g(z) \mapsto \frac{g(z)}{\alpha - x} \qquad \forall g \in X, \forall z \in S^1.$$

Consider the case $\alpha \in [-1,1]$. Suppose there exists $f_0 \in X$ such that $(\alpha Id_X - T)f_0 = 1$.

Then
$$f_0(z) = \frac{1}{\alpha - x} \qquad \text{a.e. } z \in S^1.$$

Then
$$\|f_0\|_X^2 = \int_{S^1} |f_0(z)|^2 d\lambda = \int_0^{2\pi} |f_0(e^{i\theta})|^2 d\theta = \int_0^{2\pi} \frac{1}{(\alpha - \cos\theta)^2}\, d\theta. \quad (1)$$

Since $\alpha = [-1,1]$, there exists $\theta_0 \in [0, 2\pi]$ such that $\alpha = \cos\theta_0$. Then

$$|\alpha - \cos\theta| = |\cos\theta_0 - \cos\theta| = |\theta_0 - \theta|\,|\sin\xi| \quad (\text{for some } \xi \text{ between } \theta_0 \text{ and } \theta)$$
$$\leq |\theta_0 - \theta|.$$

Then (1) implies
$$\|f_0\|_X^2 \geq \int_0^{2\pi} \frac{1}{(\theta_0 - \theta)^2}\, d\theta = \infty.$$

This is a contradiction. Therefore, there is no $f_0 \in X$ such that

$(\alpha Id_X - T)f_0 = 1$. Then $\alpha Id_X - T \notin \mathcal{L}^{\times}(X)$. We conclude that the spectrum of T is $K = \sigma(T) = [-1,1]$.

Next, we compute T as an algebra action of $C([-1,1])$ on X. For each polynomial $P(z) = a_n z^n + a_n z^{n-1} + \dots + a_1 z + a_0$, we denote

$$P(T) = a_n T^n + a_{n-1} T^{n-1} + \dots + a_1 T + a_0 Id_X \in \mathcal{L}(X).$$

Then
$$P(T)f(z) = a_n T^n f(z) + a_{n-1} T^{n-1} f(z) + \dots + a_1 Tf(z) + a_0 f(z)$$
$$= a_n x^n f(z) + a_{n-1} x^{n-1} f(z) + \dots + a_1 x f(z) + a_0 f(z)$$
$$= P(x) f(z) \qquad \forall f \in X, z \in S^1.$$

Denote $T(P) \in \mathcal{L}(X)$ to be the map

$$T(P)f(z) = P(x)f(z) \quad \forall f \in X, z \in S^1.$$

Then T can be regarded as an algebra action of $C([-1,1])$ on X as follows.

$$T : C([-1,1]) \longrightarrow \mathcal{L}(X),$$
$$T(h)f(z) = h(x)f(z) \qquad \forall h \in C([-1,1]),\ f \in X,\ z \in S^1.$$

Next, we decompose X into invariant cyclic subspaces under the action T.

Put $\xi_1, \xi_2 \in X$, $\xi_1(z) \equiv 1$, $\xi_2(z) = y$,

$$X_1 = \overline{\{T(g)\xi_1 : g \in C([-1,1])\}} \subset X,$$

$$X_2 = \overline{\{T(h)\xi_2 : h \in C([-1,1])\}} \subset X.$$

Write

$$Y_1 = \{T(g)\xi_1 : g \in C([-1,1])\} = \{z \mapsto g(x)\xi_1(z) : g \in C([-1,1])\}$$
$$= \{z \mapsto g(x) : g \in C([-1,1])\},$$
$$Y_2 = \{T(h)\xi_2 : h \in C([-1,1])\} = \{z \mapsto h(x)\xi_2(z) : h \in C([-1,1])\}$$
$$= \{z \mapsto h(x)y : h \in C([-1,1])\}.$$

