Homework Assignment 1
(due March 25)

Do two (or more) of the following six problems.

1. Let X be a Topological Linear Space (TLS) over the real numbers. Prove that for each neighborhood U of $0 \in X$ there exists a neighborhood V of $0 \in X$ such that $\bar{V} \subset U$. As usual, $\bar{V}$ denotes the closure of V.

Hint: Show that there exists an open neighborhood W of 0 such that $W - W \subset U$, and look at the sets W and $(X \setminus U) + W$.

2. Let $p \in (0,1)$ and let $L^p(0,1)$ be the set of all Lebesgue measurable functions $f\colon (0,1) \to R$ such that $I_p(f) = \int_0^1 |f(x)|^p\,dx$ is finite.
(a) Show that $L^p(0,1)$ is a linear space over R.
(b) Show that $d(f,g) = I_p(f-g)$ defines a metric on $L^p(0,1)$ with respect to which $L^p(0,1)$ is complete.
(c) Show that the metric is compatible with the linear structure, in the sense that the algebraic operations are continuous. In other words, $L^p(0,1)$ with the topology defined by d is a TLS.
(d) Show that the only non-empty open convex subset of $L^p(0,1)$ is $L^p(0,1)$ itself.
(e) Show that there are no non-trivial continuous linear functionals on $L^p(0,1)$.

Hints: For (a) use the inequality $(a+b)^p \le a^p + b^p$, which is valid for our range of $p's$ and $a, b > 0$. (As an optional part, you can prove this inequality.) For (d) use the following idea: given $f \in L^p(0,1)$, divide $(0,1)$ into small disjoint intervals $J_1, J_2, \ldots, J_n$ such that $I_p(f\chi_{J_k})$ is of order $1/n$, where χ_M is used to denote the characteristic function of M. Then write f as the convex combination $f = \frac{1}{n}\sum_j nf\chi_{I_j}$.

3. True or false? (Please give a justification for your answer.)
Let X, Y be Banach spaces over R and let $T\colon X \to Y$ be a linear mapping. Let us denote by w_X, w_Y respectively the weak topologies on X, Y. Then the following statement are equivalent:
(i) T is continuous as a linear mapping between Banach spaces X and Y.
(ii) T is continuous as a linear mapping between the Topological Linear Spaces (X, w_X) and (Y, w_Y).

In other words it is claimed that T is continuous with respect to the norm topologies if and only if it is continuous with respect to the weak topologies.

4. Let $X_1, X_2, \ldots X_n, Y$ be Banach spaces over R and let $M\colon X_1 \times X_2 \times \ldots \times X_n \to Y$ be a multi-linear mapping, i. e. a mapping which is linear in any single variable $x_1 \in X_1, \ldots, x_n \in X_n$, if the other variables are held fixed. Show that the following statements are equivalent:
(i) There exists $c \in [0,\infty)$ such that $||M(x_1,\ldots,x_n)|| \le c||x_1||\,||x_2||\ldots||x_n||$ for some any $(x_1,\ldots,x_n) \in X_1 \times \ldots \times X_n$.
(ii) M is continuous
(iii) M is separately continuous, i. e. if we fix $n-1$ of the variables $x_1 \in X_1, \ldots, x_n \in X_n$ and consider M as a function of the remaining variable, this function will be continuous.

5.* For $p \in [1,\infty]$ and $x \in R^n$ we let $||x||_p = (|x_1|^p + \ldots + |x_n|^p)^{\frac{1}{p}}$ (with the usual convention that $||x||_\infty = \max_j |x_j|$). Consider the metric on R^n defined by $d_p(x,y) = ||x-y||_p$. Recall that two metric spaces (X,d) and $(\tilde{X},\tilde{d})$ are called isometric if there exists a bijective map $\phi\colon X \to \tilde{X}$ such that $\tilde{d}(\phi(x),\phi(y)) = d(x,y)$ for all $x, y \in X$. Show that if m, n are natural numbers and $p, q \in [1,\infty]$, the metric spaces (R^m, d_p) and (R^n, d_q) are isometric if and only if $m = n$ and $p = q$. It is enough to do the case $m = n$, $p \in \{1, 2, \infty\}$ and $q = 2$ to receive full credit for the problem.

6.* Let X be a finite dimensional normed space with the following property: for each two vectors x, y of the same length (i. e. $||x|| = ||y||$) there exists a linear isometry $T\colon X \to X$ such that $Tx = y$. Show that X is a Hilbert space, i. e. the norm arises from a positive-definite quadratic form in the usual way.
(Hint: use averaging over the group of the linear isometries to construct a scalar product.)
Remark: It may be open whether the result remains valid in the category of separable Banach spaces.