Denote by $(\cdot,\cdot)$ the scalar product in X. We have

$$(z \mapsto g(x), z \mapsto h(x)y) = \int_{S^1} g(x)\overline{h(x)y}\,d\lambda = \int_{S^1} g(x)\overline{h(x)}\,y\,d\lambda$$
$$= \int_0^{2\pi} g(\cos\theta)\overline{h(\cos\theta)}\sin\theta\,d\theta \quad (\text{where } x = \cos\theta,\ y = \sin\theta)$$
$$= \int_0^{\pi} g(\cos\theta)\overline{h(\cos\theta)}\sin\theta\,d\theta + \underbrace{\int_{\pi}^{2\pi} g(\cos\theta)\overline{h(\cos\theta)}\sin\theta\,d\theta}_{\overset{\tau=2\pi-\theta}{=} \int_0^{\pi} g(\cos\tau)\overline{h(\cos\tau)}(-\sin\tau)\,d\tau}$$
$$= 0.$$

This shows that Y_1 and Y_2 are perpendicular. Then

$$Y_1 \subset Y_2^{\perp} = \overline{Y}_2^{\perp} = X_2^{\perp}.$$

Since $X_2^{\perp}$ is closed in X, $X_1 = \overline{Y}_1 \subset X_2^{\perp}$. Thus, X_1 and X_2 are perpendicular.

Now we show that $X = X_1 + X_2$. Because $X_1 + X_2$ is a direct sum of two closed subspaces of X, it is closed in X. It suffices to show that $X_1 + X_2$ is dense in X. Because the polynomials are dense in $X = L^2(S^1, \lambda)$, for each $f \in X$ and $\varepsilon > 0$ there exists a polynomial $P = P(z)$ such that $\|f - P\|_X < \varepsilon$. Write

$$\begin{aligned} P(z) &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \\ &= a_n (x+iy)^n + a_{n-1}(x+iy)^{n-1} + \cdots + a_1(x+iy) + a_0 \\ &= \sum_{0 \le k, l \le n} c_{kl} x^k y^l, \end{aligned}$$

where c_{kl}'s are complex coefficients. Because $z \in S^1$, $|z| = \sqrt{x^2+y^2} = 1$. Then $y^2 = 1 - x^2$. Then

$$\begin{aligned} P(z) &= \sum_{\substack{0 \le k \le n \\ 0 \le m \le \frac{n}{2}}} c_{k,2m} x^k y^{2m} + \sum_{\substack{0 \le k \le n \\ 0 \le m \le \frac{n-1}{2}}} c_{k,2m+1} x^k y^{2m+1} \\ &= \underbrace{\sum_{\substack{0 \le k \le n \\ 0 \le m \le \frac{n}{2}}} c_{k,2m} x^k (1-x^2)^m}_{(z \mapsto g(x))} + \underbrace{y \sum_{\substack{0 \le k \le n \\ 0 \le m \le \frac{n-1}{2}}} c_{k,2m+1} x^k (1-x^2)^m}_{(z \mapsto h(x)y)} \end{aligned}$$

Because $(z \mapsto g(x)) \in X_1$ and $(z \mapsto h(x)y) \in X_2$, we get $(z \mapsto P(z)) \in X_1 + X_2$. Hence, $X_1 + X_2$ is dense in X. We have showed that $X = X_1 \oplus X_2$.

Let $f \in C([-1,1])$. An element $\xi \in Y_1$ is of the form $\xi(z) = g(x)$ for some $g \in C([-1,1])$.

$$T(f)\xi(z) = f(x)\xi(z) = f(x)g(x).$$

Then $T(f)\xi \in Y_1$. Thus, $T(f)(Y_1) \subset Y_1$. Because $T(f) \in \mathcal{L}(X)$ and $X_1 = \overline{Y_1}$, we also have $T(f)(X_1) \subset X_1$.

An element $\eta \in Y_2$ is of the form $\eta(z) = y h(x)$ for some $h \in C([-1,1])$.

$$T(f)\eta(z) = f(x)\eta(z) = f(x)\, y\, h(x).$$

Then $T(f)\eta \in Y_2$. Thus, $T(f)(Y_2) \subset Y_2$. Because $T(f) \in \mathcal{L}(X)$ and $X_2 = \overline{Y_2}$, we get $T(f)(X_2) \subset X_2$.

Therefore, $X = X_1 \oplus X_2$ is a decomposition of X into invariant cyclic subspaces under the algebra action T. Next, we determine the measures μ_j on $[-1,1]$ associate with the cyclic action $T: C([-1,1]) \to \mathcal{L}(X_j)$ for $j \in \{1,2\}$. By definition, μ_j is the measure on $[-1,1]$ such that

$$(T(f)\xi_j, \xi_j) = \int_{[-1,1]} f(x)\, d\mu_j(x) \qquad \forall f \in C([-1,1]). \qquad (2)$$

For $j = 1$,

$$\text{LHS}(2) = \int_{S^1} T(f)\xi_1(z)\overline{\xi_1(z)}\, d\lambda = \int_{S^1} f(x)\xi_1(z)\overline{\xi_1(z)}\, d\lambda$$

$$= \int_{S^1} f(x)\, d\lambda = \int_0^{2\pi} f(\cos\theta)\, d\theta$$

$$= \int_0^{\pi} f(\cos\theta)\, d\theta + \underbrace{\int_{\pi}^{2\pi} f(\cos\theta)\, d\theta}_{= \int_0^{\pi} f(\cos\tau)\, d\tau \text{ where } \tau = 2\pi - \theta}$$

$$= 2\int_0^{\pi} f(\cos\theta)\, d\theta$$

$$= \int_{-1}^{1} \frac{2f(x)}{\sqrt{1-x^2}}\, dx.$$

This shows that μ_1 is the measure on $[-1,1]$ whose Radon-Nikodym derivative with respect to the Lebesgue measure is

$$\frac{d\mu_1}{dx} = \frac{2}{\sqrt{1-x^2}} .$$

In other words,
$$\mu_1(B) = \int_B \frac{2}{\sqrt{1-x^2}}\,dx \quad \forall \text{ Borel set } B \subset [-1,1].$$

For $j=2$,
$$LHS(2) = \int_{S^1} \pi(f)\,\xi_2(z)\,\overline{\xi_2(z)}\,d\lambda = \int_{S^1} (f(x)y)\,y\,d\lambda$$

$$= \int_0^{2\pi} f(\cos\theta)\sin^2\theta\,d\theta = \int_0^{2\pi} f(\cos\theta)(1-\cos^2\theta)\,d\theta$$

$$= \int_0^{\pi} f(\cos\theta)(1-\cos^2\theta)\,d\theta + \underbrace{\int_{\pi}^{2\pi} f(\cos\theta)(1-\cos^2\theta)\,d\theta}_{= \int_0^{\pi} f(\cos\tau)(1-\cos^2\tau)\,d\tau \text{ where } \tau = 2\pi-\theta}$$

$$= 2\int_0^{\pi} f(\cos\theta)(1-\cos^2\theta)\,d\theta$$

$$= 2\int_{-1}^{1} f(x)(1-x^2)\frac{1}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1} 2\sqrt{1-x^2}\,f(x)\,dx .$$

This shows that μ_2 is the measure on $[-1,1]$ whose Radon-Nikodym derivative with respect to the Lebesgue measure is

$$\frac{d\mu_2}{dx} = 2\sqrt{1-x^2} .$$

More explicitly,
$$\mu_2(B) = \int_B 2\sqrt{1-x^2}\,dx \quad \forall \text{ Borel set } B \subset [-1,1] .$$

We established in Lecture 04/03/2015 that there is an isomorphism (of Banach spaces) $\varphi_j : X_j \to L^2([-1,1], \mu_j)$ satisfying

$$\varphi_j(T(g)\xi_1) = g \qquad \forall g \in C([-1,1]).$$

For each $f \in C([-1,1])$, define the maps

$$\Psi_1(f) : L^2([-1,1], \mu_1) \to L^2([-1,1], \mu_1)$$
$$\Psi_1(f)\, g(x) = g(x) f(x),$$

$$\Psi_2(f) : L^2([-1,1], \mu_2) \to L^2([-1,1], \mu_2)$$
$$\Psi_2(f)\, g(x) = g(x) f(x).$$

The following diagram commutes (as established in the same lecture).

$$\begin{tikzcd} X_j \arrow[r, "T(f)"] \arrow[d, "\varphi_j"'] \arrow[dr, phantom, "\circlearrowleft"] & X_j \arrow[d, "\varphi_j"] \\ L^2(K, \mu_j) \arrow[r, "\Psi_j(f)"] & L^2(K, \mu_j) \end{tikzcd}$$

where $K = [-1,1]$.

Then the following diagram also commutes.

$$\begin{tikzcd} X = X_1 \oplus X_2 \arrow[r, "T(f)"] \arrow[d, "\varphi_1 \oplus \varphi_2"'] \arrow[dr, phantom, "\circlearrowleft"] & X = X_1 \oplus X_2 \arrow[d, "\varphi_1 \oplus \varphi_2"] \\ L^2(K, \mu_1) \oplus L^2(K, \mu_2) \arrow[r] & L^2(K, \mu_1) \oplus L^2(K, \mu_2) \end{tikzcd}$$

Denote $\varphi = \varphi_1 \oplus \varphi_2$, $\Psi(f) = \Psi_1(f) \oplus \Psi_2(f)$, $Y_j' = L^2([-1,1], \mu_j)$.

Next, we find a measure μ on $[-1,1]$ such that $Y_1 \oplus Y_2$ is isomorphic (as a Banach space) to $Y \oplus Y$ where $Y = L^2([-1,1],\mu)$. Define the maps

$$\rho_1 : [-1,1] \to [-1,0], \quad \rho_1(x) = \frac{x-1}{2},$$

$$\rho_2 : [-1,1] \to [0,1], \quad \rho_2(x) = \frac{x+1}{2}.$$

Then μ_1 induces a measure $\tilde{\mu}_1$ on $[-1,0]$ as

$$\tilde{\mu}_1(B) = \mu_1(\rho_1^{-1}(B)) \quad \forall \text{ Borel set } B \subset [-1,0].$$

Similarly, μ_2 induces a measure $\tilde{\mu}_2$ on $[0,1]$ as

$$\tilde{\mu}_2(B) = \mu_2(\rho_2^{-1}(B)) \quad \forall \text{ Borel set } B \subset [0,1].$$

Define a measure μ on $[-1,1]$ as follows.

$$\mu(B) = \tilde{\mu}_1([-1,0] \cap B) + \tilde{\mu}_2([0,1] \cap B) \quad \forall \text{ Borel set } B \subset [-1,1].$$

Define a map $\gamma : Y_1' \oplus Y_2' \to Y \oplus Y$,

$$\gamma(g_1, g_2) = \left((g_1 \circ \rho_1^{-1})\chi_{[-1,0]},\ (g_2 \circ \rho_2^{-1})\chi_{[0,1]}\right) \quad \forall g_1 \in Y_1', g_2 \in Y_2'.$$

This is a linear map.

$$\|\gamma(g_1, g_2)\|^2_{Y \oplus Y} = \int_{[-1,1]} \left(|(g_1 \circ \rho_1^{-1})\chi_{[-1,0]}|^2 + |(g_2 \circ \rho_2^{-1})\chi_{[0,1]}|^2\right) d\mu$$

$$= \int_{[-1,0]} |g_1 \circ \rho_1^{-1}|^2 \, d\mu + \int_{[0,1]} |g_2 \circ \rho_2^{-1}|^2 \, d\mu$$

$$= \underbrace{\int_{[-1,0]} |g_1(\rho_1^{-1}(x))|^2 \, d\tilde{\mu}_1(x)}_{\{1\}} + \underbrace{\int_{[0,1]} |g_2(\rho_2^{-1}(x))|^2 \, d\tilde{\mu}_2(x)}_{\{2\}}$$

$$\{1\} \overset{y = s_1^{-1}(x)}{=\!=} \int_{[-1,1)} |g_1(y)|^2 \, d\mu_1(y) = \|g_1\|^2_{Y_1'}$$

$$\{2\} \overset{y = s_2^{-1}(x)}{=\!=} \int_{[-1,1)} |g_2(y)|^2 \, d\mu_2(y) = \|g_2\|^2_{Y_2'} .$$

Then $\|\gamma(g_1, g_2)\|^2_{Y \oplus Y} = \|g_1\|^2_{Y_1'} + \|g_2\|^2_{Y_2'} = \|(g_1, g_2)\|^2_{Y_1' \oplus Y_2'}$. Thus, γ is a continuous map.

$\rightsquigarrow$ Unfortunately γ is not an isomorphism because it is not surjective. A different way to define γ is needed.